Journal of AL-Qadisiyah for computer science and mathematics Vol.9 No.2 Year 2017
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Rate of approximation of K-monotone functions in $L_{\psi,p}(I)$ space , $0<p<1$

Nada Zuhair Abd AL-Sada
Department of Mathematics, College of education of Al-Qadisiyah University
E-mail:Nadawee70@yahoo.com

Recived : 12\6\2017 **Revised : 31\7\2017** **Accepted : 22\8\2017**

Abstract: In this paper we shown that the relationship with the best algebraic approximation and K-monotone functions with bounded (i) such that $(i<k\,,i\geq 1)$ derivatives by algebraice polynomial of degree $\leq k-1$, which interpolates a K-monotone functions f in an interval I at K points, and by this worke , we are found the rate of approximation of K-monotone functions in space $L_{\psi,p}(I)\,,0<p<1$.

Key word: Monotone functions, approximation, Modulus of smoothness.

Mathematics subject classification

1.Introduction and Main results

Let $f\in L_{\psi,p}(I), I=[-b,b]$, and let p_{k-1} be algebraic polynomials of degree $\leq k-1$ which interpolates f at k points , and let $\omega_{\varphi}^{k}(f,n^{-1})_{\psi,p}$ the Ditzian-Totik modulus of smoothness of $f\in L_{\psi,p}(I), 0<p<1$, which defined by :

$$\omega_{\varphi}^{k}(f,\delta,I)_{\psi,p}=\sup_{0<h\leq\delta}\left\|\Delta_{h}^{k}(f,.)\right\|_{L_{\psi,p}(I)}$$

Where $\|.\|_{L_{\psi,p}(I)}$ denotes the weighted quasi normed space([3]) and

$$\Delta_{h}^{k}(f,x,I)_{\psi}=\Delta_{h}^{k}(f,x)_{\psi}$$

$$=\begin{cases}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\dfrac{f(x-\frac{kh}{2}-ih)}{\psi(x+\frac{kh}{2})} & ,x\pm\dfrac{kh}{2}\in I\\ 0 & o.w\end{cases}$$

is the kth symmetric difference .A functions $f:I\longrightarrow R$ is said to be k −monotone , $k\geq 1$ on $I=[-b,b]$ iff for all choices of $(k+1)$ distinct $x_0,\dots,x_k$ in I , the inequality $[x_0,\dots,x_k]f\geq 0$ …(1)
holds, where $[x_0,\dots,x_k]f=\sum_{j=0}^{k}(f(\frac{x_j}{w(x_j)})$, denotes the kth divided difference of f at $x_0,\dots,x_k$ and $w(x)=\prod_{j=0}^{k}(x-x_j)$ ([2]). Note that 1 −monotone (2 −monotone) functions the class of all k −monotone functions on I is denoted by $\Delta^{k}[I]$([5]). A function f is called weakly k −monotone if the inequility(1) is satisfied for any set of equally spaced points $x_0,\dots,x_k$ ([6]) . We let $\Delta^{1}(J)_s$ be the set of functions f which chang their monotone exactly at the points $j_i\in J_s$, and we will write $f\in\Delta^{1}$. We consider the space $L_{\psi,p}(I)$, consisting of all functions f on an interval I for which

$$\|f\|_{L_{\psi,p}(I)}^{p}=\int_{I}\left|\frac{f(x)}{\psi(x)}\right|^{p}dx<\infty\,.$$

Recall that for $f\in L_{\psi,p}(I)$ that

$$\|f\|_{L_{\psi,p}(I)}\leq 2^{\frac{1}{p}-1}\|f\|_{L_{\psi,1}(I)}\quad\dots(2)$$

That is $L_{\psi,1}(I)\subset L_{\psi,p}(I)$.

Suppose for some $k\geq 2$ that $f\longrightarrow R$ is k −monotone then $(\frac{f}{\psi})_{(x)}^{(j)}$, the derivative of order j , exists on $(-b,b)$ for $j\leq k-2$ and is $(k-j)$ −monotone ([1]).

The following theorem is the main results of this paper :

Theorem (1.1): Let $f\in\Delta^{k}[I]$, be such that $(\frac{f}{\psi})^{(i)}\in L_{\psi,p}(I)\,,i<k,i\geq 1$, then there exist a polynomial $p_n\in\prod_n$ such that

$$\|f-p_n\|_{L_{\psi,p}(I)}$$
$$\leq c(p,k)2^{\left(\frac{p-1}{p^2}\right)}n^{\frac{-k}{p}}\omega_{\varphi}^{k}(f,\delta)_{\psi,p}^{1-\frac{1}{p}}\|f^{(i)}\|_{L_{\psi,1}(I)}^{\frac{1}{p}}$$

2. Auxiliary Results:

Now the following Lemmas are crucial for the proof of theorem (1.1).

Lemma (2.1) [4]: There exist a polynomial $g_{k-1} \in \prod_{k-1}, k > 1$ interpolate f at $k > 1$ points inside an interval of $J_A = [m_0 + A|I|, m_1 - A|I|]$ where $A < \frac{1}{2}$, is a strictly positive constant then :

$$\|f - g_{k-1}(f)\|_{L_{\psi,p}(I)} \le c(p,k)\omega_\varphi^k(f, |I|, I)_{\psi,p} .$$

Lemma (2.2)[3]: For $f \in L_{\psi,p}(I)$, k>1 , 0<p<1, 0<h≤ δ , then

$$\omega_k^\varphi(f,\delta)_{\psi,p} \le c(p)\delta\omega_\varphi^{k-1}(\acute{f},\delta)_{\psi,p} .$$

Lemma (2.3)[3]: For a functions $f \in L_{\psi,p}(I)$, 0<p<1 we have

$$\omega_k^\varphi(f,\delta)_{\psi,p} \le c(p,k)\|f\|_{L_{\psi,p}(I)} .$$

Lemma (2.4) **:**Let $f \in \Delta^k[I]$, $i < k$, $i \ge 1$, then :

$$\omega_k^\varphi(f,\delta)_{\psi,p} \le c(p)n^{-i}\omega_\varphi^{k-i}(f^{(i)},\delta)_{\psi,p} .$$

Proof: By using the definition of the modulus of smoothness, keeping in mind that $\Delta_h^{k-1}f(x) \ge 0$ for $f \in \Delta^k$ and changing variables, we have

$$\Delta_{h\varphi(x)}^{k-1}(\acute{f},x)_\psi = \Delta_{h\varphi(x)}^{k-2}(\Delta_{h\varphi(x)}^1(\acute{f},x)_\psi$$

$$\|\Delta_{h\varphi(.)}^{k-1}\|_{L_{\psi,p}(I)} = \left\|\Delta_{h\varphi(.)}^{k-2}[\acute{f}\left(x+\frac{h}{2}\right) - \acute{f}\left(x-\frac{h}{2}\right)]\right\|_{L_{\psi,p}(I)}$$

$$= \|\Delta_{h\varphi(.)}^{k-2}[\acute{f}(x+\tfrac{h}{2}) - \acute{f}(x)] - [\acute{f}(x-\tfrac{h}{2}) - \acute{f}(x)]\|_{L_{\psi,p}(I)}$$

$$= \left\|\Delta_{h\varphi(.)}^{k-2}(\int_0^{\frac{h}{2}}[\acute{\acute{f}}(x+\iota) - \acute{\acute{f}}(x-\iota)])dl\right\|_{L_{\psi,p}(I)}$$

$$\le c(p)\int_0^{\frac{h}{2}}\left\|\Delta_{h\varphi(.)}^{k-2}\left[\acute{\acute{f}}(x+l) - \acute{\acute{f}}(x-l)\right]dl\right\|_{L_{\psi,p}(I)}$$

$$\le c(p)\int_0^{\frac{h}{2}}\omega_\varphi^{k-2}(\acute{\acute{f}},\delta)_{\psi,p}dl$$

$$= c(p)\tfrac{h}{2}\omega_\varphi^{k-2}(\acute{\acute{f}},\delta)_{\psi,p} \text{ , hence}$$

$$\omega_\varphi^k(f,\delta)_{\psi,p} \le c(p)n^{-2}\omega_\varphi^{k-2}(\acute{\acute{f}},\delta)_{\psi,p} \quad \ldots(3$$

Now, by lemma (2.2) and the inequality (3) for $i < k$ where $i \ge 1$ we get the result

$$\omega_\varphi^k(f,\delta)_{\psi,p} \le c(p)n^{-i}\omega_\varphi^{k-i}(f^{(i)},\delta)_{\psi,p} .$$

Lemma (2.5): Let $f \in L_{\psi,p}(I)$, then for $i < k$ where $i \ge 1$ there exists a polynomial $p_n \in \prod_n$,which is satisfies

$$\|f - p_n\|_{L_{\psi,1}(I)} \le c(p,k)n^{-k}\|f^{(i)}\|_{L_{\psi,1}(I)}$$

Proof: by Lemma (2.1) then there exist a polynomial $p_n \in \prod_{k-1}, k > 1$ interpolate f at k points which is satisfies

$$\|f - p_n(f)\|_{L_{\psi,p}(I)} \le c(p,k)\omega_\varphi^k(f,|I|,I)_{\psi,p} .$$

And by lemma (2.4) for $i < k$ where $i \ge 1$ we get

$$\|f - p_n\|_{L_{\psi,1}(I)} \le c(p,k)n^{-i}\omega_\varphi^{k-i}(f^{(i)},\delta)_{\psi,1} .$$

By lemma (2.3) we get

$$\|f - p_n\|_{L_{\psi,1}(I)} \le c(p,k)n^{-i}n^{-(k-i)}\|f^{(i)}\|_{L_{\psi,1}(I)} ,$$

hence

$$\|f - p_n\|_{L_{\psi,1}(I)} \le c(p,k)n^{-k}\|f^{(i)}\|_{L_{\psi,1}(I)}$$

Proof of theorem (1.1): by the inequality (2) then we have

$$\|f - p_n\|_{L_{\psi,p}(I)}^p \le 2^{1-\frac{1}{p}}\|f - p_n\|_{L_{\psi,1}(I)}^p$$

$$\|f - p_n\|_{L_{\psi,1}(I)}^p = \int_I \left|\frac{f-p_n}{\psi(x+\frac{kh}{2})}\right|^{p-1}\left|\frac{f-p_n}{\psi(x+\frac{kh}{2})}\right| dx$$

$$\|f - p_n\|_{L_{\psi,p}(I)}^p \le c(p)2^{(1-\frac{1}{p})}\|f - p_n\|_{L_{\psi,p}(I)}^{p-1}\|f - p_n\|_{L_{\psi,1}(I)}$$

By lemma (2.1), we get

$$\|f - p_n\|_{L_{\psi,p}(I)}^p \le c(p)2^{\left(1-\frac{1}{p}\right)}\omega_\varphi^k(f,\delta)_{\psi,p}^{p-1}\|f - p_n\|_{L_{\psi,1}(I)} .$$

By lemma (2.5)

$$\|f - p_n\|_{L_{\psi,p}(I)}^p \le c(p)2^{\left(1-\frac{1}{p}\right)}n^{-k}\omega_\varphi^k(f,\delta)_{\psi,p}^{p-1}\|f^{(i)}\|_{L_{\psi,1}(I)} .$$

Hence

$$\|f - p_n\|_{L_{\psi,p}(I)} \le c(p,k)2^{\left(\frac{p-1}{p^2}\right)}n^{\frac{-k}{p}}\omega_\varphi^k(f,\delta)_{\psi,p}^{1-\frac{1}{p}}\|f^{(i)}\|_{L_{\psi,1}(I)}^{\frac{1}{p}}$$

Where $0 < p < 1$, $i < k$ and ≥ 1 .

References

[1] Kirill A. Kopotun , Approximation of k-Monotone Functions,December 23,1996;accepted June 4,1997 .

[2] K.Kopotun, Simulataneous approximation by algebraic polynomials,Constr.Approx.12(1996),67-94.

[3] N.Z.Abd Al-Sada (2015):" On Positive and Copositive Approximation in $L_{\psi,p}(I)$ Spaces $0 < p < 1$"ph.D dissertation, Al-Mustansiriya University, College of education.

[4] N.Z.Abd Al-Sada , On Comonotony Approximation in Quasi Normed Space ,2017.

[5]P.S.Bullen , A criterion for n-convexity,Pacific J. Math.36 (1971),81-98.

[6] Z.Ciesielski,Some properties of convex functions of higher orders,Ann.polon.Math.7(1959),1-7.

قيمة تقريب الدالة المتناوبة – K في الفضاء $L_{\psi,p}(I)$ ، $0 < p < 1$

ندى زهير عبد السادة

قسم الرياضيات ، كلية التربية ، جامعة القادسية

E-mail:Nadawee70@yahoo.com

المستخلص :

في هذا البحث اشرنا الى العلاقة بين افضل تقريب والدالة المتناوبة -K ذات المشتقة(i) حيث ان $(i < k , i \geq 1)$ ، باستخدام متعدد حدود لاكرانج التي درجتها
$\geq k - 1$ ، والتي تكون فيها نقاط التقاطع بينها وبين الدالة المتناوبة f ضمن الفترة I عند k من النقاط وبهذا العمل نكون قد اوجدنا قيمة تقريب الدالة المتناوبة في الفضاء $L_{\psi,p}(I)$ المعيار $0 < p < 1$.

الكلمات المفتاحية : الدالة المتناوبة ، التقريب ، مقياس النعومة .