# Spectral method for solving fractal fractional differential equations

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## Abstract

Lately introduced fractal fractional Caputo - Fabrizio operator. By substituting the single kernel at the classic derivative of fractal fractional Caputo with the ordinary kernel This modern operator was derived. We introduce some beneficial characteristics relied on the qualifier of fractal fractional Caputo - Fabrizio. Here, we extend Caputo-Fabrizio for nonlinear fractal fractional differential equations. We apply Legendre operational matrix relied on this modern operator and then, we employ it to solve the differential equations determined in the sense of fractal fractional Caputo-Fabrizio. To show the simplicity and precision of the suggested technicality Some numerical examples are given.

**Keywords:** Fractal fractional Caputo-Fabrizio operator, Fractal fractional Caputo derivative, The fractal fractional differential equations, Operational matrix, Legendre polynomials.

## 1. Introduction

In [1], through replacement the single kernel in the classic derivative of fractal fractional Caputo with the ordinary kernel. Exponential kernel has used by the fractal fractional Caputo - Fabrizio (FFCF) operator, which is a non-single kernel FFCF have suggested the modern operator. It does not only have two various exemplifications for locative and temporal variables, but the entire impact of the memory can be pertraged else [1]. In heat convey model this modern operator has been successfully utilized [2], Freedman and nonlinear Baggs model [3], the equation of space time fractal fractional propagation [4], mathematical paradigms for an unstable Maxwell fluid flux and its thermic demeanor in a micro-pipe [5], mass-spring-damper system [6], and fractal fractional Maxwell liquid [7]. In this research, to the trouble determined in the sense of FFCF operator some existent analytical and numerical methods to solve fractal fractional calculus trouble have been expanded, amongst them is the paper of Morales-Delgado et.al. In [8], to detect the essential solution for the fractal fractional advection propagation equation with the exporter the authors employ integral converts where the derivative is considered in FFCF sense. Nevertheless, since FFCF operator is comparatively modernistic, there are still comparatively bounded works conducted to gain the authoritative, precision and simple solving for the fractional calculus trouble determined in FFCF operator.

Moreover, in solving several fractal fractional calculus troubles operational matrix (OPM) method relied on perpendicular function was successfully utilized which are acquainted in classic sense of fractal fractional Caputo. The method minimizes like these troubles to solve a system of algebraic equations, thence extremely simplify the trouble. In this research field the major contribution starts with the seminal paper concerning Legendre wavelets OPM through Yousefi and Razzaghi [9] and OPM relied on Legendre polynomials in [10]. to solve changeable arrangement fractal fractional differential equations this OPM process has been expanded as in [11]. Nevertheless, there are still no OPM related processes to solve the troubles determined in FFCF operator. Thus, in this paper, to represent the FFCF operator we derive the modern OPM relied on shifted Legendre polynomials (SLP), which is achieved by first deriving the formula for ${}^{FFCF}D^{\omega,\beta}(z-a)^{\beta}$ of a public derivative order $n<\omega<n+1$. Thereafter, by pursuing the work of Dehghan and Saadatmandi [10], we derive the OPM relied on SLP to solve troubles in FFCF sense. It is the first time that the OPM is used for solving the problem in FFCF sense. The goal for this paper is to solve nonlinear fractal fractional Caputo Fabrizio by using OPM relied on Legendre polynomials.

The article is arranged as the following: Section 2 shortly clarify fractal fractional Caputo and FFCF fractional derivative. In section 3 the Legendre OPM of fractal fractional derivative is gained. In section 4 the suggested method is utilized to many examples. Also, a conclusion is presented in section 5.

## 2. Basic Concepts

### 2.1. Caputo and FFCF fractal derivative

The fractal fractional derivative of Caputo left-sided ${}^{C}\boldsymbol{D}^{\boldsymbol{\omega},\boldsymbol{\beta}}$ of a function $\boldsymbol{y}(\boldsymbol{z})\in\boldsymbol{Y}^{\mathbf{1}}(\mathbf{0},\boldsymbol{b})$ with $\mathbf{0}<\boldsymbol{\omega}<\mathbf{1}$, is acquainted as

$$ {}^{C}\boldsymbol{D}^{\boldsymbol{\omega},\boldsymbol{\beta}}\boldsymbol{y}(\boldsymbol{z})=\frac{\mathbf{1}}{\boldsymbol{\Gamma}(\mathbf{1}-\boldsymbol{\omega})}\int_{\mathbf{0}}^{\boldsymbol{z}}\frac{\boldsymbol{dy}(\boldsymbol{\tau})}{\boldsymbol{d\tau}^{\boldsymbol{\beta}}}(\boldsymbol{z}-\boldsymbol{\tau})^{-\boldsymbol{\omega}}\boldsymbol{d\tau} \qquad (\mathbf{1}) $$

The fractal fractional derivative of fractal fractional Caputo is considered to be one of the most beneficial definitions of the fractal fractional derivatives utilized in several fields of engineering and science. Nevertheless, the modern qualifier proposed of FFCF presumes two various representations for the temporal and locative variable. Indeed, they purported that the classic definition which is given by eq. (1) seems to be especially appropriate for mechanistic phenomenon, concerning with elasticity, exertion, harm and electromagnetic hysteresis. it looks more suitable to use the novel FFCF operator when these influences are not exist [1].

**Definition 2.2.** In [1], FFCF submitted the novel operator through substituting the single Kernel $(z-\tau)^{-\omega}$with $e^{\frac{-\omega(z-\tau)}{1-\omega}}$and $\frac{1}{\Gamma(1-\omega)}$ with $\frac{M(\omega)}{1-\omega}$ in Eq (1) to acquire:

For $0<\omega<1$ , $a \in [-\infty, z)$ and $y(z) \in Y^1(0,b)$ , $b>a$ the FFCF operator or more accurately the left-sided FFCF operator of $y(z)$ is acquainted as:

$$ {}^{C}\boldsymbol{D}^{\boldsymbol{\omega},\boldsymbol{\beta}}\boldsymbol{y}(\boldsymbol{z})=\frac{\boldsymbol{M}(\boldsymbol{\omega})}{\mathbf{1}-\boldsymbol{\omega}}\int_{a}^{z}\frac{\boldsymbol{d}\boldsymbol{y}(\boldsymbol{\tau})}{\boldsymbol{d}\boldsymbol{\tau}^{\boldsymbol{\beta}}}\boldsymbol{e}^{\frac{-\boldsymbol{\omega}(\boldsymbol{z}-\boldsymbol{\tau})}{\mathbf{1}-\boldsymbol{\omega}}}\boldsymbol{d}\boldsymbol{\tau} \qquad (\mathbf{2}) $$

Where the normalization function is $M(\omega)$ for example $M(0)=M(1)=1$

Here $\boldsymbol{\omega}$ denotes the fractal fractional order, $\beta$ denotes the fractal order and the integral has power law kernel and,

$$ \frac{\boldsymbol{d}\boldsymbol{y}(\boldsymbol{\tau})}{\boldsymbol{d}\boldsymbol{z}^{\boldsymbol{\beta}}}=\lim_{z\to\tau}\frac{\boldsymbol{y}(\boldsymbol{z})-\boldsymbol{y}(\boldsymbol{\tau})}{\boldsymbol{z}^{\boldsymbol{\beta}}-\boldsymbol{\tau}^{\boldsymbol{\beta}}} = \frac{\mathbf{1}}{\boldsymbol{\beta}\boldsymbol{\tau}^{\boldsymbol{\beta}-\mathbf{1}}}\frac{\boldsymbol{d}}{\boldsymbol{d}\boldsymbol{\tau}}\boldsymbol{y}(\boldsymbol{\tau}) $$

In [1], the definition of Eq. (2) is expanded by FFCF also for the state of $\boldsymbol{n}<\boldsymbol{\omega}<\boldsymbol{n}+\mathbf{1}$ with the additional presumption that $y^{(k)}(\omega)=0, k=1,2,\dots,n$:

$$ {}^{FFCF}\boldsymbol{D}_{a^{+};z}^{\boldsymbol{\omega},\boldsymbol{\beta}}\boldsymbol{y}(\boldsymbol{z}) = {}^{FFCF}\boldsymbol{D}_{a^{+};z}^{\boldsymbol{\upsilon}}(\boldsymbol{D}^{\boldsymbol{n}}\boldsymbol{\tau}(\boldsymbol{z})) $$

$$ =\frac{\boldsymbol{M}(\boldsymbol{\upsilon})}{\mathbf{1}-\boldsymbol{\upsilon}}\int_{a}^{z}\boldsymbol{y}^{(\boldsymbol{n}+\mathbf{1})}\,\boldsymbol{e}^{\frac{-\boldsymbol{\upsilon}(\boldsymbol{z}-\boldsymbol{\tau})}{\mathbf{1}-\boldsymbol{\upsilon}}}\boldsymbol{d}\boldsymbol{\tau} \qquad (\mathbf{3}) $$

$$ \boldsymbol{y}^{(\boldsymbol{n}+\mathbf{1})}(\boldsymbol{Z})=\boldsymbol{D}^{(\boldsymbol{n}+\mathbf{1})}\boldsymbol{y}(\boldsymbol{z})=\boldsymbol{D}^{\lceil\boldsymbol{\omega}\rceil}\boldsymbol{y}(\boldsymbol{z}) $$

Where $\upsilon, n$ are the decimal part and integer part of $\omega \in \mathbb{R}_{+}$, respectively.

**Note:** We let $\omega=n+\upsilon$, where $\upsilon$ is the fractal fractional part and $n$ to indicate the floor( $\omega$) or $\lfloor\omega\rfloor$ (i.e., integer part). Also, $\lceil\omega\rceil$ is utilized to indicate the ceil $(\omega)$.

## 2.2 Some Properties of The Shifted Legendre Polynomials

The notable Legendre polynomials are acquainted with this interval [-1,1] and can be resolved with the guide of the accompanying repeat formulation [10]:

$$ \boldsymbol{\mathcal{L}}_{\delta+1}(\boldsymbol{t})=\frac{\mathbf{2}\boldsymbol{\delta}+\mathbf{1}}{\boldsymbol{\delta}+\mathbf{1}}\,\boldsymbol{t}\,\boldsymbol{\mathcal{L}}_{\delta}(\boldsymbol{t})-\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}+\mathbf{1}}\,\boldsymbol{\mathcal{L}}_{\delta-1}(\boldsymbol{t}) \quad , \quad \boldsymbol{\delta}=\mathbf{1},\mathbf{2},\dots \qquad (\mathbf{4}) $$

Where $\mathcal{L}_0(t) = 1$ and $\mathcal{L}_1(t) = t$. For utilizing these polynomials on the interval $z \in [0,1]$ we limit which is named SLP through presenting the alteration of variable $t = 2z - 1$. Let the SLP $\mathcal{L}_\delta(2z-1)$ indicated through $P_\delta(z)$ . Then $P_\delta(z)$ can be acquired as the following:

$$\boldsymbol{P_{\delta+1}(z) = \frac{(2\delta+1)(2z-1)}{(\delta+1)} P_\delta(z) - \frac{\delta}{\delta+1} P_{\delta-1}(z)}, \quad \boldsymbol{\delta = 1, 2, \dots} \qquad \mathbf{(5)}$$

Where $P_0(z) = 1$ and $P_1(z) = 2z - 1$ . The analytic form of the SLP $P_\delta(z)$ of degree $\delta$ is given by

$$\boldsymbol{P_\delta(z) = \sum_{s=0}^{\delta} (-1)^{\delta+s} \frac{(\delta+s)!}{(\delta-s)!} \frac{z^s}{(s!)^2}} \qquad \mathbf{(6)}$$

Notice that $P_\delta(0) = (-1)^\delta$ and $P_\delta(1) = 1$ . The orthogonality condition is

$$\boldsymbol{\int_0^1 P_\delta(z)\, P_\eta(z)\, dz = \begin{cases} \frac{1}{2\delta+1} & \delta = \eta \\ 0 & \delta \neq \eta \end{cases}} \qquad \mathbf{(7)}$$

A function $g(z)$ square-integrable in $[0,1]$ may be expressed in terms of SLP

$$g(z) = \sum_{\eta=0}^{\infty} c_\eta\, P_\eta(z)$$

where the coefficients $c_\eta$ are presented through

$$\boldsymbol{c_\eta = (2\eta+1) \int_0^1 g(z)\, P_\eta(z)\, dz \qquad \eta = 1, 2, \dots}$$

Practically speaking, only the first $(N+1)-$ terms SLP are consider. So, we have

$$\boldsymbol{g(z) = \sum_{\eta=0}^{N} c_\eta\, P_\eta(z) \; = C^T \emptyset(\mathbf{z})}$$

where the shifted Legendre vector $\emptyset(\mathrm{z})$ and the shifted Legendre coefficient vector $C$ are presented by

$$\boldsymbol{C^T = [\, c_0\,, \dots, c_N]}$$
$$\boldsymbol{\emptyset(\mathbf{z}) = [\, P_0(z)\,, P_1(z)\,, \dots, P_N(z)]^{\,T}} \qquad \mathbf{(8)}$$

The derivative of the vector $\emptyset(\mathrm{z})$ can be expressed through

$$\boldsymbol{\frac{d\emptyset(\mathbf{z})}{dz} = D^{(1)} \emptyset(\mathbf{z})} \qquad \mathbf{(9)}$$

where $D^{(1)}$ is the $(\mathrm{N}+1) \times (\mathrm{N}+1)$ OPM of derivative presented through

$$\boldsymbol{D^{(1)} = (d_{\delta\eta}) = \begin{cases} 2(2\eta+1), & for\ \eta = \delta - s \ \begin{cases} s = 1, 3, \dots, N \ \ if\ N\ odd \\ s = 1, 3, \dots, N-1 \ \ if\ N\ even \end{cases} \\ 0 & \qquad otherwise \end{cases}}$$

For instance, for even $N$ we have

$$\boldsymbol{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \dots & 2N-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \dots & 0 & 2N-1 & 0 \end{pmatrix}$$

## 3. OPM for fractal fractional order differential equation

By utilizing (9). can be written the higher derivative as follows [13]:

$$\frac{d^n \emptyset(z)}{dz^n} = (D^{(1)})^n \emptyset(z) \qquad (10)$$

Where $n \in N$ and the superscript symbol, in $D^{(1)}$ , indicate matrix powers. Thus

$$D^{(1)} = (D^{(1)})^n \qquad (11)$$

**Theorem 1**. Let $n < \omega < n+1$, for a presented integer $\beta \geq \lceil \omega \rceil$ ,the FFCF operator of order $\omega \neq \lceil \omega \rceil$ of $(\tau - a)^{\beta}$ is presented as [17]

$$\begin{aligned} &{}^{FFCF}D^{\omega,\beta}_{a^+;\tau}(\tau-a)^{\beta} \\ &= \frac{M(v)\Gamma(\beta+1)}{1-v}\left[\left(\sum_{\delta=0}^{\beta-n-1} \frac{(-1)^{\delta}(\tau-a)^{\beta-n-1-\delta}}{\Gamma(\beta-n-\delta)\left(\frac{v}{1-v}\right)^{\delta+1}}\right) \right. \\ &\left. + \frac{(-1)^{\beta-n}}{\left(\frac{v}{1-v}\right)^{\beta-n}} e^{\frac{-v(\tau-a)}{1-v}}\right] \qquad (12) \end{aligned}$$

**Theorem 2**. Let $\phi_{\mathcal{L}}(z)$ be the shifted Legendre vector acquainted in (6) and also assume $n < \omega < n+1$ , then,

$${}^{FFCF}D^{\omega,\beta}\phi_{\mathcal{L}}(z) = P^{\omega,\beta}\phi_{\mathcal{L}}(z) \qquad (13)$$

where $P^{\omega,\beta}$ is the $N \times N$ OPM of FFCF operator of order $\omega$ acquainted as [13]:

$$P^{\omega,\beta} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,0,s} & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,1,s} & \dots & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\lceil\omega\rceil,N-1,s} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{s=\lceil\omega\rceil}^{\delta} \theta_{\delta,0,s} & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\delta,0,s} & \dots & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{\delta,N-1,s} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{s=\lceil\omega\rceil}^{N-1} \theta_{N-1,0,s} & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{N-1,0,s} & \dots & \sum_{s=\lceil\omega\rceil}^{\lceil\omega\rceil} \theta_{N-1,N-1,s} \end{bmatrix} \qquad (14)$$

where $\theta_{\delta,\eta,s}$ is presented through

$$\begin{aligned} &\theta_{\delta,\eta,s} \\ &= \frac{(2\eta+1)M(v)}{1-v}\sum_{\iota=0}^{\eta}\frac{(-1)^{\delta+\eta+\iota}(\delta+s)!\,(\eta+\iota)!}{(\delta-s)!\,s!\,(\eta-\iota)!\ \iota!}\left[\frac{(-1)^{1-\lceil\omega\rceil}}{\gamma^{s-\lceil\omega\rceil+\iota+2}}\right. \\ &+ \sum_{r=0}^{\iota}\frac{(-1)^{\lceil\omega\rceil}e^{-y}}{(\iota-r)!\ \gamma^{s-\lceil\omega\rceil+r+2}} \\ &\left. + \sum_{r=0}^{s-\omega}\frac{(-1)^{s+r}}{\iota!\,\Gamma(s-\lceil\omega\rceil-r+1)\gamma^{r+1}(s-\lceil\omega\rceil-r+\iota+1)}\right] \qquad (15) \end{aligned}$$

**where**

$$\boldsymbol{\gamma = \frac{\upsilon}{1-\upsilon}} \qquad (\mathbf{16})$$

$\boldsymbol{\delta = \lceil \omega \rceil, \dots, N}$, $\quad \boldsymbol{\eta = 0, 1, 2, \dots, N-1}$

### 3.1. For nonlinear fractal fractional order differential equation

Consider the nonlinear multi-order fractal fractional differential equation

$$\boldsymbol{D^{\omega,\beta} g(z) = Y(z, g(z), D^{\beta_1} g(z), \dots, D^{\beta_s} g(z)} \quad (\mathbf{17})$$

With initial condition

$$\boldsymbol{g^{(\delta)}(0) = d_\delta \ , \delta = 0, \dots, n} \qquad (\mathbf{18})$$

Where $n < \omega \leq n+1$, $0 < \beta_1 < \beta_2 < \cdots < \beta_s < \omega$, and $D^{\omega,\beta}$ indicates the fractal fractional derivative of Caputo of order $\omega$. It ought to be noticed that $Y$ can be nonlinear in generic. To utilizing SLP for this problem, we firstly approximate $g(z)$, $D^{\omega,\beta} g(z)$ *and* $D^{\beta_\eta} g(z)$ for $\eta = 0, \dots, s$ . Through replacing these equations in Eq. (17) we obtain

$$\boldsymbol{C^T D^{(\omega,\beta)} \phi(z)} \\ \boldsymbol{\simeq Y(z, C^T \phi(z), C^T D^{(\beta_1)} \phi(z), \dots, C^T D^{(\beta_s)} \phi(z)} \quad (\mathbf{17})$$

Also, we get

$$g(0) = C^T \phi(0) = d_0,$$

$$g^{(\lambda)}(0) = C^T D^\delta \phi(0) = d_\delta$$

firstly calculates Eq. (19) at $(N-n)$ points, to find the solution $g(z)$We utilize the first $(N-n)$ roots of shifted Legendre of $P_{N+1}(z)$ for suitable collocation points. Together these equations with Eq. (20) generate $(N+1)$ nonlinear equations which disbanded by utilizing the iterative method of Newton.

## 4. Numerical Examples

In this part, of nonlinear fractional differential equations with left-sided CF operators some numerical examples are solved by utilizing the enforcement of the recently derived OPM for left-sided CF operator.

**Example 1.** We next consider the following nonlinear

$$\boldsymbol{D^3 g(z) + {}^{FFCF}D^{\omega,\beta} g(z) + g^2(z) = z^4} \ ,$$
$$\boldsymbol{g(0) = g'(0) = 0, \quad g''(0) = 2}$$

$y(z) = z^2$ is the exact solution of this problem and $N = 3$

We solved the above problem

$$\boldsymbol{C^T D^3 \emptyset(z) + C^T D^{\omega,\beta} \emptyset(z) + [C^T \emptyset(z)]^2 - z^4 = 0} \quad (\mathbf{21})$$

| Abs. Error of $\beta$ | | | |
|---|---|---|---|
| $z$ | 2.95 | 2.98 | 2.99 |
| 0.1 | 4.270668598 $e^{-10}$ | 5.07711699 $e^{-11}$ | 6.75841409 $e^{-11}$ |
| 0.2 | 3.41653491 $e^{-9}$ | 4.06169291 $e^{-10}$ | 5.40673194 $e^{-10}$ |
| 0.3 | 1.15308053 $e^{-8}$ | 1.37082135 $e^{-9}$ | 1.82477205 $e^{-9}$ |
| 0.4 | 2.73322793 $e^{-8}$ | 3.24935432 $e^{-9}$ | 4.32538563 $e^{-9}$ |
| 0.5 | 5.338335797 $e^{-8}$ | 6.34639516 $e^{-9}$ | 8.44801881 $e^{-9}$ |
| 0.6 | 9.22464426 $e^{-8}$ | 1.09665708 $e^{-8}$ | 1.45981765 $e^{-8}$ |
| 0.7 | 1.46483934 $e^{-7}$ | 1.74145083 $e^{-8}$ | 2.31813636 $e^{-8}$ |
| 0.8 | 2.18658234 $e^{-7}$ | 2.59948346 $e^{-8}$ | 3.46030851 $e^{-8}$ |
| 0.9 | 3.11331743 $e^{-7}$ | 3.70121766 $e^{-8}$ | 4.92688458 $e^{-8}$ |

**Table 1:** The Absolute errors for different value of $\omega = 2.5$ , for example (1)

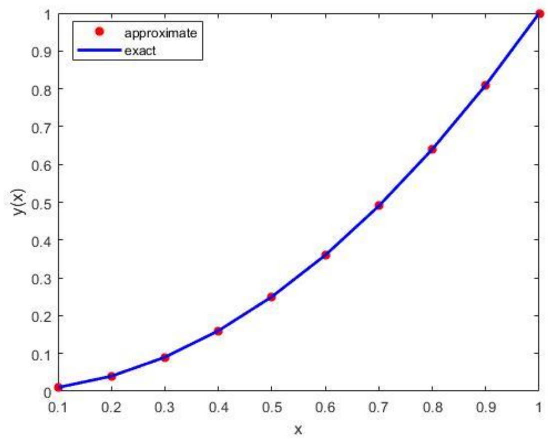


**Figure 1:** The exact solution and approximate solution of $\omega = 2.5$ , for example (1) which is the exact solution of this problem.

**Example 2.** We next consider the following nonlinear

$$D^4 g(z) + {}^{FFCF}D^{\omega,\beta} g(z) + g^3(z) = z^9 \quad ,$$

$$g(0) = g'(0) = 0, \quad g''(0) = 2$$

$y(z) = z^3$ is the exact solution of this problem and $N = 4$

We solved the above problem.

$$\boldsymbol{C^T D^4 \emptyset(z) + C^T D^{\omega,\beta} \emptyset(z) + [C^T \emptyset(z)]^3 - z^9 = 0} \quad (\mathbf{21})$$

| **Abs. Error of $\beta$** | | | |
|---|---|---|---|
| $z$ | **3.95** | **3.98** | **3.99** |
| 0.1 | 5.11456466 $e^{-11}$ | 2.82267774 $e^{-11}$ | 1.97688371 $e^{-11}$ |
| 0.2 | 8.18330370 $e^{-10}$ | 4.51628530 $e^{-10}$ | 3.16301594 $e^{-10}$ |
| 0.3 | 4.14279752 $e^{-9}$ | 2.28636944 $e^{-9}$ | 1.60127685 $e^{-9}$ |
| 0.4 | 1.3093286 $e^{-8}$ | 7.22605649 $e^{-9}$ | 5.06082560 $e^{-9}$ |
| 0.5 | 3.19660303 $e^{-8}$ | 1.76417395 $e^{-8}$ | 1.23555313 $e^{-8}$ |
| 0.6 | 6.62847604 $e^{-8}$ | 3.65819109 $e^{-8}$ | 2.56204296 $e^{-8}$ |
| 0.7 | 1.22800702 $e^{-7}$ | 6.77725063 $e^{-8}$ | 4.74650089 $e^{-8}$ |
| 0.8 | 2.09492576 $e^{-7}$ | 1.15616904 $e^{-7}$ | 8.09732096 $e^{-8}$ |
| 0.9 | 3.355666 $e^{-7}$ | 1.85195924 $e^{-7}$ | 1.29703425 $e^{-7}$ |

**Table 2:** The Absolute errors for different value of $\omega = 3.5$ , for example (2)

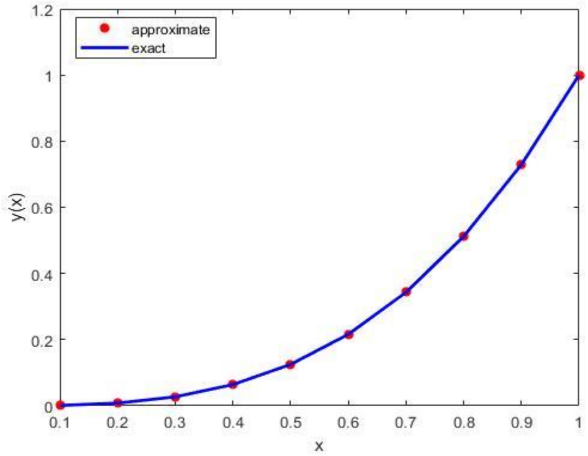


**Figure 2:** The exact solution and approximate solution of $\omega = 3.5$ , for example (2)

## 5. Conclusion

To solve non-linear FFCF differential new equation OPM has been utilized. Some numerical examples appear that the method is soft to employ and giving aloft fineness. The new OPM for the operator of FFCF inherits the gorgeous advantage from the well known OPM for the fractal fractional derivative of fractal fractional Caputo. The method minimizes the trouble in the sense of FFCF to those which are related to solve a system of algebraic equations, subsequently extremely simplify this trouble.

## 6. References


1. Caputo, M., and Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications, 1(2), 73-85.
2. Abdulhameed, M., Vieru, D., and Roslan, R. (2017). Modeling electro-magneto-hydrodynamic thermo-fluidic transport of biofluids with new trend of fractional derivative without singular kernel. Physica A: Statistical Mechanics and its Applications, 484, 233-252.
3. Atangana, A., and Koca, I. (2016). On the new fractional derivative and application to nonlinear Baggs and Freedman model. J. Nonlinear Sci. Appl, 9(5), 2467-2480.
4. Gómez-Aguilar, J. F. (2017). Space–time fractional diffusion equation using a derivative with nonsingular and regular kernel. Physica A: Statistical Mechanics and its Applications, 465, 562-572.
5. Abdulhameed, M., Vieru, D., and Roslan, R. (2017). Magnetohydrodynamic electroosmotic flow of Maxwell fluids with Caputo–Fabrizio derivatives through circular tubes. Computers and Mathematics with Applications, 74(10), 2503-2519.
6. Al-Salti, N., Karimov, E., and Sadarangani, K. (2016). On a differential equation with Caputo-Fabrizio fractional derivative of order \$1<\beta\leq 2\$ and application to mass-spring-damper system. arXiv preprint arXiv:1605.07381.
7. Gao, F., and Yang, X. J. (2016). Fractional Maxwell fluid with fractional derivative without singular kernel. Thermal Science, 20(suppl. 3), 871-877.
8. Mirza, I. A., and Vieru, D. (2017). Fundamental solutions to advection–diffusion equation with time-fractional Caputo–Fabrizio derivative. Computers and Mathematics with Applications, 73(1), 1-10.
9. Razzaghi, M., and Yousefi, S. (2001). The Legendre wavelets operational matrix of integration. International Journal of Systems Science, 32(4), 495-502.
10. Saadatmandi, A., and Dehghan, M. (2010). A new operational matrix for solving fractional-order differential equations. Computers and

mathematics with applications, 59(3), 1326-1336.

11. Bhrawy, A. H., and Zaky, M. A. (2015). Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation. Nonlinear Dynamics, 80, 101-116.
12. Saadatmandi, A., and Dehghan, M. (2010). A new operational matrix for solving fractional-order differential equations. Computers and mathematics with applications, 59(3), 1326-1336.
13. Loh, J. R., Isah, A., Phang, C., and Toh, Y. T. (2018). On the new properties of Caputo–Fabrizio operator and its application in deriving shifted Legendre operational matrix. Applied Numerical Mathematics, 132, 138-153.