A New SEPARATION AXIOM $ii - T_{\frac{1}{4}}$

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Abstract.

This research aims to continue the investigation of $ii - T_{\frac{1}{4}}$ spaces, specifically their behavior when producing products. As a result, we may easily design non-$ii - T_{\frac{1}{4}}$spaces as well as $ii - T_{\frac{1}{4}}$ spaces that aren't$ii - T_{\frac{1}{2}}$. Furthermore, use new axioms of separation, which are, $ii - T_{\frac{1}{4}}$ by using $ii -$ open sets. We begin in section two with some definitions and theories that can be used in arriving at the new axioms of separation. Also, as an explanation of the $ii -$open set and some of its characteristics. We define some closed sets that explain the relationship between the new axioms of separation, including the $ii\lambda -$ closed set. We also give some examples to illustrate the new axioms of separation. Furthermore, we prove its relationship to the axioms of separation.$T_{\frac{1}{4}}$. We also learn about the important characteristics of these axioms. We know the new separation axioms are called $ii - T_{\frac{1}{4}}^{c}$ spaces and clarify the relationship between them and $ii - T_{\frac{1}{4}}$space and$ii - T_{\frac{1}{2}}$ space. We can illustrate the relationship between the separation axioms of type $ii -$through a diagram and then explain and list the theorems that show the domain of products. $ii - T_{\frac{1}{4}}$.

Keywords: $ii\lambda -$ closed set, $ii - T_{\frac{1}{4}}$ space,$ii - T_{\frac{1}{4}}^{c}$ space, $ii - T_1$ space, $iig -$closed set.

1. Introduction

Arenas, Dontchev, and Ganster [1] established the concept of a $\lambda -$ closed set in a topological space in their study of generalised continuity in 1997. If $A = L \cap F$, where L is a V- set, a subset A of a space (X,t) is called ii- closed. A subset A of a space (X,τ) is called $\lambda-$closed if $A = L \cap F$, where $L$is a $\Lambda-$set, i.e. $L$is an intersection of open sets, and $F$is closed. Using $\lambda-$closed sets, the authors in [1] characterized T_0 spaces as those spaces where each singleton is $\lambda-$closed, and $T_{\frac{1}{2}}$spaces as those spaces where every subset is $\lambda-$closed. The notion of a $T_{\frac{1}{2}}$space has been introduced by Levine in [2]. Dunham [3] showed that a space (X,τ) is $T_{\frac{1}{2}}$if and only if each singleton is open or closed. One of the most important examples of $T_{\frac{1}{2}}$spaces is the digital line or Khalimsky line $(\mathbb{Z},K)$ (see e.g. [4]). The digital line is the set of integers $\mathbb{Z}$ with the topology κhaving $S = \{\{2m-1,2m,2m+1\}: m \in \mathbb{Z}\}$as a subbase. Clearly,$(\mathbb{Z},K)$ fails to be T_1. However, each singleton of the form $\{2m\}$is closed and each singleton of the form $\{2m-1\}$is open. It should be observed that $(\mathbb{Z},K)$ is even a $T_{\frac{3}{4}}$space (see [5]). In [1], $T_{\frac{1}{4}}$ is a new separation axiom introduced by the authors. They emphasized that the $T_{\frac{1}{4}}$ space class lies squarely between the T_0space, and $T_{\frac{1}{2}}$ space classes, and that those spaces are ones in which each finite set is $\lambda-$closed. Throughout this paper, we will introduce new class axioms by using $ii-$ open set.A subset $A$of X is called $ii-$open set [6] if there exist an open set Gin the topology$\tau$of X, such that: $G \neq \phi, X, A \subset cl(A \cap G)$ and $int(A) = G$, the complement of an $ii-$ open set is an $ii-$ closed set.

The goal of this paper is to continue the investigation of $ii-T_{\frac{1}{4}}$spaces, specifically to look into their behavior when forming products. As a result, we may easily construct non-$ii-T_{\frac{1}{4}}$spaces as well as $ii-T_{\frac{1}{4}}$ spaces that aren't$ii-T_{\frac{1}{2}}$.

2 Preliminaries.

Throughout this paper (X,τ) or simply Xrepresent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$of X,$cl(A)$,$int(A)$, $cl_{ii}(A)$ and A^cdenote the closure of A, the interior of A, $ii-$closure of $A$and the complement of Arespectively. Let us recall the following definitions, which are useful in the sequel.

Definition. 2.1

A subset $A$of a space (X, τ) is called a

- Semi-open set [7] if $A \subset cl(int(A))$.
- A closed generalized set (in short$g-$ closed) if $cl(B) \subset G$at any time$B \subset G$ and $G$is open [2].
- A set (Λ) [1] is defined to be the set $\cap \{O \in (X, \tau) - A \subset O\}$.
- A $\lambda-$ closed set if $A = L \cap F$such that $L$is (Λ) set and $F$is closed set and $(\lambda)^c$represent λ −open set [1].
- $\delta-$ closed set if $B = \delta cl(B)$, where $\delta cl(B) = \{x \in X - int\, c\, l(G) \cap B \neq \emptyset\}$, $x \in G$and $G \in \tau$ [8]
- $\delta-$ closed generalized set (in short $\delta g-$ closed) if $\delta cl(B) \subset G$at any time $B \subset G$ in addition $G \in \tau$ [8].

Definition. 2.2

The intersection of all open set supersets of $A$is the kernel of a set A, represented by $\hat{A}$ [9].

Definition. 2.3

The ii −interior [6] of a subset $A$of $X$is the union of all ii −open set of Xcontained in $A$and is denoted by $int_{ii}(A)$. The subset $A$is called ii −open set [6] if $A = int_{ii}(A)$. The complement of a ii −open set is called ii −closed set. Alternatively, a subset $A$of a space (X, τ) is called ii −closed set [6] if $A = cl_{ii}(A)$, when $cl_{ii}(A) = \{x \in X: int(cl(G)) \cap A \neq \phi, G \in \tau$and $x \in G\}$.

Theorem. 2.4 [9]

The following conditions are equivalent for a subset $A$of a topological space(X, τ):

- $A$is $\lambda-$ closed set.
- $A = L \cap cl(A)$, where $L$is a (Λ) set.
- $A = \hat{A} \cap cl(A)$.

Theorem. 2.5 [9]

The following conditions are equivalent for a subset $A$of a topological space(X, τ):

- Every $\lambda-$ closed set is a locally closed set.
- Each (Λ) set is a closed λ −set.

Lemma. 2.6 [9]

Let (X, τ) be a topological space and let $A \subset (X, \tau)$ then $A$is $g-$ closed set if and only if $cl(A) \subset \hat{A}\ cl(A) \subset \hat{A}$.

Definition. 2.7

A space (X, τ) is called a:

- $T_{\frac{3}{4}}$ −space [5] if each δg −closed set in it is δ −closed.
- $y \notin F$ $T_{\frac{1}{2}}$ −space [3] if each $g-$ closed set in it is closed set.
- $T_{\frac{1}{4}}$ −space [1] if each finite subset $F \subset X$ and each point $y \notin F$ there is a subset that exists $A \subset X$ such that $F \subset A, y \notin A$, and A is open or closed.
- Semi T_1 −space [10] if for any $x, y \in X$ such that $x \neq y$ there exists a semi –open set $M(x)$ which contains x but not y, and an semi open set $N(y)$ which contain y but not x.

3 $ii - T_{\frac{1}{4}}$ space.

In this section we will talk about new separation axioms using the ii −open set.

Definition. 3.1

The topological space (X, τ) is said to be $\boldsymbol{ii - T_{\frac{1}{4}}}$ if each finite subset $F \subset X$ and each $y \notin F$, there is a subset $A \subset X$ that exists such that $F \subset A, y \notin A$, and A is $ii -$ (open or closed) set.

Example. 3.2

Let $X = \{a, b, c\}$, and $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

So, the ii −open sets are $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

We note that X is the $\boldsymbol{ii - T_{\frac{1}{4}}}$ space because it fulfills the above definition.

Proposition. 3.3

Every $T_{\frac{1}{4}}$ −space is $\boldsymbol{ii - T_{\frac{1}{4}}}$.

Proof.

let (X, τ) be a $T_{\frac{1}{4}} -$ space this mean each finite subset $F \subset X$ and each point $y \notin F$ there is a subset that exists $A \subset X$ such that $F \subset A, y \notin A$, and A is open or close, but every open set is $ii -$ open. This mean (X, τ) is $ii - T_{\frac{1}{4}}$ space.

Remark. 3.4

The converse of the above proposition is not true as shown in the following example.

Example. 3.5

Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$, by definition of $ii -$ open we have : $ii -$ open sets are: $\phi, X, \{b\}, \{a, b\}, \{b, c\}$. This mean (X, τ) is $ii - T_{\frac{1}{4}}$ space but it is not $T_{\frac{1}{4}} -$ space.

Definition. 3.6

Let (X,τ) be a topological space, and let $A \subset (X,\tau)$ then $A$is $ii\lambda$ −closed set if $A = L \cap F$, where $L$is a (Λ) set and $F$is ii −closed set. The complement of an

$ii\lambda$ −close set is an $ii\lambda$ −open set.

Proposition. 3.7

The following are equivalents for a space(X,τ):

- (X,τ) is a $ii - T_{\frac{1}{4}}$ space .
- There exists a subset for each locally finite subset $F \subset X$ and each point $y \notin F$.

Proof.

1) →2). Suppose that $F \subset X$be ii −locally finite and let $y \notin F$. We're done if $F$is finite. So, let's pretend $F$is infinite. $A = X - U$is the needed set if $y$has an open neighborhood $U$with an empty intersection with F. Otherwise$y \in cl(F)$, and because $F$is $ii -$ locally finite, there exists an open neighborhood $y$of y where $U \cap F$ is finite, say$\{x_1, x_2, \dots, x_k\}$, and$y \in cl_{ii}\{x_i\}$, for some x_i. Now choose any of them$x \in F - \{x_1, x_2, \dots, x_k\}$. Then $\{x_1, x_2, \dots, x_k\}$ is a finite set that does not contain$y$. Since (X,τ) is $ii - T_{\frac{1}{4}}$ A subset must exist $A_x \subset X$such that$\{x_1, x_2, \dots, x_k\} \subset A_x, y \notin A_x,$ and $A_x$is ii −open or closed. Since $y \in cl_{ii}\{x_i\} \subset cl_{ii}(A_x), A_x$cannot be ii −closed set, thus $A_x$must be open. If $A = \cup \{A_x : x \in F - \{x_1, \dots, x_k\}\}$ then $A$is an ii −open set containing Fsuch that$y \notin A$, and then we're done.

2) →1). Since any finite subset is ii-locally finite subset, this is obvious.

Definition. 3.8

The topological space (X,τ) is said to be $ii - T^c_{\frac{1}{4}}$ if for each at most countable subset $F \subset X$and each point$y \notin F$, there exists is a subset $A \subset X,$ such that $F \subset A, y \notin A,$ and $A$is $ii -$ (open or closed)set.

Remark. 3.9

Every $ii - T^c_{\frac{1}{4}}$ space is $ii - T_{\frac{1}{4}}$space. But the converse is not true as shown in the following example.

Example. 3.10

Let X be a collection of nonnegative integers in the topological space(X,τ), where $U \in \tau$if and only if $U = \phi$ or $0 \in U$While $X - U$is finite. Which is a $ii - T_{\frac{1}{4}}$ space by definition 3.1. Now let $F =$

$(X-0)$. Then $F$is countable and $0 \notin F$. Since $\{0\}$is neither ii −open nor ii −closed, (X, τ) cannot be $ii - T^c_{\frac{1}{4}}$.

Definition. 3.11

A subset $A$of (X, τ) is called iig −closed set if $cl_{ii}(A) \subset U$whenever $A \subset U$and Aisii −open.

4 Applications.

We introduce the following definition.

Definition. 4.1`

A topological space (X, τ) is called.

- $ii - T_{\frac{1}{2}}$ space if every iig −closed set is ii −closed set.
- $ii - T_1$ space if for each two distinct points $x, y \in X$, there exist ii −open sets U, Vsuch that $x \in U, y \in V$and $U \cap V = \phi$.
- $ii - T_0$ space if for each two distinct points $x, y \in X$, there exist ii −open set Usuch that $x \in U, y \notin U$or vice versa.

Theorem. 4.2

The following conditions are equivalent for a subset $A$of a topological space(X, τ).

- $X$is a $ii - T_{\frac{1}{4}}$ space.
- Each finite subset of $X$is $ii\lambda$ −closed set.

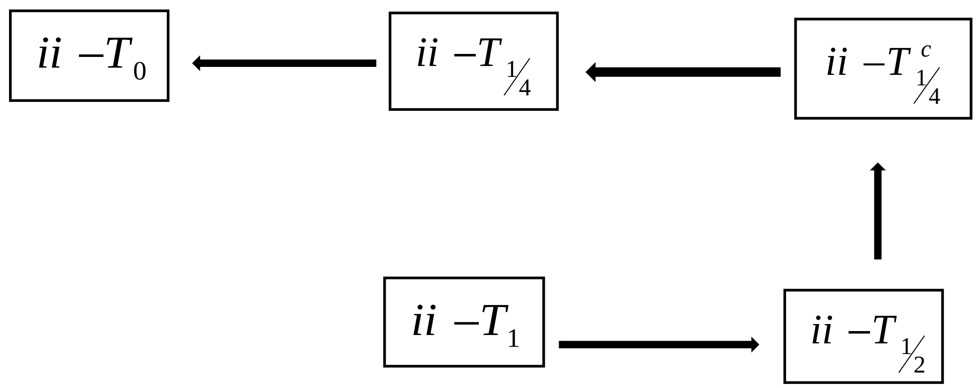
Proof.

1) →2) let $F \subset X$ be a subset of Xthat is finite. Since $X$is $ii - T_{\frac{1}{4}}$, then for each point $y \notin F$, there exists is a subset $A \subset X$, such that $F \subset A, y \notin A$, and $A$is $ii -$ (open or closed) set. Let Cbe the intersection of all $ii -$ closed sets A and be the intersection of all ii −open sets A. Clearly, $L$is (Λ) set and $C$is $ii -$ closed. Note that $F = L \cap C$. This demonstrates that $F$is $ii\lambda$ −closed set.

2)→ 1) Let Fbe a finite subset of X, and $y \notin F$. By (2) $F = L \cap C$, where $L$is (Λ) set and $C$is $ii -$ closed. If C does not include y, then $X - C$is an ii −open set containing y, and then we're done. If Ccontains y, then $y \notin L$ As a result, for some ii −open set U that contains F, we have $y \notin U$. therefore $X$is $ii - T_{\frac{1}{4}}$.

Remark. 4.3

The following diagram shows the relationship between the new axioms of separation.



Theorem. 4.4

The following conditions are equivalent for a subset $A$of a topological space(X,τ).

- $X$is $ii-T_{\frac{1}{2}}$ space.
- Every singleton of $X$is $ii\lambda-$ closed.

Proposition. 4.5.

let (X,τ) be $ii-T_{\frac{1}{4}}$ space. Then each subspace of $X$is a $ii-T_{\frac{1}{4}}$ space.

5 $ii-T_{\frac{1}{4}}$ space's products.

In this section we give the products for the new separation axioms that we defined previously.

Proposition. 5.1

Let (X,τ) and (Y,τ^*) be a topological space such that $X\times Y$are $ii-T_{\frac{1}{4}}$. After then, both spaces are available (X,τ) and (Y,τ^*) are $ii-T_{\frac{1}{4}}$, and at least one of the space must be$ii-T_1$.

Proof.

Since (X,τ) and (Y,τ^*) are homeomorphic to subspaces of $X\times Y$, As a result of Proposition 4.5, both spaces must be $ii-T_{\frac{1}{4}}$.

Assume that neither (X,τ) nor (Y,τ^*) are $ii-T_1$ spaces. Then there exist $x_1,x_2\in X, x_1=x_2$ such that $x_2\in cl_{ii}\{x_1\}$, and $y_1,y_2\in Y, y_1\neq y_2$ such that $y_2\in cl_{ii}\{y_1\}$. Assume that $F=\{(x_1,y_2),(x_1,y_1),(x_2,y_2)\}\subset X\times Y$. Then $(x_2,y_1)\notin F$. There is, according to theory a set $A\subset X\times Y$, such that $F\subset A$, $(x_2,y_1)\notin A$, and $A$is $ii-$ (open or closed).

If $Aii-$closed , this mean if $(X\times Y)-A$is $ii-$open, there exist $ii-$open sets $U_2\subset X, V_1\subset Y$ such that $x_2\in U_2, y_1\in V_1$and $(U_2\times V_1)\cap F=\phi$. however, since $x_2\in cl_{ii}\{x_1\}$, we have $x_1\in U_2$, and so $(x_1,y_1)\in U_2\times V_1$, a contradiction.

If $A$is $ii-$open, After that, there's $ii-$open set $U_2 \subset X, V_2 \subset Y$such that $x_2 \in U_2, y_2 \in V_2$and $U_2 \times V_2 \subset A$. Since $y_2 \in cl_{ii}\{y_1\}$.We have $y_1 \in V_2$, and thus $(x_2, y_1) \in A$, a contradiction. As a result, at least one of the spaces is (X, τ) or (Y, τ^*) must be $ii - T_1$.

Proposition. 5.2.

Let $X = \prod_{i \in I} x_i$ be a topological space's product $(x_i, \tau_i), i \in I$. if $X$is $ii - T_{\frac{1}{4}}$ then all spaces (x_i, τ_i) are $ii - T_{\frac{1}{4}}$ and there is just one factor that isn't space $ii - T_1$.

Proof.

Since each (x_i, τ_i) is homeomorphic to subspaces of X, and therefore it is $ii - T_{\frac{1}{4}}$ by Proposition 4.5. $(\mathbb{Z}, \kappa)$, Assume the following: (x_i, τ_i) and $(x_j, \tau_j), i \neq j$, are not $ii - T_1$. Since $(X_i \times X_j)$ is homeomorphic to subspaces of X, it must be $ii - T_{\frac{1}{4}}$ by Proposition 4.5. However, this is a contradiction to Proposition 5.1.

It's simple to come up with examples of $ii - T_1$ spaces that aren't $ii - T_{\frac{1}{4}}$ using our previous findings. For instance, we have

Example. 5.3

The product of two copies of the digital line$(\mathbb{Z}, \kappa)$, fails to be $ii - T_{\frac{1}{4}}$ in the digital plane$\mathbb{Z}^2$.

Proposition. 5.4

If (X, τ) is $ii - T_{\frac{1}{4}}$and (Y, τ^*) i $ii - T_1$. Then $X \times Y$is $ii - T_{\frac{1}{4}}$.

Proof.

Let $F = \{(x_1, y_1), \dots, (x_n, y_n)\} \subset X \times Y$and let $(x_0, y_0) \notin F$. The natural projections are denoted by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Let $F^* = \{(x_i, y_i) \in F : y_i = y_0\}$ and $F^{**} = \{(x_i, y_i) \in F : y_i \neq y_0\}$. Then $F = F^* \cup F^{**}, x_0 \notin p_1(F^*)$ and$y_0 \notin p_2(F^{**})$. . Since $p_1(F^*) \subset X$ is finite and(X, τ) is $ii - T_{\frac{1}{4}}$, there is a subset $A \subset X$such that $p_1(F^*) \subset A, x_0 \notin A$and $A$is $ii-$open or closed. If $A$is $ii-$closed in (X, τ), then $A \times \{y_0\}$ is $ii-$closed in $X \times Y$such that $F^* \subset A \times \{y_0\}, (x_0, y_0) \notin A \times \{y_0\}$. Also, $X \times p_2(F^{**})$ is $ii-$closed in $X \times Y$, $F^{**} \subset X \times p_2(F^{**})$ and $(x_0, y_0) \notin X \times p_2(F^{**})$. As a result, there is an $ii-$closed set$B \subset X \times Y$, namely$B = (A \times \{y_0\}) \cup (X \times p_2(F^{**}))$, such that $F \subset B$and $(x_0, y_0) \notin B$.

If $A$is $ii-$open in(X,τ), then $A\times Y$ is $ii-$open in $X\times Y$such that $F^{*}\subset A\times Y$, $(x_0,y_0)\notin A\times Y$. Also, $X\times(Y-\{y_0\})$ is $ii-$open in $X\times Y$, $F^{**}\subset X\times(Y-\{y_0\})$and $(x_0,y_0)\notin X\times(Y-\{y_0\})$. As a result, there is an $ii-$open set $B\subset X\times Y$, namely$B=(A\times Y)\cup(X\times(Y-\{y_0\}))$, such that $F\subset B$ and $((x_0,y_0)\notin B$. This demonstrates that $X\times Y$is $ii-T_{\frac{1}{4}}$.

Proposition. 5.5

Let $X=\prod_{i\in I}X_i$ a topological space's product (x_i,τ_i), $i\in I$. If (x_i,τ_i) is $ii-T_{\frac{1}{4}}$ for some $i\in I$ and all other spaces (x_j,τ_j), $i\neq j$ are $ii-T_1$, then $X$is $ii-T_{\frac{1}{4}}$.

Proof.

Clearly X is homeomorphic to$X_i\times(\prod_{j\neq i}X_j)$. By Proposition. 5.4, it follows that $X$is $ii-T_{\frac{1}{4}}$, since $\prod_{j\neq i}X_j$is $ii-T_1$.

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