A Note on Normal and n-Normal Operators

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Abstract

This paper is devoted to the study of normal operator on a Hilbert space H. Normal and n-normal operators from a given operator are obtained. Some properties of the normal and n-normal operators are investigated and some examples are also given.

Keywords: normal operators, n-normal operators.

1. Introduction

Here and hereafter, $B(H)$, $NB(H)$ and $nNB(H)$ denote, respectively, to the algebra of all bounded, normal and n-normal linear operators acting on a complex Hilbert space H and An operator $T \in B(H)$ is called normal operator if $TT^* = T^*T$, self adjoint if $T = T^*$, projection if $T^2 = T^* = T$, n-normal operator if $T^nT^* = T^*T^n$, invertible with inverse S if there exists $S \in B(H)$ such that $ST = I = TS$, where $I \in B(H)$ is the identity operator.

The properties of normal and n-normal operators and operators related with them were extensively studied by many authors. For example, Patel and Ramanujan (1981) for normal operators, Naoum and Nassir 2007 for pseudo-normal operators , Jibril 2008 for n-power normal operators, Jibril (2010) for the operators satisfying $T^{*2}T^2 = (T^*T)^2$, Alzuraiqi and Patel (2010) for n-normal operators, Nassir (2010) for quasi-posinormal operators and Sid Ahmed (2011) for n-power quasi-normal operators satisfying $T^n|T|^2 = |T|^2T^n$. Recently, Panayappan and Sivaman (2012) introduced n-binormal operators and studied some basic properties of them and Panayappan (2012) studied n-power class (Q) operators for which $T^{*2}T^{2n} = (T^*T^n)^2$.

2. The Main Results

In the following proposition, we obtain a normal operator from given invertible normal operator.

Proposition 2.1. If $T \in NB(H)$ with inverse T^{-1} then $T^*T^{-1} \in NB(H)$ and $T^{-1}T^* \in NB(H)$.

Proof: For proving $T^*T^{-1} \in NB(H)$, it is sufficient to prove

$(T^*T^{-1})(T^*T^{-1})^* = (T^*T^{-1})^*\ (T^*T^{-1})$.

Observe that

$$\begin{aligned}(T^*T^{-1})(T^*T^{-1})^* &= T^*T^{-1}(T^{-1})^*(T^*)^* = T^*T^{-1}T^{*-1}\,T \\ &= T^*(T^*T)^{-1}T = T^*(TT^*)^{-1}T = (T^{-1}T)^*T^{-1}\,T = I\end{aligned}$$

and

$(T^*T^{-1})^*\ (T^*T^{-1}) = T^{-1^*}TT^*T^{-1} = T^{-1^*}T^*TT^{-1} = (TT^{-1})^*TT^{-1} = I.$

then

$(T^*T^{-1})(T^*T^{-1})^* = (T^*T^{-1})^*\ (T^*T^{-1})$.

Hence $T^*T^{-1} \in NB(H)$.

Similarly, we can prove that $T^{-1}T^{*} \in NB(H)$. ■

Remark 2.1 Another way to prove Proposition 2.1 is given in the following. Clearly, if $T \in NB(H)$ then $T^{*} \in NB(H)$ and $T^{-1} \in NB(H)$. Since T^{*} and T^{-1} are commuting normal operators then $T^{*}T^{-1}$ and $T^{-1}T^{*}$ are normal operators (see Gheondea (2009)).

Example 2.1. Let $T: R^2 \to R^2$, where $T(x,y) = (y,-x)$. It is easy to see that $T^{*}: R^2 \to R^2$, where $T^{*}(x,y) = (-y,x)$. Since

$$TT^{*}(x,y) = T(-y,x) = (x,y) = T^{*}(y,-x) = T^{*}T(x,y)$$

Then T is a linear operator. It can be seen that $T^{-1}: R^2 \to R^2$, where $T^{-1}(x,y) = (-y,x)$. Now

$$T^{-1}T^{*}(x,y) = T^{-1}(-y,x) = (-x,-y) = T^{*}(-y,x) = T^{*}T^{-1}(x,y)$$

Hence $T^{-1}T^{*}$ and $T^{*}T^{-1}$ are linear operators. ■

Proposition 2.1 states that $T \in NB(H)$ is the sufficient condition for $T^{*}T^{-1} \in NB(H)$. However this is not the necessary condition as in the following example.

Example 2.1. Let $T: R^2 \to R^2$, where $T(x,y) = (2y,x)$. Note that $T^{*}(x,y) = (y,2x)$ and $T^{-1}(x,y) = \left(y, \frac{1}{2}x\right)$. Since $TT^{*}(x,y) = (4x,y)$ and $T^{*}T(x,y) = (x,4y)$ then T is not normal. Let $S_1 = T^{-1}T^{*}$ and $S_2 = T^{*}T^{-1}$.

It is easy to see that

$S_1(x,y) = \left(2x, \frac{1}{2}y\right) = S_1^{*}(x,y)$ and $S_2(x,y) = \left(\frac{1}{2}x, 2y\right) = S_2^{*}(x,y)$

i.e. S_1 and S_2 are self adjoint operators and hence S_1 an S_2 are normal operators. ■

Remark 2.2 It is known (see, Alzuraiqi and Patel (2010)) that every normal operator is n-normal for every n but the converse is not true. So that if $T \in NB(H)$ then $T^{*}T^{-1} \in nNB(H)$ but this result need not to be true in case of $T \in nNB(H)$. In the following theorem, we study the case when T is an n-normal operator.

Theorem If $T \in nNB(H)$ then $T^{*}T^{-1}$ and $T^{-1}T^{*}$ are n-normal operators.

Proof. The idea of some of the following proof are borrowed from (see, Now

$$\begin{aligned} T \in nNB(H) &\Rightarrow T^n \in NB(H) \\ &\Rightarrow (T^n)^{*} = (T^{*})^n \in NB(H) \text{ and } (T^{-1})^{*} = (T^{*})^{-1} \in NB(H) \\ &\Rightarrow T^{*}, T^{-1} \in nNB(H). \end{aligned}$$

Now, since T^{*} and T^{-1} are commuting normal operators then $(T^{*})^n$ and $(T^{-1})^n$ are commuting normal operators. The result follows using Theorem 2.8 of Alzuraiqi and Patel (2010). ■

Example Let $T(x,y) = (ix + 2y, -iy)$. Since $T^{*}(x,y) = (ix, 2x - iy)$ and $T^{-1}(x,y) = (-ix - 2y, iy)$ then T is 2-normal but not normal. Let $S(x,y) = T^{*}T^{-1}(x,y) = (x - 2iy, -2ix - 3y)$. Since $S^{*}(x,y) = (x + 2iy, +2ix - 3y)$ then $SS^{*} \neq S^{*}S$ and hence $T^{*}T^{-1}$ is not normal. It is easy to show that $(T^{*}(x,y))^2 = (-x,-y)$ and $(T^{-1}(x,y))^2 = (-x,-y)$ are normal which implies that $T^{*}(x,y)$ and $T^{-1}(x,y)$ are 2-normal operators. Since $(T^{*})^2$ and $(T^{-1})^2$ are commuting 2-normal operator then $(T^{*})^2(T^{-1})^2 = I$ and $(T^{-1})^2(T^{*})^2 = I$ are n-normal operators (see Alzuraiqi and Patel (2010)). ■

Corollary $T \in NB(H)$ iff $T^{*}T^{-1}T = TT^{*}T^{-1}$.

Proof: The proof is straightforward and is hence is omitted. ■

Theorem 2.4. If $T \in NB(H)$ and with inverse T^{-1} then

$$(T^*T^{-1})^n(T^*T^{-1})^*(T^*T^{-1}) = (T^*T^{-1})^*(T^*T^{-1})^{n+1}$$

Proof:

$$\begin{aligned}(T^*T^{-1})^n(T^*T^{-1})^*(T^*T^{-1}) &= (T^*T^{-1})^n T^{-1^*}TT^*T^{-1} \\ &= (T^*T^{-1})^n T^{-1^*}T^*TT^{-1} \\ &= (T^*T^{-1})^n I \\ &= I(T^*T^{-1})^n \\ &= T^{-1^*}T^*TT^{-1}(T^*T^{-1})^n \\ &= (T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})^n \\ &= (T^*T^{-1})^*(T^*T^{-1})^{n+1} \blacksquare\end{aligned}$$

Theorem 2.6. Let $T \in B(H)$. Then T normal operator iff $T^*T^{-1}T^* = (T^*)^2T^{-1}$

Proof.

Suppose T normal operator. Since T^{-1} normal operator then

$T^*T^*T^{-1^*}T^{-1}T^* = T^*T^*T^{-1}T^{-1^*}T^* = T^*T^*T^{-1}I = T^*T^*T^{-1}$

$\Rightarrow T^*T^{-1}T^* = T^*T^*T^{-1}$

Suppose that $T^*T^{-1}T^* = T^*T^*T^{-1}$

Since $T^*T^{-1}T^* = T^*T^*T^{-1} \Rightarrow T^{*^{-1}}T^*T^{-1}T^* = T^{*^{-1}}T^*T^*T^{-1}$

$\Rightarrow IT^{-1}T^* = IT^*T^{-1} \Rightarrow T^{-1}T^*T = T^*T^{-1}T \Rightarrow T^{-1}T^*T = T^*I$

$\Rightarrow TT^{-1}T^*T = TT^* \Rightarrow IT^*T = TT^* \Rightarrow T^*T = TT^* \Rightarrow T$ is a normal operator $\blacksquare$

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