# On Moduli of Smoothness of Bernstein's Polynomials in $L_p(\mu)$

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**Abstract.**

Let $f$ be bounded $\mu-$measurable function, that is $f \in L_p(\mu), 1 \le p < \infty$.The main results of this paper describes some properties that defined on the weighted Ditzian-Totik modulus of smoothness on $f$ by using Bernstein's polynomial.

## 1. Introduction and main results

Let $L_p(\mu)$ , $1 \le \mathrm{p} < \infty$ consists of all $\mu-$measurable function $f$ for which $\|f\|_{p,\mu} < \infty$ , where

$$(1.1) \qquad \|f\|_{p,\mu} = \left(\int |f|^p d\mu\right)^{\frac{1}{p}} ,$$

and let $f \in L_p(X)$ , $\mathrm{X} = [0,1]$ , then we denote

$$(1.2) \qquad L_p(X) = \left\{ f : \|f\|_p = \left( \int_X |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}$$

and $f$ defined on $X$ (a.e.).

Also , let $f$ be bounded $\mu-$measurable function , we denoted the Bernstein's polynomials of $f$ by

$$(1.3) \qquad B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n})\, P_{k,n}(x), \text{ where } \quad P_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

for all positive integers $n$ ,and $\mu$ be Lebsegue measure on $X$ ,see [9].

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Let us consider the family of locally global norms in the forms for $\delta > 0$ and $1 \le p < \infty$, then

(1.4) $$\|f\|_{\delta,p} = \left\{ \int_X \left( \sup\{ |f(u)| : u \in N(x,\delta) \} \right)^p dx \right\}^{\frac{1}{p}},$$

also ,if $f \in L_p(\mu)$ ,then we define the locally global norms of $f$ by

Let $\phi$ be a function such that , $\phi(u,\delta) = \delta\varphi(x) + \delta^2$

(1.5) $$\|f\|_{\delta,p,\mu} = \left\{ \int_X \left( \sup\{ |f(u)| : u \in N(x,\delta) \} \right)^p d\mu(x) \right\}^{\frac{1}{p}}$$

(1.6) $$\|f\|^{\varphi}_{\delta,p,\mu} = \left\{ \int_x \left( \sup\{ |f(u)| : u \in N(x,\phi(x,\delta)) \} \right)^p d\mu(x) \right\}^{\frac{1}{p}}$$

where , $N(x,\delta) = \{ u \in X : |x-u| \le \delta \}$ , $\delta \in R^+$.

We will use the moduli of smoothness which are connected with difference of higher order, that is the $rth$ symmetric difference of $f$ is given by

(1.7) $$\Delta_h^k(f,x) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - \frac{kh}{2} + ih), & x \mp \frac{kh}{2} \in X \\ 0 & , \text{ o.w.} \end{cases}$$

Then the $rth$ usual modulus of smoothness of $f \in L_p(\mu)$ is defined by

(1.8) $$\omega_k(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\| \Delta_h^k(f,x) \right\|_{p,\mu},$$

and the Ditzian-Totik modulus of smoothness of $f$ is defined by

(1.9) $$\omega_k^{\varphi}(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\| \Delta_{h\varphi(.)}^k(f,x) \right\|_{p,\mu},$$

where ,in this applications the $\varphi$ usually used $\varphi(.) = \varphi(x) = (x(1-x))^{\frac{1}{2}}$ for $x \in [0,1]$.
The weighted Ditzian-Totik modulus of smoothness of $f$ is defined by

(1.10) $$\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\| \varphi^r(x) \Delta_{h\varphi(.)}^k(f,x) \right\|_{p,\mu},$$

where , $k,r$ denoted nonnegative integers and $k + r > 0$.

By [5] For a function $f \in L_p(X)$, $1 \le p < \infty$ ,then we have

(1.11) $$\omega_{k,r}^{\varphi}(f,\delta)_p \approx \tilde{K}_{k,r,\varphi}(f,\delta)_p$$

where , $\tilde{K}_{k,r,\varphi}$ is the weighted Ditzian-Totik $\tilde{K}$-functional defined by

$$\tilde{K}_{k,r,\varphi}(f,\delta)_p = \inf_{\substack{P_n \in \Pi_n \\ n = \left[\frac{1}{t}\right]}} \left\{ \left\| \varphi^r(x)(f - P_n) \right\|_p + \delta^k \left\| \varphi^k P_n^{(k)} \right\|_p \right\}.$$

Let $C^{\ell}(X)$ denoted the set of $\ell-$times continuously differentiable functions on $[0,1]$.

Also , for $1 \le p < \infty$ ,then the Sobolev space $W_p^\ell$ is a collection of all functions $f$ denoted on $X$ ,such that , $f^{(\ell-1)}$ is absolutely continuous and $f^{(\ell-1)} \in L_p(X)$.

**Our main results are the following:**

***Theorem*** 1.1

Let $f$ be bounded $\mu-$measurable function defined on $X$ ,and $k+r>0$ .then we have

(1.12) $$\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu} \le C\,\delta^k \left\| f^{(k)} \right\|_{p,\mu},$$

with the equivalence constants depending only on $r$ and $p$.

***Theorem*** 1.2

Let $f \in L_p(\mu)$ , $1 \le p < \infty$ , $\delta > 0$,then we have

(1.13) $$\left\| B_n(f) - f \right\|_{\delta,p,\mu} \le C\,\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu},$$

with the constants depending only on $r$ and $p$.

***Theorem*** 1.3

For every $r,n \in N$ and $f \in L_p(\mu) \cap C^{\ell}(X)$,with $1 \le p < \infty$ ,$0 < \ell < r$ and $k \ge 1$, $\delta > 0$,we have

(1.14) $$\omega_{k+r-\ell,\ell}^{\varphi}(f^{(\ell)},\delta)_{p,\mu} \le C\,2^{-\ell}\,\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu},$$

with the constants depending only on $r,k$ and $p$.

***Theorem*** 1.4

Let $k \ge 1, n \ge 1, 0 \le \delta \le 1/n$ and $1 \le p < \infty$,for any Bernstien's polynomials $B_n(f)$,then we have

(1.15) $$\omega_{k,r}^{\varphi}(B_n(f),\delta)_{p,\mu} \le C\,\delta^k \left\| \varphi^k P_{k,n}(x) \right\|_{p,\mu},$$

with the constants depending only on $r,k$ and $p$.

## 2.Basic Results

In this section we mention some basic results ,which will be used to prove the main results.

***Lemma*** 2.1 [6]

Let $f \in L_p(X)$ , $1 \le p < \infty$, $\alpha,\beta > 0$, $h \in R$,then we have

i. $\Delta_h^{\alpha}(\Delta_h^{\beta}(f,x)) = \Delta_h^{\alpha+\beta}(f,x)$ for almost every $\alpha$,

ii. $\left\| \Delta_h^{\alpha+\beta}(f,x) \right\|_p \le C(\alpha) \left\| \Delta_h^{\beta}(f,x) \right\|_p$.

***Lemma*** 2.2 [1]

Let $f$ be a bounded $\mu-$measurable function and $1 \le p < \infty$, then we have

(2.1) $$\|f\|_p \le C(p) \|f\|_{p,\mu}.$$

***Lemma*** 2.3 [2]

For $k \in N$ and $1 \le p < \infty$,there exists constant C with depends only on $k$ ,so that ,for any $f \in L_p[-1,1]$ and $n \ge k-1$there exists a polynomial $P_n \in \Pi_n$,such that

(2.2) $$\|f - P_n\|_p \le C(k)\,\omega_k^{\varphi}(f, n^{-1})_p,$$

***Lemma*** 2.4 [7]

Let $n, r \in N$, $n \in N_0$ , $0 < \ell \le r$ and let $f \in C^{\ell-1}[-1,1]$, then for any $1 \le p < \infty$ and $\delta > 0$,we have

(2.3) $$\omega_{k+r}^{\varphi}(f, \delta)_p \le C(\ell, k, p)\,\omega_{k,r}^{\varphi}(f^{(\ell)}, \delta)_p,$$

in particular , in case $k = 0$ ,then

(2.4) $$\omega_{\ell}^{\varphi}(f, \delta)_p \le C(\ell, p)\,\delta^{\ell}\left\|\varphi^{\ell}\ f^{(\ell)}\right\|_{p,\mu}.$$

***Lemma*** 2.5 [1]

Let $f$ be a bounded $\mu-$measurable function then for $1 \le p < \infty$, we have:

(i) $$\|f\|_{\delta,p} \le C(p)\|f\|_{\delta,p,\mu},$$

(ii) $$\|f\|_{\delta,p,\mu} \le C(p)\|f\|_{p,\mu}.$$

***Lemma*** 2.6 [2]

Let $B_n(f)$ be a Bernstein's polynomial and $k \ge 0$, then

(2.5) $$\Delta_h^k(B_n(f), x) = \sum_{i=0}^{k} \frac{P_{k,n}}{(2i)!} h^{k+ih} \alpha_{k+2i}^{2i},$$

(2.6) $$\Delta_{h\varphi(.)}^k(B_n(f), x) = \sum_{i=0}^{k} \frac{\varphi^{k+2i}(x) P_{k,n}(x)}{(2i)!} h^{k+ih} \alpha_{k+2i}^{2i}.$$

***Lemma*** 2.7 [3]

For $p \ge 1$ and $\alpha_i > 0$ then we have

(2.7) $$\left|\sum_{i=0}^{n} \alpha_i \beta_i\right|^p \le \sum_{i=0}^{n} \alpha_i |\beta_i|^p \text{ ,where } \sum_{i=0}^{n} \alpha_i = 1.$$

***Lemma*** 2.8 [1]

Let $\mu$ be a non-decreasing function on $R$, satisfying: $\mu(y) - \mu(x) =$ constant, and $1 \le p < \infty$, We put: $\omega_\mu(\delta) = \sup\limits_{0 < y - x \le \delta} (\mu(y) - \mu(x)), \delta > 0$ ,and

$\left(\frac{1}{n}\sum_{k=0}^{n-1} \max\limits_{x \in I_k} |P_n|^p\right)^{\frac{1}{p}} \le C(p)\|P_n\|_p$ ,where $P_n$ is algebraic polynomials of degree at most $n$ and $I_k = \left[\frac{k}{n}, \frac{k+1}{n}\right]$. then:

(2.8) $$\|P_n\|_{p,\mu} \le C(p)\left(n\omega_\mu\left(\frac{1}{n}\right)\right)^{\frac{1}{p}}\|P_n\|_p$$

***Lemma*** 2.9 (Minkowsk's Inequality) [8]

If $p \ge 1$ and $f, g \in L_p(\mu)$, then $f+g \in L_p(\mu)$ and

(2.9) $$\left[\int_X |f+g|^p d\mu\right]^{1/p} \le \left[\int_X |f|^p d\mu\right]^{1/p} + \left[\int_X |g|^p d\mu\right]^{1/p}.$$

***Lemma*** 2.10 [4]

For $f \in L_p$, $1 \le p \le \infty$ then we have

(i) $\|f\|_p \le \|f\|_{\delta,p} \le \|f\|_{\delta,\infty} \le \|f\|_\infty$,

(ii) $\|f\|_p \le \|f\|^{\Phi}_{\delta,p} \le \|f\|^{\Phi}_{\delta,\infty} \le \|f\|_\infty$.

***Lemma*** 2.11 [10]

For $k, n \in N, 0 \le k \le n$ then we have

(2.10) $\|P_{k,n}\|^{\varphi}_{1/n,1} \le \frac{c}{n}$ , where $P_{k,n} = \binom{n}{k} x^k (1-x)^{n-k}$ , $x \in X$ .

***Lemma*** 2.12 [1]

For $f \in L_p(\mu)$,$1 \le p < \infty$ and $n \in N$ ,then we have

(i) $\|f\|_{1/n,p,\mu} \le \|f\|^{\Phi}_{1/\sqrt{n},p,\mu}$,

(ii) $\|f\|^{\Phi}_{1/n,p,\mu} \le \|f\|^{\Phi}_{1/\sqrt{n},p,\mu}$.

***Lemma*** 2.13

Let $f$ be a bounded $\mu-$measurable function on $X$ ,and $g_n \in \Pi_n \cap L_p(\mu)$,then we have

(2.11) $\|f - g_{k,n}\|_{\delta,p,\mu} \le C(r,k,p) n^{1/p} 2^r \ \omega^{\varphi}_{k,r}(f,\delta)_{p,\mu}$.

***Proof***:

By using Lemma 2.5(ii) and equations (2.8) , (2.2),(1.9) and since $\varphi^r(x) < \frac{1}{2^r}$ ,and for $g_{k,n} \in W^k_p(X)$,then we have

$$\|f - g_{k,n}\|_{\delta,p,\mu} \le \|f - g_{k,n}\|_{p,\mu} \le C(p) n^{1/p} \|f - g_{k,n}\|_p$$

$$\le C(k,p)n^{1/p}\ \omega_k^{\varphi}(f,\delta)_p$$

$$=C(k,p)n^{1/p}\sup_{0<h\le\delta}\left\|\Delta^k_{h\varphi(.)}(f,x)\right\|_p$$

$$\le C(k,p)n^{1/p}\,2^r\sup_{0<h\le\delta}\left\|\varphi^r\Delta^k_{h\varphi(.)}(f,x)\right\|_p$$

$$\le C(r,k,p)n^{1/p}\,2^r\sup_{0<h\le\delta}\left\|\varphi^r\Delta^k_{h\varphi(.)}(f,x)\right\|_{p,\mu}$$

$$\le C(r,k,p)n^{1/p}\,2^r\,\omega^{\varphi}_{k,r}(f,\delta)_{p,\mu}\ ♣$$

***Lemma*** 2.14

Let $f$ be a bounded $\mu-$measurable function on $X$ ,for $1\le p<\infty$,then we have

(2.12) $$\|B_n(f)\|_{\delta,p,\mu}\le C\|f\|_{\delta,p,\mu}$$

***Proof***:

From using (1.5) ,Lemma 2.11(i), (2.9),(2.7),(1.4),(2.10) and Lemma 2.5(i) ,then we have

$$\|B_n(f)\|_{\delta,p,\mu}\le\left[\int_X\left(\sup\left\{\left|\sum_{k=0}^{n}f\left(\frac{k}{n}\right)P_{k,n}(t)\right|:t\in N(x,\delta)\right\}\right)^p d\mu(x)\right]^{1/p}$$

$$\le\left[\int_X\left(\sup\left\{\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|P_{k,n}(t):t\in N(x,\delta)\right\}\right)^p d\mu(x)\right]^{1/p}$$

$$\le\left(\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|^p\right)^{1/p}\left(\int_X\left(\sup\left\{P_{k,n}(t):t\in N(x,\delta)\right\}\right)^p d\mu(x)\right)^{1/p}$$

$$\le\left(\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|^p\right)^{1/p}\left\|P_{k,n}\right\|^{\varphi}_{1/\sqrt{n},p,\mu}\le\left(\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|^p\right)^{1/p}\left\|P_{k,n}\right\|^{\varphi}_{1/n,p,\mu}$$

$$\le\left(\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|^p\right)^{1/p}\left\|P_{k,n}\right\|^{\varphi}_{1/n,p,\mu}\le\frac{c(p)}{n}\left(\sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|^p\right)^{1/p}$$

$$\le C(p)\|f\|_{\delta,p}\le C(p)\|f\|_{\delta,p,\mu}\ ♣$$

***Lemma*** 2.15

Let $f$ and $g$ be bounded $\mu-$measurable functions for $1\le p<\infty$, then we have

(2.13) $$\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu}\le\omega_{k,r}^{\varphi}(f-g,\delta)_{p,\mu}+\omega_{k,r}^{\varphi}(g,\delta)_{p,\mu}.$$

***Proof***:

By using (1.10) , (1.1) , (1.7) and (2.9) ,then

$$\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu}=\omega_{k,r}^{\varphi}(f-g+g,\delta)_{p,\mu}$$

$$=\sup_{0<h\le\delta}\left\|\varphi^{r}(.)\Delta_{h\varphi(.)}^{k}(f-g+g,x)\right\|_{p,\mu}$$

$$=\sup_{0<h\le\delta}\left\{\int_X\left|\varphi^{r}(.)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(f-g+g)(x-\frac{kh\varphi}{2}+ih\varphi)\right|^{p}d\mu(x)\right\}^{1/p}$$

$$\le\sup_{0<h\le\delta}\left\{\int_X\left|\varphi^{r}(.)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(f-g)(x-\frac{kh\varphi}{2}+ih\varphi)\right|^{p}d\mu(x)\right\}^{1/p}$$

$$+\sup_{0<h\le\delta}\left\{\int_X\left|\varphi^{r}(.)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}g(x-\frac{kh\varphi}{2}+ih\varphi)\right|^{p}d\mu(x)\right\}^{1/p}$$

$$\le\sup_{0<h\le\delta}\left\{\int_X\left|\varphi^{r}(.)\Delta_{h\varphi(.)}^{k}(f-g,x)\right|^{p}d\mu(x)\right\}^{1/p}+\sup_{0<h\le\delta}\left\{\int_X\left|\varphi^{r}(.)\Delta_{h\varphi(.)}^{k}(g,x)\right|^{p}d\mu(x)\right\}^{1/p}$$

$$\le\omega_{k,r}^{\varphi}(f-g,\delta)_{p,\mu}+\omega_{k,r}^{\varphi}(g,\delta)_{p,\mu}$$ ♣

## ***3.Proof of Theorems***

***Proof of Theorem 1.1***

We may assume the Bernstein's polynomial $B_n(f)\in W_p^k(X)$,and we introduce a $\tilde{\mathrm{K}}_{k,r,\varphi}-$functional, by using (1.11) then

$$\tilde{\mathrm{K}}_{k,r,\varphi}(f,\delta)_p=\inf_{\substack{P_n\in\Pi_n\\ n=\left[\frac{1}{\delta}\right]}}\left\{\left\|\varphi^{r}(x)(f-P_n)\right\|_p+\delta^{k}\left\|\varphi^{k}P_n^{(k)}\right\|_p\right\}$$

also, since $\varphi^{k}(x)\le\frac{1}{2^k}$ , $\forall x\in X$ ,and by (1.10)

$$\omega_{k,r}^{\varphi}(f,\delta)_p\approx\tilde{\mathrm{K}}_{k,r,\varphi}(f,\delta)_p$$

then we have

$$\omega_{k,r}^{\varphi}(f,\delta)_p \le C(r)\left(\|(f-B_n(f))\|_p + \delta^k \left\|\varphi^k(.)B_n^{(k)}(f)\right\|_p\right)$$

also, by (2.2) and (2.4) , then

$$\begin{aligned}\|f-B_n(f)\|_p &\le C(k)\omega_k^{\varphi}(f,\delta)_p \\ &\le C(k,p)\delta^k \left\|\varphi^k f^{(k)}\right\|_p \\ &\le C(k,p)2^{-r}\delta^k \left\|f^{(k)}\right\|_p\end{aligned}$$

hence by using (2.1)

$$\begin{aligned}\omega_{k,r}^{\varphi}(f,\delta)_p &\le C(k,p)2^{-r}\delta^k \left\|f^{(k)}\right\|_p \\ &\le C(k,p)2^{-r}\delta^k \left\|f^{(k)}\right\|_{p,\mu}\end{aligned}$$

so that ,by using (2.8),then we have

$$\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu} \le \omega_{k,r}^{\varphi}(f,\delta)_p$$

hence, we complete proof of Theorem 1.1 ♣

***Proof of Theorem 1.2***

Suppose that any polynomial $g_n \in \Pi_n \cap L_p(\mu), 1 \le p < \infty, \delta > 0$ then we have

$$\begin{aligned}\|B_n(f)-f\|_{\delta,p,\mu} &= \|B_n(f)-f-B_n(g_n)+B_n(g_n)-g_n+g_n\|_{\delta,p,\mu} \\ &\le \|B_n(f)-B_n(g_n)+\|_{\delta,p,\mu} + \|B_n(g_n)-g_n\|_{\delta,p,\mu} + \|f-g_n\|_{\delta,p,\mu} \\ &\le \|B_n(f-g_n)+\|_{\delta,p,\mu} + \|B_n(g_n)-g_n\|_{\delta,p,\mu} + \|f-g_n\|_{\delta,p,\mu}\end{aligned}$$

hence , by using (2,12), then we have

$$\|B_n(f)-f\|_{\delta,p,\mu} \le C(p)\|f-g_n\|_{\delta,p,\mu} + \|B_n(g_n)-g_n\|_{\delta,p,\mu}$$

So that

$$\begin{aligned}\|B_n(g_n)-g_n\|_{\delta,p,\mu} &= \|B_n(g_n)-f+f-g_n\|_{\delta,p,\mu} \\ &\le \|B_n(g_n)-f\|_{\delta,p,\mu} + \|f-g_n\|_{\delta,p,\mu}\end{aligned}$$

also , by using (2.11) and Lemma 2.5(ii) ,then

$$\begin{aligned}\|B_n(f)-f\|_{\delta,p,\mu} &\le C(p)\left(\|B_n(g_n)-f\|_{p,\mu} + \|f-g_n\|_{p,\mu}\right) \\ &\le C(r,k,p)2^{-r}n^{1/p}\omega_{k,r}^{\varphi}(f,\delta)_{p,\mu}\ ♣\end{aligned}$$

***Proof of Theorem 1.3***

Since $f \in L_p(\mu) \cap C^{\ell}(X)$ and $B_n(f)$ is Bernstein's polynomial defined on $X$ , $0 < \ell < r$ and since $\varphi(x) \le \frac{1}{2}$ ,for all $x \in X$ ,also by using Lemma 2.1(ii), (2.8) and (1.10) ,then we have

$$\omega^{\varphi}_{k+r-\ell,\ell}(f^{(\ell)},\delta)_{p,\mu} \le \omega^{\varphi}_{k+r-\ell,\ell}(f^{(\ell)},\delta)_{p}$$
$$= \sup_{0<p\le\delta}\left\|\varphi^{r}(.)\Delta^{k+r-\ell}_{h\varphi(.)}(f^{(\ell)},x)\right\|_{p}$$
$$= \sup_{0<p\le\delta}\left\|\varphi^{r}(.)\Delta^{k+r-\ell}_{h\varphi(.)}(f^{(\ell)}-B_n(f),x)\right\|_{p}$$
$$\le C(r,k)2^{-\ell}\left\|f^{(\ell)}-B_n(f)\right\|_{p} \le C(r,k)2^{-\ell}\left\|f^{(\ell)}-B_n(f)\right\|_{\delta,p}$$
$$\le C(r,k)2^{-\ell}\left\|f^{(\ell)}-B_n(f)\right\|_{\delta,p,\mu}$$

now, by using Theorem 1.2 ,then we will finished the proof of this theorem ♣

***Proof of Theorem 1.4***

In view (2.5) and (2.6) ,we can assume $0 \le t \le t_0 = \min\left(\frac{1}{Dk^2n},\frac{1}{\sqrt{D}k^2n}\right)$ , where $D = 3^{1/n}\,p$ ,so that $\sum_{i=1}^{\infty}\frac{1}{D^{2ip}} = \frac{1}{2}$ and recalling $|\alpha_i| < \frac{k}{2}$ and by (1.3),then for any $0 \le h \le t \le t_0$ and since $\varphi^r(x) \le 2^{-r}$ ,then we have

$$\frac{\left\|\varphi^{k+i}(x)P_{k,n}(x)\right\|_{p,\mu}}{(i)!}h^{k+i}|\alpha_{k+i}|^{i} \le Cn(k+i)\frac{\left\|\varphi^{k+i-1}(x)P_{k,n}(x)\right\|_{p,\mu}}{Dk^2n}h^{k+i-1}\frac{k|\alpha_{k+i}|^{i-1}}{2}$$
$$= \frac{\left\|\varphi^{k+i-1}(.)P_{k,n}(.)\right\|_{p,\mu}}{(i-1)!D}\frac{(k+i)h^{i-1}|\alpha_{k+i}|^{i-1}}{2ik}$$
$$\le \frac{\left\|\varphi^{k+i-1}(.)P_{k,n}(.)\right\|_{p,\mu}}{(i-1)!D}h^{i-1}|\alpha_{k+i}|^{i-1}$$
$$\le \frac{\left\|\varphi^{k}(.)P_{k,n}(.)\right\|_{p,\mu}}{D}$$

where, we have used the fact $k+i \le 2ik$ for $i,k \ge 1$.
Now, we have

$$\left\|\varphi^{r}(.)\Delta^{k}_{h\varphi(.)}(B_n(f),x)\right\|_{p,\mu} \le \frac{h^k}{2^r}\left[\sum_{i=0}^{k}\frac{\left\|\varphi^{k+2i}(x)P_{k,n}(x)\right\|_{p,\mu}}{(2i)!}h^{k+2i}|\alpha_{k+2i}|^{2i}\right]$$
$$\le \frac{h^k}{2^r}\left\|\varphi^{k}(x)P_{k,n}(x)\right\|_{p,\mu}\left(1+\sum_{i=1}^{k}\frac{1}{D^{2ip}}\right)$$
$$\le \frac{3h^k}{2^{r+1}}\left\|\varphi^{k}(x)P_{k,n}(x)\right\|_{p,\mu} \le C(r,k,p)h^k\left\|\varphi^{k}(x)P_{k,n}(x)\right\|_{p,\mu}.$$

Therefore ,by (1.9),then

$$\sup_{0<h\le\delta}\left\|\varphi^{r}(x)\Delta^{k}_{h\varphi(.)}(B_n(f),x)\right\|_{p,\mu} \le C(r,k,p)\sup_{0<h\le\delta}\left\{h^k\left\|\varphi^{k}(x)P_{k,n}(x)\right\|_{p,\mu}\right\}$$

Hence ,

(3.1) $$\omega_{k,r}^{\varphi}(B_n(f),\delta)_{p,\mu} \le C\delta^k \left\|\varphi^k(x)P_{k,n}(x)\right\|_{p,\mu}$$

where , the constant $C$ dependent on $r,k$ and $p$ ♣

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**المستخلص:**

الهدف من هذا البحث هو دراسة بعض الخواص الاساسية لمقياس النعومة المرجّح لكل من دتيزين وتوتك $\omega_{k,r}^{\varphi}$ للدوال المقيدة والقابلة للقياس - $\mu$ في الفضاءات $L_p(\mu)$ ، $1 \le p < \infty$ باستخدام متعددة حدود برنشتاين الجبرية .