Solving Singular Perturbed Boundary Value Problems

By using Semi-Analytic Method

Dr.Khalid. Mindeel. M. Al-Abrahemee
Department of Mathematics, College of Education / University of AL-Qadisiyha
(Email: khalid_math98@yahoo.com)

Abstract: In this paper, we suggest sufficied a method for solving a class of singularly perturbed boundary value problems (SPPBVP). The method is proposed the semi analytic, modest problem preparation and ready computer implementation. That is, we concerned with constructing polynomial solutions to two point second order of singularly perturbed problems of ordinary differential equation. A semi-analytic technique using two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0,1]. Numerical linear and non linear examples are given to illustrate the method. It is observed that the present method converges to the exact solution very well.

Keywords: Singular perturbation problems, Two-point boundary-value problems, Osculatory interpolation.

Mathematical subject classification : 34K28

1.Introduction

Singularly perturbed boundary value problems (SPBVPs) are common in applied sciences and engineering. They often occur in, for example, fluid dynamics, quantum mechanics, chemical reactions, electrical networks, etc. A well known fact is that the solution of such problems has a multiscale character, i.e. there are thin transition layers where the solution varies very rapidly, while away from the layers the solution behaves regularly and varies slowly. For a detailed discussion on the analytical and numerical treatment of such problems one may refer to the books of O'Malley [1]; Doolan et al. [2]; Roos et al.[3]; and Miller et al. [4]. Numerically, the presence of the perturbation parameter leads to difficulties when classical numerical techniques are used to solve such problems, this is due to the presence of the boundary layers in these problems; see for example O'Malley [5]. Even in the case when only the approximate solution is required, finite difference schemes and finite element methods produced unsatisfactory results; see Samarski [6]. It was shown in [7,8] that the results of using classical methods are also unsatisfactory even when a very fine grid is used. Therefore, the numerical treatment of singular perturbation problems presents some major computational difficulties.

2. Singularly Perturbed Problems

The term "perturbation problem" is generally used in mathematics when one deals with the following situation: There is a family of problems depending on a small parameter ε>0, which we denote by P_ε,when ε= 0, we have the reduced problem P_o. We want to study the relationship between the solution of $P_\in$and the solution of P_o under appropriate assumptions .The perturbation problem PP, may consist of an ordinary differential equation, or a system of differential equations, doing with some given conditions, such as boundary conditions. The general form of the 2nd order singularly perturbed problems (SPPs) are:

$$\begin{cases} \varepsilon y'' = f(x, y, y'), \quad x \in [a, b], \quad 0 < \varepsilon \ll 1 \\ \text{boundary conditions} \end{cases} \tag{1}$$

such as boundary conditions are : $y(a) = A$, $y(b) = B$, where f are n-dimensional vector functions, x is a scalar variable in a given interval. A perturbation problem (1) is called a singular perturbation problem if $\varepsilon \to 0$, the solution $y_\varepsilon(x)$converges to $y_0(x)$only in some interval of x, but not throughout the entire interval, thus giving rise to an "boundary layers" phenomena at both end-points. [9]

There is no loss in generality in taking a = 0 and b = 1, and we will sometimes employ this slight simplification. In this paper we introduce a new technique for the qualitative and quantitative analysis of singular perturbation problems SPPBVP using two-points polynomial interpolation .

3. Osculatory Interpolation [10]

Given $\{x_i\}, i = 1, \ldots k$ and values $f_i^{(0)}, \ldots, f_i^{(ri)}$,where ri are nonnegative integers and $f_i = f(x_i)$.We want to construct a polynomial $P(x)$ such that .

$$P^j(x_i) = f_i^{(j)}, \; for \; i = 1, \ldots, k \; and \; j = 0, \ldots, ri.$$

Such a polynomial is said to be an osculatory interpolating polynomial of a function f and The degree of $P(x)$ is at most $\sum_{i=1}^{k}(r_i + 1) - 1$.

In this paper we use two-points osculatory interpolation for singular perturbation problems. Essentially this is a generalization of interpolation using Taylor polynomials and for that reason osculatory interpolation is sometimes referred to as two-point Taylor interpolation. The idea is to approximate a function $y(x)$ by a polynomial $P(x)$ in which values of $y(x)$ and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of $P(x)$.

And we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0,1],wherein a useful and succinct way of writing a osculatory interpolate $P_{2n+1}(x)$ of degree $2n + 1$ was given for example by Phillips [11] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n}\{y^{(j)}(0)q_j(x) + (-1)^j y^{(j)}(1)q_j(1-x)\} \qquad (2)$$

$$q_j(x) = \left(\frac{x^j}{j!}\right)(1-x)^{n+1}\sum_{s=0}^{n-j}\binom{n+s}{s}x^s = \frac{Q_j(x)}{j} \qquad (3)$$

so that (2) with (3) satisfies :

$$y^{(r)}(0) = P_{2n+1}^{(r)}(0)\,, y^{(r)}(1) = P_{2n+1}^{(r)}(1), \quad r = 0,1 \ldots, n$$

implying that $P_{2n+1}(x)$ agrees with the appropriately truncated Taylor series for $y(x)$ about x = 0 and x = 1.The error on [0, 1] is given by :

The osculatory interplant for $P_{2n+1}(x)$ may converge to $y(x)$ in [0,1] irrespective of whether the intervals of convergence of the constituent series intersect or are disjoint .The important consideration here is whether $R_{2n+1} \to 0$ as $n \to \infty$ for all x in [0,1]. In the application to the boundary value problems in this paper such convergence with n is always confirmed numerically .We observe that (2) fits an equal number of derivatives at each end point but it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval.

Finally we observe that (2) can be written directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as :

$$P_{2n+1}(x) = \sum_{j=0}^{n}\{a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \qquad (4)$$

4. Illustration of the Method

In this section we describe solution of SPPs using two-points polynomial interpolation .

.To illustrate the method, we will consider the 2nd order SPPs:

$$\varepsilon y'' + f\,(x\,, y\,, y') = 0 \qquad (5)$$

$$g_i\,(\,y(0)\,, y(1)\,, y'(0), y'(1)\,) = 0\,, i = 1\,,2 \qquad (6)$$

where g_1, g_2 are in general nonlinear functions of their arguments and g_1 and g_2 are given in three kinds [12] :

1- $y(0) = a_0\,, y(1) = b_0$ …… (6a), and we say this kind Dirichlet condition (value specified).

2- $y'(0) = a_1\,, y'(1) = b_1 \ldots .(6b)$, and we say this kind Neumann condition (Derivative specified).

3- $c_0 y'(0) + c_1 y(0) = a, d_0 y'(1) + d_1 y(1) = b$ …. (6c), where c_0, c_1, d_0, d_1 are all positive constants not all are zero but c_1, d_0 are equal to zero or c_0, d_1 are equal to zero and we say this kind Mixed condition (Gradient & value) .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (5)–(6), or an alternative formulation of it, by a P_{2n+1} which enables any unknown boundary values or derivatives of y(x) to be computed . The first step therefore is to construct the P_{2n+1} . To do this we need the Taylor coefficients of y (x) at x = 0 :

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \qquad (7a)$$

into (5)and equate coefficients of powers of x. The resulting system of equations can be solved to obtain a_i (a_0, a_1)for all $i \geq 2$. Also we need the Taylor coefficients of y (x) at x = 1. Using MATLAB throughout we simply insert the series forms :

$$y = b_0 + b_1(x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i$$

(7b) into (5) and equate coefficients of powers of (x−1). The resulting system of equations can be solved to obtain b_i (b_0, b_1) for all $I \geq 2$.The notation implies that the coefficients depend only on the indicated unknowns a_0, a_1, b_0, b_1. The algebraic manipulations needed for this process .We are now in a position to construct a $P_{2n+1}(x)$ from (7) of the form (2) and use it as a replacement in the problem (5)–(6). Since we have only the four unknowns to compute for any n we only need to generate two equations from this procedure as two equations are already supplied by the boundary conditions (6). An obvious way to do this would be to satisfy the equation (5) itself at two selected points $x = c_1$, $x = c_2$ in [0,1] so that the two required equations become :

$$\varepsilon P''_{2n+1}(c_i) + f\{P'_{2n+1}(c_i), P_{2n+1}(c_i), c_i\} = 0 \quad , i = 1,2 \qquad (8)$$

An alternative approach is to recast the problem in an integral form before doing the replacement. Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n. We therefore use this alternative formulation throughout this paper although we should keep in mind that the procedure based on (8) is a viable option and shares many common features with the approach outlined below. Of the many ways we could provide an integral formulation we adopt the following. We first integrate the formula (5) twice where $a_0 = y(0)$ and $a_1 = y'(0)$ and putting x = 1 then gives :

$\varepsilon b_1 - a_1 + \int_0^x f(y(s), y'(s), s)ds = 0$ (9)

and

$\varepsilon b_1 - a_0 - a_1 + \int_0^x (1-s) f(y(s), y'(s), s)ds = 0$

(10)

where $b_0 = y(1)$ and $b_1 = y'(1)$.

The precise way we make the replacement of y(x) with a $P_{2n+1}(x)$ in (9) and (10) depends on the nature of f(y ,y',x) and will be explained in the examples which follow. In any event the important point to note is that once this replacement has been made, the equations (6), (9) and (10) constitute the four equations we require to determine the set $\{a_0, b_0, a_1, b_1\}$. As we shall see the fact that the number of unknowns is independent of the number of derivatives fitted represents perhaps the most important feature of the method.

5. Numerical results

To demonstrate the applicability of the method we have applied it on the linear and nonlinear singular perturbation problems. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. Also, we test the accuracy of obtained solutions computing the mean square error (M.S.E).

5.1 The linear example

Example 1

Consider the following 2nd order singular perturbed boundary-value problem (S.P.Ps) : $\varepsilon^2 y'' - y + 1 = 0$ with Dirishlit BC: $y(0) = 0$, $y(1) = 2$ $x \in [0,1]$. The analytic solution :

$$y = 1 + \frac{-1-e^{\frac{-1}{\epsilon}}}{1-e^{\frac{-2}{\epsilon}}} - e^{\frac{-t}{\epsilon}} + \frac{1+e^{\frac{-1}{\epsilon}}}{1-e^{\frac{-2}{\epsilon}}} e^{\frac{-(1-x)}{\epsilon}} \quad \text{s.t}$$

ε=0.03 [13]

Here (9) and (10) become

$0.0009\ (b_1 - a_1)1 - \int_0^1 y(s)\, ds = 0$ (11)

and

$0.5018 - a_1 - \int_0^1 (1-s) y(s)\, ds = 0$ (12)

The coefficients : a_2 ,b_2, a_3 , b_3 ,…can be found from (7a) and (7b) .A initio inclusion of the boundary conditions of the problem has reduced the number of unknowns to two, namely $\{a_1, b_1\}$, which are computed by solving (11) and (12) with y(s) replaced by a $P_{2n+1}(s)$ and. If the value of n = 4 we will get polynomial of degree nine, which represents the resolution of the singular perturbation problem which are as follows

P_9 = (8712453704582529 x^9)/4398046511104-(2450045339036563x^8)/274877906944+(2376848378417523x^7)/137438953472-(1301316542224843x^6)/68719476736+(7067365088926747x^5)/549755813888-(6148040127986429x^4)/1099511627776+(6828527416826277x^3)/4398046511104 -(2326170035692319 x^2)/8796093022208 +(7038451686753859x)/281474976710656.

The results for n = 4 are displayed in Table 1. We can see that there is clear convergence with n to the 'exact' values which are obtained using MATLAB boundary value software. Figure 1 gives the accuracy of the method.

Example 2

Consider the following 2nd order homogeneous singular perturbed boundary-value problem (S.P.Ps) :

$\varepsilon y'' + \left(1 - \frac{x}{2}\right) y' - \frac{1}{2} y = 0$, $x \in [0,1]$ with Dirishlit BC $y(0) = 0$, $y(1) = 1$ and analytic solution : $y = \frac{1}{2-x} - \frac{1}{2}\, e^{(-(x - \frac{x^2}{4})/\varepsilon}$ [14] Such that $\varepsilon = 10^{-3}$, If the value of n = 4 we will get polynomial of degree nine, which represents the resolution of the singular perturbation problem which are as follows

P_9=(5770898234988837x^9)/131072-(4126284522071755x^8)/65536+(5854518678880257x^7)/262144-(3911524096275751x^6)/1048576+(5866447649961765x^5)/16777216-(5251378322827223x^4)/268435456+(5646797071639953x^3)/8589934592-(7008388454598599x^2)/549755813888+82378570341061x)/2199023255552.

The results of solution given in the following table :

5.2 The non-linear example

Example 3

Consider the following 2nd order nonlinear singular perturbed boundary-value problem

$$\varepsilon y'' + 2y' + e^{y} \qquad \text{s.t } x \in [0,1]$$

with Mixed BC's $y(0) = 0$, $y'(1) = -0.5$.

The exact solution:

$y = \log\left(\frac{2}{1+x}\right) - \log(2)\, e^{\frac{-2x}{\epsilon}}$ [16] Such that $\varepsilon = 10^{-5}$. If the value of n = 4 we will get polynomial of degree nine, which represents the resolution of the singular perturbation problem which are as follows

***P*$_9$**=-(6339216228566459x^9)/281474976710656 + (1587385008841237x^8)/17592186044416 -(165257470682931x^7)/1099511627776+(4754109614451443x^6)/35184372088832-(1251853984166609x^5)/17592186044416+(3135285418890191x^4)/140737488355328-(1129098090681425x^3)/281474976710656+(424920472391205x^2)/1125899906842624-(8041888101053153x)/576460752303423488.

Example 4

Consider the following 2nd order nonlinear singular perturbed boundary-value problem

$\varepsilon y'' + yy' - y$ with Dirishlit BC's: $y(0) = -1$, $y(1) = 3.9995$, x∈[0,1]

and the analytic solution :

$$y = x + c_1 \tanh(c_1 \frac{\frac{x}{\epsilon} + c_2}{2}) \quad [17]$$

$$s.t\ c_1 = 2.9995\ \ and$$

$$c_2 = \frac{1}{c_1}\log(\frac{c_1 - 1}{c_2 + 1})\ \ s.t\ \varepsilon = 10^{-7}$$

. If the value of n = 3 we will get polynomial of degree nine, which represents the resolution of the singular perturbation problem which are as follows

$\boldsymbol{P_7}$=(435411227217891x^7)/2199023255552-(7490497929480767x^6) /8796093022208+(6751639456538077x^5)/4398046511104-(6609553554849531x^4)/ 4398046511104+(7584009124791225x^3) /8796093022208-(2547607450080331x^2)/ 8796093022208+ (7556089753371941x

/140737488355328 – 1

The results of solution given in the following table :

References

[1] R. E. O'Malley, Singular Perturbation Methods for Ordinary Differential Equations, SpringerVerlag,NewYork,1991.

[2] E. P. Doolan, J. J. H. Miller and W. H. A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, Ireland, 1980.

[3] H. G. Roos, M. Stynes and L.Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer, Berlin, 1996.

[4] J. J. H. Miller, E. O'Riordan and G. I. Shishkin , Fitted Numerical Methods for Singular Perturbation Problems,World Scientific, Singapore, 1996.

[5] R. E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.

[6] A. Samarski, Theory of difference schemes, Moscow, Nauka,1980.

[7] G. I. Shishkin, Grid approximation of singularly perturbed boundary value problems or quasi-linear parabolic equations in the case of complete degeneracy in spatial variables, Sov J. Numer. Anal.Math.Model,61-243,(1991).

[8] G. I. Shishkin, A finite difference scheme on a priori adapted meshes for a singularly perturbed parabolic convection–diffusion equation, Numer. Math. Theor Methods Appl,**1**(2008), 34-214.

[9] JianzhongMeng, 1997, Some Singular Singularly Perturbed roblems, Calgary, Alberta.

[10] V. Girault and L. R. Scott, Hermite Interpolation of Non smooth Functions Preserving Boundary Conditions ,Mathematics of Computation, Vol. 71, No.239, pp.1043-1074, 2002.

[11] G .M.Phillips , "Explicit forms for certain Hermite approximations ",BIT 13 (1973) 177-180.

[12] R.J. LeVeque, Finite Difference Methods for Differential Equations ,University of Washington ,2006 .

[13] Jianzhong Meng , Some Singular Singularly Perturbed roblems , Calgary, Alberta, September, 1997.

[14] Nayfeh, A. H. *Perturbation Methods*, Wiley, New York ,(1979)

[15] Wen- Zhi Zhang, Pei-yan Huang, Coupling of high order multiplication perturbation method and reduction method for variable coefficient singular perturbation problems,Springer-Verlag,2014.

[16] P. Padmajaa,* and Y. N. Reddy , A Numerical Patching Method for solving Singular Perturbation Problems Via Padé Approximates , 2013.

[17] Kevorkian, J. and Cole, J. D..“*Perturbation methods in Applied Mathematics*” .pringer-Verlag, New York, 1981.

TABLE 1: The result of the methods for n= 4 of example1

x	**Analytic solution $y_a(x)$**	**a_1= -2.00000000312880**	
		b_1= -1.45223411074369e-09	
		P_9 (N=4)	**E(x) = | $P_9 - y_a(x)$ |**
0.0	**0**	**0**	**0**
0.1	**٠.٩٦٤٣٢٦٠٠٦٦٥٢٨٤١**	**0.960737796991716**	**0.00358820966112500**
0.2	**٠.٩٩٨٧٢٧٣٦٦٢٠١٢٨٣**	**0.998727366201283**	**0**
0.3	**٠.٩٩٩٩٥٤٦٠٠١٤٣٧٦٧**	**0.999954600143766**	**1.11022302462516e-15**
0.4	**٠.٩٩٩٩٩٨٣٨٢٤٦٤٣٦١**	**0.999995906986901**	**١.١١٠٢٢٣٠٢٤٦٢٥١٦e-16**
0.5	**١**	**1.00000000000000**	**1.11022302462516e-16**
0.6	**١.٠٠٠٠٠١٦١٧٥٣٥٦٣٩**	**1.00000541840825**	**2.47547746057286e-06**
0.7	**١.٠٠٠٠٤٥٣٩٩٨٥٦٢٣**	**1.00004539985623**	**0**
0.8	**١.٠٠١٢٧٢٦٣٣٧٩٨٧٢**	**1.00112012903621**	**0.000152504762511052**
0.9	**١.٠٣٥٦٧٣٩٩٣٣٤٧١٦**	**١.٠٣٥٦٧٣٩٩٣٣٤٧١٦**	**0**
1.0	**٢**	**٢.٠٠٠٠٠٠٠٠٠٠٠٠٠٠**	**0**
M.S.E= 1.1726e-06			

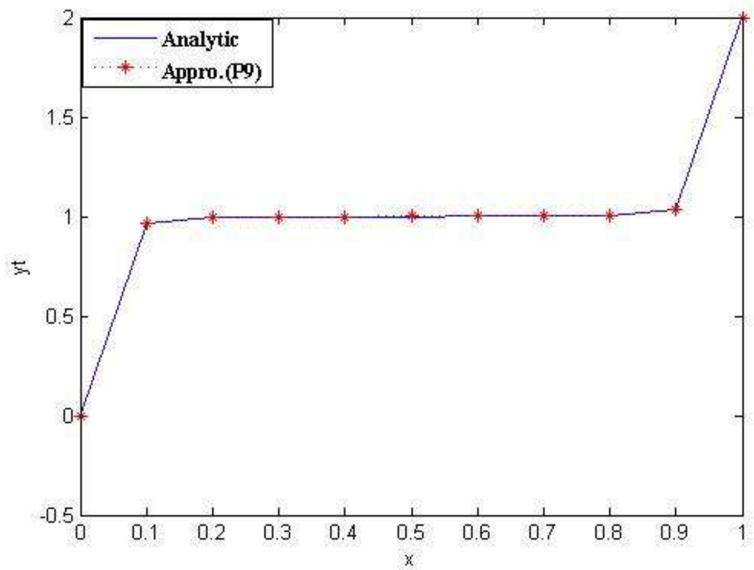

Figure1:Comparison between the exact solution and semi-analytic method P_9

TABLE 2: The result of the methods for n= 4 of example 2

x	Analytic solution $y_a(x)$	a_1= 1.10528545600124		present solution in [15]
		b_1= 1		
		P_9 (N=4)	$E(x) = \lvert P_9 - y_a(x) \rvert$	
0.0	0	0	0	0.000 000 0
0.01	0.502489288196842	0.502488758543014	5.29653828684751e-07	0.502 489 3
0.02	0.505050503911542	0.505061429949489	1.09260379473897e-05	0.505 050 5
0.03	0.507614213197911	0.507597320985607	1.68922123041648e-05	0.507 614 2
0.04	0.510204081632653	0.510205722940375	1.64130772173365e-06	0.510 204 1
0.05	0.512820512820513	0.512829978545462	9.46572494886500e-06	0.512 820 5
0.06	0.515463917525773	0.515471671230134	7.75370436068013e-06	0.515 463 9
0.07	0.518134715025907	0.518136222284390	1.50725848346855e-06	0.518 134 7
0.08	0.520833333333333	0.520830073497412	3.25983592175394e-06	0.520 833 3
0.09	0.523560209424084	0.523560495118079	2.85693995349945e-07	0.523 560 2
0.10	0.526315789473684	0.526335512722622	1.97232489379529e-05	0.526 315 8
M.S.E= 8.721894117502932e-11				

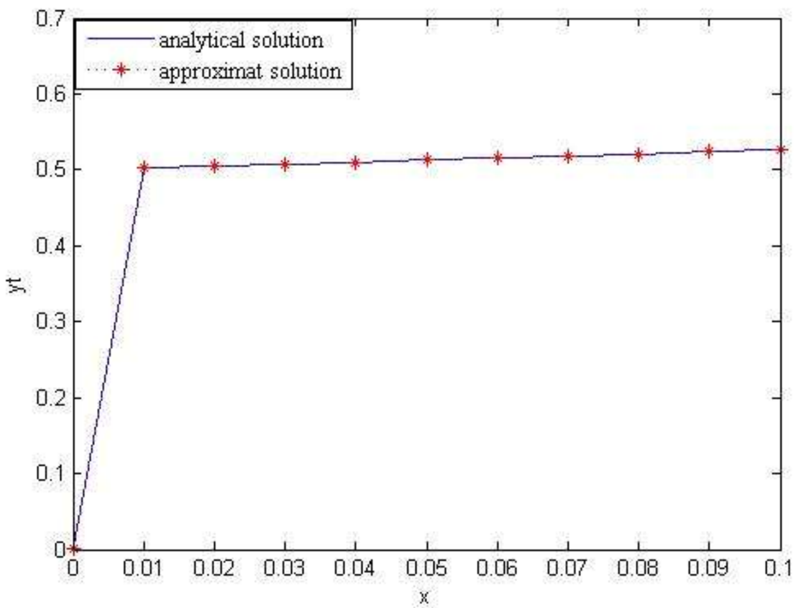

Figure2:Comparison between the exact solution and semi-analytic method P_9

TABLE 3: The result of the methods for n= 4 of example3

x	Analytic solution $y_a(x)$	$a_1 = 0$, $b_0 = 0$	
		P_9 (N=4)	$E(x) = \lvert P_9 - y_a(x) \rvert$
0.0	0	0	0
0.1	0.597837000755620	0.597825695848629	1.13049069915272e-05
0.2	0.510825623765991	0.510863642279602	3.80185136111821e-05
0.3	0.430782916092454	0.430663211975091	0.000119704117363617
0.4	0.356674943938732	0.356818341946612	0.000143398007879825
0.5	0.287682072451781	0.287605751003049	7.63214487315977e-05
0.6	0.223143551314210	0.223012881327226	0.000130669986984244
0.7	0.162518929497775	0.162529616078207	1.06865804320455e-05
0.8	0.105360515657826	0.105437615771765	7.71001139389105e-05
0.9	0.0512932943875505	0.0512916273469943	1.66704055614558e-06
1.0	0	0	0
M.S.E = 5.883446236369177e-09			

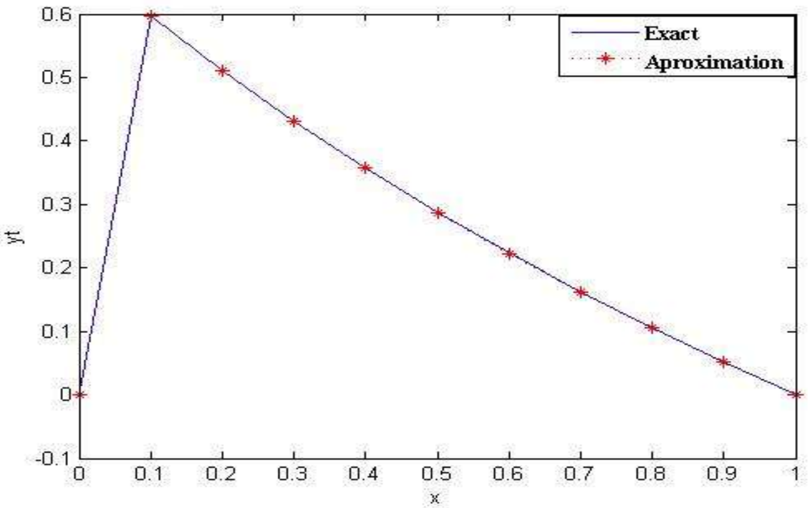

Figure3:Comparison between the exact solution and semi-analytic method P_9

TABLE 4: The result of the methods for n= 3 of example4

x	**Analytic solution $y_a(x)$**	**a_1=5.99940860402658**		Numerical method in [16]
		b_1=8.99877132674263		
		P_9 (N=4)	**E(x) = $\lvert P_7 - y_a(x) \rvert$**	
0.0	**-1**	**-1**	**0**	**-1.000000**
0.1	**3.09950000000000**	**3.09950037405570**	**3.74055701524156e-7**	**3.0988336**
0.2	**3.19950000000000**	**3.19950054404607**	**5.44046066863757e-7**	**3.1988096**
0.3	**3.29950000000000**	**3.29949792067516**	**2.07932483542450e-6**	**3.2988017**
0.4	**3.39950000000000**	**3.39951024523336**	**1.02452333554659e-5**	**3.3987978**
0.5	**3.49950000000000**	**3.49948916543168**	**1.08345683234035e-5**	**3.4987953**
0.6	**3.59950000000000**	**3.59949445307403**	**5.54692596788087e-6**	**3.5987937**
0.7	**3.69950000000000**	**3.69949921685082**	**7.83149178840148e-7**	**3.6987927**
0.8	**3.79950000000000**	**3.79950299240116**	**2.99240115753108e-6**	**3.7987916**
0.9	**3.89950000000000**	**3.89950351334945**	**3.51334945358772e-6**	**3.8987911**
1.0	**3.99950000000000**	**3.99950000000000**	**0**	**3.9987905**
M.S.E=2.663488578194586e-11				

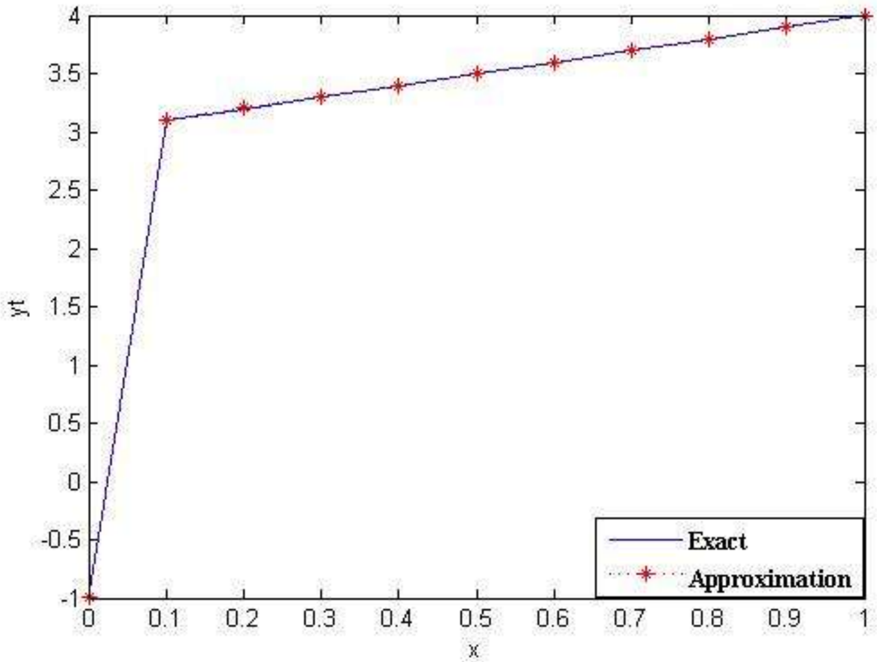

Figure4:Comparison between the exact solution and semi-analytic method P_7

الطـريقة شــبه التحـليلية لحـــل مسائـل الاضطراب الشاذة ذات القيم الحدودية

خالد منديل محمد الآبراهيمي
قسم الرياضيات ـ كلية التربية ـ جامعة القادسية

المستخلص:

في هذا البحث، طريقة جديدة لحل مسائل الاضطراب الشاذة ذات القيم الحدودية. صممت هذه الطريقة بحيث تكون سهلة الاستخدام ، وذات أعداد جيد للمسألة.حيث اقترحت الحل كمتعددة حدود لحل مسائل الاضطراب الشاذة ذات القيم الحدودية من الرتبة الثانية للمعادلات التفاضلية الاعتيادية .استخدمت التقنية الشبه تحليلية باستخدام نوع من أنواع الاندراج وهو الاندراج التماسي لعدد من مشتقات نقاطي نهاية الفترة المذكورة وقورنت النتائج مع الحل باستخدام الطرائق التقليدية الأخرى وأثبتنا من خلا مجموعة من الأمثلة الخطية والغير خطية بأن الطريقة المقترحة هي الأسرع و الأدق .