Strongly Goldie Extending Modules

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Abstract

A submodule N of a right R-module M is said to be strongly large in case for any $m \in M$, $s \in R$ with $ms \neq 0$, there exists an $r \in R$ such that $mr \in N$ and $mrs \neq 0$. In this paper, we introduce strongly $\mathcal{G}$-extending modules which are particular $\mathcal{G}$-extending modules, and investigate their properties and characterizations. An R-module M is called strongly $\mathcal{G}$-extending if for each submodule X of M there exists a direct summand D of M such that $X \cap D$ is strongly large in both X and D. Some sufficient conditions for a direct sum of strongly $\mathcal{G}$-extending modules to be strongly $\mathcal{G}$-extending are obtained. Examples to illustrate this concept are given.

Kew words: Strongly large submodules; SL-closed submodules; Strongly extending modules; $\mathcal{G}$-extending modules; Strongly $\mathcal{G}$-extending modules; Strongly $\mathcal{G}^+$-extending modules.

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1 Introduction

Throughout all rings are associative with identity, unless indicated otherwise, and all modules are unital right R-modules, R denotes such a ring. A submodule N of a right R-module M is a large submodule (briefly $N \leq_e M$) if, $N \cap K \neq 0$ for each nonzero submodule K of M [8]. And, a submodule N of M is said to be strongly large (briefly $N \leq_{sl} M$) in case for any $m \in M$, $s \in R$ with $ms \neq 0$, there exists an $r \in R$ such that $mr \in N$ and $mrs \neq 0$ [16]. Notice that every strongly large submodule of a module is always large submodule but not conversely. In [2], Akalan, Birkenmeier and Tercan, consider the following relations on the set of submodules of a module M: (i) $X\alpha Y$ if and only if there exists $A \leq M$ such that $X \leq_e A$ and $Y \leq_e A$; (ii) $X\beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$. Note that β is an equivalence relation. Moreover, an R-module M is called extending (also called CS) if, for each $X \leq M$ there exists a direct summand D of M such that $X\alpha D$ [5]. Also, a module M is said to be Goldie extending (briefly $\mathcal{G}$-extending) if, for each $X \leq M$ there exists a direct summand D of M such that $X\beta D$ [2]. It is clear that every extending module is $\mathcal{G}$-extending. Following [15], a module M is called strongly extending if every submodule X of M, there exists a decomposition $M = K \oplus L$ such that X is a strongly large in K.

In this paper, our aim is to introduce and study strongly $\mathcal{G}$-extending modules by using concept of strongly largeness. In section 2, we introduce the relations ω and ρ on the set of submodules of a module, as following: (i) $X\omega Y$ if and only if there exists $A \leq M$ such that $X \leq_{sl} A$ and $Y \leq_{sl} A$; (ii) $X\rho Y$ if and only if $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$.Since every strongly large is always large, then clear that the relations ω and ρ imply relations α , β respectively. Notice ρ is an equivalence relation. We give some elementary properties of these relations. For a module M, we prove that M is strongly extending if and only if for each $X \leq M$ there exists a direct summand D of M such that $X\omega D$. In section 3, we introduce the concept of strongly $\mathcal{G}$-extending modules, and give some properties of such modules. An R-module M is called strongly $\mathcal{G}$-extending if for each $X \leq M$ there exists a direct summand D of M such that $X\rho D$. Various characterizations of such modules are given. Also, we define another concept of modules called strongly $\mathcal{G}^{+}$-extending, where an R-module M is said to be strongly $\mathcal{G}^{+}$-extending if every direct summand of M is strongly $\mathcal{G}$-extending. It is shown that a strongly $\mathcal{G}$-extending module satisfying (C_3) is strongly $\mathcal{G}^{+}$-extending. Some relations between strongly $\mathcal{G}$-extending modules and other concept of modules are obtained. In section 4, we discuss various sufficient conditions on a module under which the direct sum of strongly $\mathcal{G}$-extending modules is strongly $\mathcal{G}$-extending. We prove that, if M_1 and M_2 are strongly $\mathcal{G}$-extending modules with $M = M_1 \oplus M_2$ is a duo SL-direct sum module, then M is strongly $\mathcal{G}$-extending, where for a module M, $\{A_\alpha\}$ and $\{B_\alpha\}$ be collections of submodules of M. Then M is called a SL-direct sum module if for each α, $A_\alpha \leq_{sl} B_\alpha$ implies $\oplus A_\alpha \leq_{sl} \bigoplus B_\alpha$.

2 Basic Properties Of Relations $\boldsymbol{\omega}$ And $\boldsymbol{\rho}$

In this section we define the relations ω and ρ on the set of submodules of a module. Many of properties of such relations are given. We give some characterizations of these relations.

Now, we presented the following definition.

Definition 2.1. For a module M, considered the following relations on the set of submodules of M :

(i) $X\omega Y$ if and only if there exists $A \leq M$ such that $X \leq_{sl} A$ and $Y \leq_{sl} A$;

(ii) $X\rho Y$ if and only if $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$.

A submodule N of a module M is called SL-closed if, N has no proper strongly large extensions in M [16]. Ungor; Halicioglu; Kamal and Harmanci in [16], introduce Theorem 4.8, " if M is a module and $N \leq M$. Then there exists $H \leq M$ such that N is a strongly large submodule in H and H is a SL-closed in M ". In this case, a submodule H is called SL-closure of N and it is not necessarily unique. However, Ghawi; T.Y in [7], presented the following, an R-module M is called a SL-UC module if every submodule of M has a unique SL-closure.

Remarks 2.2.

(i) It is clear that ω is reflexive and symmetric, but may not be transitive.

(ii) The relation ρ is an equivalence relation.

Proof. It is clear that ρ is reflexive and symmetric. Now, to prove that ρ is transitive. Let X , Y and L be submodules of M such that $X\rho Y$ and $Y\rho L$. Let $x \in X$ and $r \in R$ such that $xr \neq 0$. Since $X \cap Y \leq_{sl} X$, so there exists $s_1 \in R$ such that $xs_1 \in X \cap Y$; $xs_1 r \neq 0$. Since $Y \cap L \leq_{sl} Y$ and $xs_1 \in Y$, there exists $s_2 \in R$ such that $xs_1 s_2 \in Y \cap L$ and $xs_1 s_2 r \neq 0$. Put $t = s_1 s_2$, this means there exists $t \in R$ such that $xt \in X \cap L$ and $xtr \neq 0$. Thus $X \cap L \leq_{sl} X$. By a similar way, $X \cap L \leq_{sl} L$. Hence $X\rho L$ and ρ is transitive. ■

(iii) If X and Y are submodules of M such that $X\omega Y$, then $X\rho Y$.

Proof. Let $X, Y \leq M$ and $X\omega Y$, then there exists a submodule A of M such that $X \leq_{sl} A$ and $Y \leq_{sl} A$, thus by [16, Lemma 2.9] $X \cap Y \leq_{sl} A$. So $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$; that is $X\rho Y$.■

(iv) For a module M and $X \leq M$. $X\rho M$ if and only if $X \leq_{sl} M$. Also, $X\rho\{0\}$ if and only if $X = 0$.

(v) Let M be a module such that $X_1\rho Y_1$ and $X_2\rho Y_2$,then $(X_1 \cap X_2)\rho(Y_1 \cap Y_2)$ where X_1, X_2, Y_1 and Y_2 are submodules of M.

Proof. Since $X_1\rho Y_1$ and $X_2\rho Y_2$,so $X_1 \cap Y_1 \leq_{sl} X_1$ and $X_1 \cap Y_1 \leq_{sl} Y_1$, also $X_2 \cap Y_2 \leq_{sl} X_2$ and $X_2 \cap Y_2 \leq_{sl} Y_2$. Then by [16, Lemma 2.9],$(X_1 \cap X_2) \cap (Y_1 \cap Y_2)$ is strongly large in both $X_1 \cap X_2$ and $Y_1 \cap Y_2$, and hence $(X_1 \cap X_2)\rho(Y_1 \cap Y_2)$. ■

(vi) Let M be a module such that X_i, Y_i are submodules of M, for all $i \in (I\ finite\ set)$. If $X_i\rho Y_i$ then $(\bigcap_{i\in I} X_i)\ \rho\ (\bigcap_{i\in I} Y_i)$ for all $i \in I$.

Proof. It is straightforward. ■

In the next, we give a condition under which ω is transitive.

Proposition 2.3. Let M be an R-module. Then ω is transitive if and only if M is a SL-UC module.

Proof. Assume that ω is transitive. Let $X \leq M$ and let D_1, D_2 be two SL-closed submodules of M such that $X \leq_{sl} D_1$ and $X \leq_{sl} D_2$. Since $D_1 \leq_{sl} D_1$ and $D_2 \leq_{sl} D_2$, so we have $D_1\omega X$ and $X\omega D_2$, thus $D_1\omega D_2$, so there exists $A \leq M$ such that $D_1 \leq_{sl} A$ and $D_2 \leq_{sl} A$. But D_1 and D_2 are both SL-closed, hence $D_1 = A = D_2$.

Conversely, let M be a SL-UC module and let N, K and L be submodules of M such that $N\omega K$ and $K\omega L$. Then there exists submdules A and B of M such that $N \leq_{sl} A$, $K \leq_{sl} A$, $K \leq_{sl} B$ and $L \leq_{sl} B$. Assume $N \leq_{sl} X$, $K \leq_{sl} Y$ and $L \leq_{sl} P$, where $X, Y$and P are SL-closed in M. By [16, Lemma 2.9], $N \cap K \leq_{sl} A$ and $K \cap L \leq_{sl} B$. Hence, $N \cap K \leq_{sl} N \leq_{sl} X$, $N \cap K \leq_{sl} K \leq_{sl} Y$, $K \cap L \leq_{sl} K \leq_{sl} Y$ and $K \cap L \leq_{sl} L \leq_{sl} P$. But M is a SL-UC module, then $X = Y = P$ this is implies $N \leq_{sl} X$ and $L \leq_{sl} X$. Therefore $N\omega L$ and hence ω is transitive. ■

Proposition 2.4. A module M is SL-UC if and only if $\omega = \rho$.

Proof. Assume that M is a SL-UC module. By Remarks 2.2(iii), the relation ω implies ρ. Now, let X and Y be submodules of M such that $X\rho Y$, then $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$. Let H_1 and H_2 be two SL-closed submodules of M such that $X \leq_{sl} H_1$ and $Y \leq_{sl} H_2$, then $X \cap Y \leq_{sl} H_1$ and $X \cap Y \leq_{sl} H_2$, but M is SL-UC, $H_1 = H_2$. Thus $X\omega Y$ and hence $\omega = \rho$.

Conversely, since ρ is transitive and $\omega = \rho$, then ω is transitive, so by previous Proposition, M is a SL-UC module. ■

The next result presented in [16].

Proposition 2.5. Let M , N be R-modules and $\varphi: M \to N$ be an R-monomorphism. Then

(i) If $A \leq_{sl} B$ in M, then $\varphi(A) \leq_{sl} \varphi(B)$ in $Im\varphi$.

(ii) If $X \leq_{sl} N$, then $\varphi^{-1}(X) \leq_{sl} M$.

Now, we introduce the following Proposition.

Proposition 2.6. Let M , N be an R-modules and $\varphi: M \to N$ be an R-monomorphism. Then

(i) If $A\rho B$, then $\varphi(A)\ \rho\ \varphi(B)$ where $A, B \leq M$.

(ii) If $X\rho Y$,then $\varphi^{-1}(X)\ \rho\ \varphi^{-1}(Y)$ where $X, Y \leq N$.

Proof. It follows by Proposition 2.5. ■

The condition of monomorphism of Proposition 2.6 (*i*) is necessarily, as the following example shows.

Example 2.7. Consider the natural epimorphism $\pi: Z \oplus Z_2 \to Z \oplus Z_2 / Z \oplus (0)$. Let $A = (1, \bar{0})Z$ and $B = (1, \bar{1})Z$, then $A \cap B = (2, \bar{0})Z$ is a strongly large submodule in both A and B, so $A\rho B$. But $\pi(A) = Z \oplus (0) / Z \oplus (0) = 0$ is not related to $\pi(B)$ by ρ . Note that π is not monomorphism.

Let M be an R-module. We define $Z_S(M) = \{m \in M: r_R(m) \leq_{sl} R_R\}$,where $r_R(m)$ means the annihilator set of m in R . If R is a commutative ring, then $Z_S(M)$ is a submodule of M called a strongly singular submodule. An R-module M is called strongly singular if $Z_S(M) = M$, and M is called non-strongly singular if $Z_S(M) = 0$ [16].

We shall present the following Proposition. Before, we need the following Lemma.

Lemma 2.8. Let R be a commutative ring and let A, B and C be R-modules. If there exists a short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ such that $Imf \leq_{sl} B$, then C is a strongly singular module. In particular, if $A \leq_{sl} B$ then B/A is strongly singular.

Proof. Assume that a short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$. For $b \in B$, we can given a homomorphism $h: R \to B$ defined by $h(r) = br$ for all $r \in R$. Since h is R-homomorphism and $Imf \leq_{sl} B$, then by Proposition 2.5(ii), $h^{-1}(Imf) \leq_{sl} R_R$. Put $I = \{r \in R: br \in Imf\}$, thus $BI \leq Imf = Kerg$, hence $\big(g(b)\big)I = 0$ and so $g(b) \in Z_S(C)$. Thus $C = Img \subseteq Z_S(C)$. ■

Proposition 2.9. Let R be a commutative ring and M be an R-module. If A , B are submodules of M such that $A\rho B$, $A + B/A$ and $A + B/B$ are strongly singulars.

Proof. Since $A\rho B$, then $A \cap B \leq_{sl} A$ and $A \cap B \leq_{sl} B$, then by previous Lemma, $A/A \cap B$ and $B/A \cap B$ are strongly singulars, this implies $A + B/A$ and $A + B/B$ are strongly singulars, by 2nd isomorphisms theorem. ■

Let M be an R-module. M is said to satisfy condition (∗) in case if, for each $m(\neq 0) \in M$ and $r_1, r_2 \in R$, if $r_i \notin r_R(m)$ for some $(i = 1,2)$ and $r_1 R r_2 \leq r_R(m)$, then $r_j = 0$ for $j \neq i$ [16].

The following Proposition is appeared in [16, Prop. 3.8].

Proposition 2.10. Let M be an R-module satisfy the condition (∗).Then M/K is a strongly singular R-module, for all strongly large submodule K of M.

Remark 2.11. If M is an R-module satisfy the condition $(*)$ and $N \leq M$, then it is easy to see that N also satisfies condition $(*)$.

Proposition 2.12. Let M be an R-module satisfies condition $(*)$, A and B are submodules of M. If $A\rho B$, then $A+B/A$ and $A+B/B$ are strongly singulars.

Proof. Assume that $A\rho B$, then $A \cap B \leq_{sl} A$ and $A \cap B \leq_{sl} B$. Since M satisfy the condition $(*)$, so by above Remark both of A and B is so. Thus by Proposition 1.10, $A/A \cap B$ and $B/A \cap B$ are strongly singulars, and so by 2^{nd} isomorphisms theorem, $A+B/A$ and $A+B/B$ are strongly singulars. ■

Using an argument similar to that used in the proof of Remarks 2.2(v), Propositions (2.6), (2.9) and (2.12), one can get the following three Propositions for the relation ω.

Proposition 2.13. Let M and N be R-modules. Then

(i) If $X_1\omega Y_1$ and $X_2\omega Y_2$, then $(X_1 \cap X_2)\omega(Y_1 \cap Y_1)$ where X_1, X_2, Y_1 and Y_2 are submodules of M.

(ii) If $\varphi: M \to N$ is an R-monomorphism and $A\ \omega B$, then $\varphi(A)\ \omega\ \varphi(B)$,also $X\omega Y$ implies that $\varphi^{-1}(X)\ \omega\ \varphi^{-1}(Y)$, where $A, B \leq M$ and $X, Y \leq N$.

Proposition 2.14. Let R be a commutative ring and M be an R-module. If A , B are submodules of M such that $A\omega B$, then C/A and C/B are strongly singulars, for some submodule C of M.

Proposition 2.15. Let M be an R-module satisfy the condition $(*)$, A and B are submodules of M. If $A\omega B$, then C/A and C/B are strongly singulars, for some submodule C of M.

Remark 2.16. Let M be an R-module and let A_1, A_2, B_1 and B_2 are submodules of M. If $A_1\rho B_1$ and $A_2\rho B_2$, so it is not necessarily $(A_1 + A_2)\ \rho\ (B_1 + B_2)$, for example, consider the Z-module $Z\oplus Z_2$, let $A_1 = A_2 = (2, \bar{0})Z$, $B_1 = (1, \bar{0})Z$ and $B_2 = (1, \bar{1})Z$. It is easy to see that $A_1\rho B_1$ and $A_2\rho B_2$. But $A_1 + A_2 = (2, \bar{0})Z$ is not related to $B_1 + B_2 = Z\oplus Z_2$ by ρ, since $(A_1 + A_2) \cap (B_1 + B_2) = (2, \bar{0})Z$ is not strongly large in $B_1 + B_2 = Z\oplus Z_2$. In fact, $(2, \bar{0})Z$ is not large in $Z\oplus Z_2$.

Recall that an R-module M is called strongly extending if every submodule is strongly large in a direct summand of M. (Equivalently, a module M is strongly extending if and only if every SL-closed submodule of M is a direct summand) [15].

In fact, every direct summand is a SL-closed submodule but not conversely. It is Clear that, every strongly extending module is extending.

Now, the next result gives characterization for strongly extending modules.

Proposition 2.17. Let M be an R-module. Then M is a strongly extending module if and only if for each submodule X of M, there exists a direct summand D of M such that $X\omega D$.

Proof. Suppose that M is a strongly extending R-module. Let $X \leq M$, then there exists a direct summand D of M such that $X \leq_{sl} D$, but $D \leq_{sl} D$ this implies $X\omega D$.

Conversely, assume $A \leq M$, so by assumption, there exists a direct summand K of M such that $A\omega K$ this imply, there exists a submodule D of M

such that $A \leq_{sl} D$ and $K \leq_{sl} D$. We claim that D is a direct summand of M. Since K is a direct summand, then $K \oplus L = M$ for some submodule L of M. But $M = K + L \subseteq D + L$, hence $D + L = M$. Now, let $x (\neq 0) \in D \cap L$, then $x \in D$ and $x \in L$. Since $K \leq_{sl} D$ implies $K \leq_{e} D$, so there exists $r \in R$ such that $xr (\neq 0) \in K$, but $xr \in L$, hence $xr (\neq 0) \in K \cap L$ which is a contradiction. Thus $D \cap L = 0$, and so $D \oplus L = M$, this mean D is a direct summand of M such that $A \leq_{sl} D$. ■

3 Characterizations And Properties Of Strongly $\mathcal{G}$-extending Modules

In this section, we introduce and investigate notion of strongly $\mathcal{G}$-extending modules and give some basic properties. Several of characterizations about this concept are presented. Also, we give relations between such modules and other classes of modules.

Now, we present the following definition.

Definition 3.1. An R-module M is called strongly Goldie extending (briefly strongly $\mathcal{G}$-extending) if for each $X \leq M$, there exists a direct summand D of M such that $X \rho D$.

Remarks and Examples 3.2.

(i) Every strongly extending module is strongly $\mathcal{G}$-extending, but not conversely, as the following example shows: let M denote Z-module $Q \oplus Z_p$ where p is a prime, so M is strongly $\mathcal{G}$-extending. But it is not extending, see [2, Example 3.20] hence M is not strongly extending.

(ii) Since, every strongly large submodule is large, then every strongly $\mathcal{G}$-extending module is $\mathcal{G}$-extending, and hence from (i), every strongly extending module is $\mathcal{G}$-extending. But the converse is not true in general, for example: the Z-module $Z_2 \oplus Z_4$ is $\mathcal{G}$-extending, since it is extending, but it is not strongly extending, see [15, Example 3.7].

A nonzero R-module module M is said to be SL-uniform if, every nonzero submodule of M is strongly large [7]. Clearly, every SL-uniform module is a uniform module.

(iii) Every SL-uniform module is strongly $\mathcal{G}$-extending, since, for a SL-uniform module M, if $X \leq M$, X is strongly large, and hence $X \rho M$, by Remarks 2.2(iv). But the converse need not be true in general, for example: it is clear that Z_6 as Z-module is strongly $\mathcal{G}$-extending, but it is not SL-uniform. In fact, every proper submodule of Z_6 is not large, and so it is not strongly large.

(iv) It is clear that every semisimple module is strongly $\mathcal{G}$-extending. In fact, every semisimple module is strongly extending. But the converse is not true in general, as example: the Z-module $Z \oplus Z$ is strongly extending, see [15, Example 3.10], so it is strongly $\mathcal{G}$-extending, but it is not semisimple.

(v) Every integral domain is strongly $\mathcal{G}$-extending.

Proof. Let R be an integral domain and I be a nonzero ideal of R. We claim that I is strongly large. Let $x \in R$ and $s \in R$ such that $xs \neq 0$. For any $a \in I$, $xa \in I$. Also, $xas \neq 0$ (since, if $xas = 0$ and $xs \neq 0$ with R has no zero divisors, then $a = 0$ which is a contradiction). So $I \leq_{sl} R$, this implies R is SL-uniform, and hence by (iii), the result is obtained. ■

In particular, Z_Z is strongly $\mathcal{G}$-extending.

The next two Propositions gives condition under which the concepts of strongly extending and strongly $\mathcal{G}$-extending modules are coincide.

Proposition 3.3. Let M be a SL-UC module. Then M is strongly extending if and only if M is strongly $\mathcal{G}$-extending.

Proof. Follows by Propositions (2.4) and (2.17). ■

Proposition 3.4. Let R be a commutative ring and M be a non-strongly singular R-module. Then M is strongly extending if and only if M is strongly $\mathcal{G}$-extending.

Proof. ⇒) Clear.

⇐) Assume that M is a strongly $\mathcal{G}$-extending R-module. Let $X \leq M$, so there exists a submodule A of M such that $X\rho A$ and $A \oplus B = M$ for some submodule B of M. By Proposition 2.9, $X + A/A$ is strongly singular. On the other hand, M is a non-strongly singular module, so is $M/A \cong B$, but $(X + A)/A \leq M/A$ then $(X + A)/A$ is non-strongly singular, this implies that $X + A = A$, thus $X \leq A$. Hence $X = X \cap A \leq_{sl} A$. Therefore M is strongly extending. ■

Proposition 3.5. Let M be an indecomposable R-module. Then M is strongly $\mathcal{G}$-extending if and only if M is SL-uniform.

Proof. Assume that M is a strongly $\mathcal{G}$-extending R-module. Let X be a nonzero submodule of M, so there exists a direct summand D of M such that $X\rho D$, that is; $X \cap D \leq_{sl} X$ and $X \cap D \leq_{sl} D$. Since M is an indecomposable R-module, either $D = 0$ or $D = M$. If $D = 0$, then $\{0\} \leq_{sl} X$ which is a contradiction. Thus $D = M$, and so $X \leq_{sl} M$. Therefore M is SL-uniform.

Conversely, follows by Rem.and.Ex. 3.2 (iii).■

B.Ungor and S.Halicioglu, in [15, Examples 3.3(1)], show that a module in which every nonzero submodule is strongly large is strongly extending (this means; every SL-uniform module is strongly extending).

However, we have the following Corollary.

Corollary 3.6. Let M be an indecomposable R-module. Then the following statements are equivalent.

(i) M is a strongly extending R-module.

(ii) M is a strongly $\mathcal{G}$-extending R-module.

(iii) M is a SL-uniform R-module.

The following two results are appeared in [16, Lemma 2.22; Prop. (5.1) and (5.7)].

Lemma 3.7. Let M be an R-module with the condition $(*)$. Then N is a large submodule of M if and only if N is a strongly large in M.

Let M be a module over integral domain R. Then the set $T(M) = \{m \in M: mr = 0 \text{ for some } r(\neq 0) \in R\}$ is called a torsion submodule of M. If $T(M) = M$, then M is called torsion, and it is called torsion free whenever $T(M) = 0$ [12].

Proposition 3.8. Let M be a prime (or torsion free) R-module. Then every large submodule of M is strongly large.

However, by pervious two results, we can get the following two Propositions directly.

Proposition 3.9. Let M be an R-module with the condition $(*)$. Then M is $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}$-extending.

Proposition 3.10. Let M be a prime (or torsion free) R-module. Then M is $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}$-extending.

Now, we consider the following condition $(\boldsymbol{d})$ for some submodule N of a module M :

- For all $D \leq^{\oplus} M$, $D \cap N$ is a direct summand of N… $(\boldsymbol{d})$

Proposition 3.11. Let M be a strongly $\mathcal{G}$-extending R-module and $N \leq M$. If N satisfies the condition $(\boldsymbol{d})$, then N is strongly $\mathcal{G}$-extending.

Proof. Let X be a submodule of N. Since M is a strongly $\mathcal{G}$-extending module, so there exists a direct summand D of M such that $X\rho D$. By condition $(\boldsymbol{d})$, $D \cap N$ is a direct summand of N. On the other hand, $X \cap D \leq_{sl} X$ and $X \cap D \leq_{sl} D$, then by [16, Lemma 2.9], $X \cap (D \cap N) \leq_{sl} X \cap N = X$ and $X \cap (D \cap N) \leq_{sl} D \cap N$. This mean that $X\rho(D \cap N)$, and hence N is a strongly $\mathcal{G}$-extending R-module. ■

Recall that an R-module M is said to have the direct summand intersection property (briefly SIP) if, the intersection of any two direct summands of M is a direct summand [17].

Corollary 3.12. Let M be a module has the SIP. If M is strongly $\mathcal{G}$-extending, then every direct summand of M is strongly $\mathcal{G}$-extending.

Proof. It is obvious. ■

Let M be a module and $A \leq M$. A complement for A in M is any submodule B of M which is maximal with respect to the property $A \cap B = 0$ [8]. Following [16], let N and L be submodules of a module M with $N \cap L = 0$. Then L is called an SL-complement of N in M if, L is an SL-closed submodule of M and $N \oplus L \leq_{sl} M$. From the fact that every direct summand is SL-closed, hence every direct summand is SL-complement.

The following result is appeared in [16, Prop. 4.19].

Proposition 3.13. Let M be an R-module and $N, L \leq M$. If L is an SL-complement of N in M, then L is a complement of N in M.

Proposition 3.14. Let M be an R-module, consider the following statements.

(i) M is a strongly $\mathcal{G}$-extending R-module ;

(ii) For each $Y \leq M$, there exists $X \leq M$ and a direct summand D of M such that $X \leq_{sl} Y$ and $X \leq_{sl} D$, where $D \oplus D' = M$ for some $D' \leq M$;

(iii) For each $Y \leq M$, there exists an SL-complement L of Y and an SL-complement K of L such that $Y\rho K$ and each homomorphism $\varphi: K \oplus L \to M$ extends to a homomorphism $\bar{\varphi}$ from M to M.

Then $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$. If $r_R(D) + r_R(D') = R$, then (i) through (iii) are equivalent.

Proof. $(i) \Rightarrow (ii)$ Let $Y \leq M$, then by (i), there exists a direct summand D of M such that $Y\rho D$, that is; $Y \cap D \leq_{sl} Y$ and $Y \cap D \leq_{sl} D$. Put $X = Y \cap D$, hence $X \leq M$ such that $X \leq_{sl} Y$ and $X \leq_{sl} D$.

$(iii) \Rightarrow (i)$ Assume that $X \leq M$, then by (iii), there exists an SL-complement L of X and an SL-complement K of L such that $X\rho K$, so by Proposition 3.13, K is a complement of L and L is a complement of X, hence by [13, Lemma 2], K is a direct summand of M. Therefore M is strongly $\mathcal{G}$-extending.

$(ii) \Rightarrow (iii)$ Let $Y \leq M$, then by (*ii*), there exists a direct summand *D* of *M* such that $Y \cap D \leq_{sl} Y$ and $Y \cap D \leq_{sl} D$, $D \oplus D' = M$ for some $D' \leq M$. It is clear that D is an SL-complement of D' in *M*. Now, since D' is a direct summand of *M*, then D' is SL-closed in *M*. On the other hand, $Y \cap D \leq_{sl} D$ and $r_R(D) + r_R(D') = R$, then by [7, Lemma 2.1.33], $(Y \cap D) \oplus D' \leq_{sl} M$, thus $Y \oplus D' \leq_{sl} M$. Let $x (\neq 0) \in Y \cap D'$, then $x \in Y$ but $Y \cap D \leq_{sl} Y$, that is; $Y \cap D \leq_e Y$, so there exists $r \in R$ such that $xr(\neq 0) \in Y \cap D$, then $xr(\neq 0) \in D \cap D'$ which is a contradiction, thus $Y \cap D' = 0$. Therefore D' is an SL-complement of *Y*. By taking, $D = K$ and $D' = L$, we get the result. ■

Following [14], a module *M* is said to be satisfy the C_{11}-condition if, every submodule of *M* has a complement which is a direct summand of *M*.

Proposition 3.15. Let *M* be an *R*-module, consider the following conditions.

(i) M is a strongly extending *R*-module ;

(ii) M is a strongly $\mathcal{G}$-extending *R*-module ;

(iii) For each $X \leq M$, then there exists an SL-complement *D* of *X* such that $D \leq^{\oplus} M$, and hence M has C_{11} ;

(iv) For each semisimple submodule X of *M*, $X \leq_{sl}$ (*direct summand*) in *M*.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$.

Proof. (*i*) implies (*ii*), clear. Assume that (*ii*), let $X \leq M$. Since M is strongly $\mathcal{G}$-extending, then there exists submodules D' and D of *M* such that $M = D \oplus D'$ and $X \rho D'$. It is clear that D is an SL-complement of *X* and *D* is a direct summand of *M*. Since SL-complement is complement, hence M has C_{11}, so (*iii*) hold.

$(ii) \Rightarrow (iv)$Suppose *X* is a semisimple submodule of *M*. By (*ii*), there exists a direct summand *D* of M such that $X \rho D$, that is; $X \cap D \leq_{sl} X$, $X \cap D \leq_{sl} D$, so $X \cap D \leq_e X$ but X is semisimple, then $X \cap D = X$, and hence $X \leq_{sl} D$. ■

Proposition 3.16. Let M be an *R*-module such that $Soc(A) \leq_{sl} A$ for each $A \leq M$. Then *M* is strongly $\mathcal{G}$-extending if and only if for every semisimple submodule is strongly large in a direct summand.

Proof. Assume that a condition holds. Let $X \leq M$, then $Soc(X)$ is a semisimple submodule of *M*, then there exists a direct summand *D* of *M* such that $Soc(X) \leq_{sl} D$. By hypothesis, $Soc(X) \leq_{sl} X$, thus $Soc(X) \leq X \cap D$ which is a submodule in both *X* and *D*, so $X \cap D \leq_{sl} X$ and $X \cap D \leq_{sl} D$, that is; $X \rho D$ and hence *M* is strongly $\mathcal{G}$-extending. The converse, follows from Proposition 3.15. ■

A submodule *N* of a module *M* is called fully invariant if, $\varphi(N) \subseteq N$ for all endmorphisms φ of *M* [18]. An *R*-module *M* is called duo if, every submodule of *M* is a fully invariant submodule [11]. A submodule *N* of a module *M* is said to be distributive if, for each submodules A, B of *M*, $N \cap (A + B) = (N \cap A) + (N \cap B)$. An *R*-module M is called distributive if, all it is submodules are distributive [6].

Proposition 3.17. Let M be an *R*-module and $N \leq M$ is fully invariant. If *M* is a strongly $\mathcal{G}$-extending module then *N* is strongly $\mathcal{G}$-extending.

Proof. Let $X \leq N \leq M$. Since M is strongly $\mathcal{G}$-extending, then there exists submodules D, D' of M such that $X \rho D$ and $M = D \oplus D'$. Consider the projection maps $\pi_1: M \to D$, $\pi_2: M \to D'$.For any $x \in N$, $x = a + b$ where $a \in D$, $b \in D'$, and so $\pi_1(x) = a$, $\pi_2(x) = b$. Since *N* is fully invariant and $i_1 \circ \pi_1 \in End(M)$, hence $a = \pi_1(x) = i_1 \circ \pi_1(x) \in i_1 \circ \pi_1(M) \cap N$, that is; $a \in \pi_1(M) \cap N$, where $i_1: D \to M$ is an inclusion map. By a similar way, we have $b \in \pi_2(M) \cap N$. Therefore $x = a + b \in (\pi_1(M) \cap N) \oplus (\pi_2(M) \cap N)$, and hence $N = (D \cap N) \oplus (D^{'} \cap N)$. Since $X\rho D$, then by some steps of proof of Proposition 3.11, $X\rho(D \cap N)$ and $D \cap N$ is a direct summand of *N*. Thus *N* is strongly $\mathcal{G}$-extending. ■

Proposition 3.18. Let *M* be an *R*-module and $N \leq M$ is distributive. If *M* is strongly $\mathcal{G}$-extending then *N* is strongly $\mathcal{G}$-extending.

Proof. It is easy to check. ■

However, the next result follows directly from Propositions (3.17) and (3.18).

Corollary 3.19. Let *M* be a duo (or distributive) *R*-module. If *M* is a strongly $\mathcal{G}$-extending module then every submodule of *M* is so.

Recall that an *R*-module *M* is called multiplication if, for every submodule *N* of *M* there exists an ideal *I* of *R* such that $N = MI$ [4].

It is clear that every multiplication modules is duo. Hence we have:

Corollary 3.20. Let *M* be a multiplication module. If *M* is strongly $\mathcal{G}$-extending then every submodule of *M* is so.

Proposition 3.21. Let *R* be a commutative ring and *M* be a strongly $\mathcal{G}$-extending *R*-module with $N \leq M$. If M/N is non-strongly singular, then *N* is a direct summand of *M*.

Proof. Since *M* is a strongly $\mathcal{G}$-extending module and $N \leq M$, so there exists a direct summand D of *M* such that $N\rho D$, then by Proposition 2.9, $N + D/N$ is strongly singular. Since M/N is non-strongly singular and $N + D/N \leq M/N$,so clear that $N + D/N$ is a non-strongly singular module. But $D/N \cap D \cong N + D/N$ by (2nd isomorphism theorem), thus $D/N \cap D$ is strongly singular and non-strongly singular, this implies that $D/N \cap D$ is the zero submodule, and hence $N \cap D = D$. Since D is a direct summand of *M*, then D is SL-closed, but $D = N \cap D \leq_{sl} N$ then $D = N$. Thus N is a direct summand of *M*. ■

By Proposition 2.12, we can get the following result.

Proposition 3.22. Let *M* be a strongly $\mathcal{G}$-extending *R*-module satisfy the condition (*), and $N \leq M$. If M/N is non-strongly singular, then *N* is a direct summand of *M*.

Proof. Analogous proof of previous Proposition. ■

Recall that an *R*-module *M* is said to have the SL-closed intersection property (briefly SLCIP) if, the intersection of any two SL-closed submodules of *M* is again SL-closed [7].

The following Proposition illustrate a connection between strongly $\mathcal{G}$-extending, SL-UC and SLCIP (SIP) modules.

Proposition 3.23. Let *M* be a strongly $\mathcal{G}$-extending *R*-module. Then *M* is a SL-UC module if and only if *M* has the SLCIP (also SIP).

Proof. Assume that *M* is a SL-UC module. Since *M* is a strongly $\mathcal{G}$-extending module, then by Proposition 3.3, *M* is strongly extending. Let *K* and L be direct summands of M, so by [15, Th. 3.25] K and L are strongly extending. Since $K \cap L \leq K$, there exists a direct summand C of K such that $K \cap L \leq_{sl} C$. Similarly, $K \cap L \leq_{sl} D$ for some direct summand D of L. Thus C and D denote the SL-closures of $K \cap L$, but M is a SL-UC module, so $C = D$. It follows that $C = D \subseteq K \cap L$, and hence $C = D = K \cap L$. But C, D are direct summands of M, then $K \cap L$ is a direct summand of M and M has the SIP. Since M is a strongly extending module, so M has the SLCIP, by [7, Prop. 2.1.42]. The converse, follows by [7, Th. 2.1.12]. ■

Now, we consider the following definition.

Definition 3.24. A module M is called strongly $\mathcal{G}^+$-extending provided that every direct summand of M is strongly $\mathcal{G}$-extending.

Proposition 3.25. Every strongly extending module is strongly $\mathcal{G}^+$-extending.

Proof. Suppose that M is a strongly extending module, and let N be a direct summand of M. By [15, Th. 3.25] N is also strongly extending, and hence N is strongly $\mathcal{G}$-extending, by Rem. and Ex. 3.2(i). Thus M is strongly $\mathcal{G}^+$-extending. ■

The converse need not be true in general, see Rem.and.Ex. 3.2(i). In fact, the direct summands of $Q \oplus Z_p$ as Z-module are only trivial summands which are strongly $\mathcal{G}$-extending.

Remarks 3.26.

(i) Let M be an R-module has the SIP. Then M is strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}^+$-extending.

Proof. It follows by Corollary 3.12. ■

(ii) Let M be a duo (or distributive) R-module. Then M is strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}^+$-extending.

Proof. It follows by Corollary 3.19. ■

(iii) Let M be a multiplication R-module. Then M is strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}^+$-extending.

Proof. It follows by Corollary 3.20. ■

(iv) Let M be a free module over a PID. Then M is a strongly $\mathcal{G}$-extending module if and only if M is a strongly $\mathcal{G}^+$-extending module.

Proof. By [3, Cor. 1.1.6] M has the SIP. Hence, the result is obtained from (i). ■

(v) Let R be a commutative ring. Then R is strongly $\mathcal{G}$-extending if and only if R is strongly $\mathcal{G}^+$-extending.

Proof. Since R is a commutative ring, then R has the SIP. Thus, the result is obtained by (i). ■

Recall the following conditions for a module M:

(C_2) If a submodule A of M is isomorphic to a summand of M, then A is a summand of M ;

(C_3) If M_1 and M_2 are two direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M.

Now, we shall give another condition under which the concepts of strongly $\mathcal{G}$-extending and strongly $\mathcal{G}^+$-extending are coincide.

Theorem 3.27. Let M be a module satisfying C_3. Then M is strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}^+$-extending.

Proof. Assume that M is a strongly $\mathcal{G}$-extending R-module. Let N be a direct summand of M, so $M = N \oplus K$ for some submodule K of M. Consider a projection map $\pi: M \to N$. Let $X \leq N$ in M, then there exists a direct summand Y of M such that $X \rho Y$, that is; $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$. To prove that $K \cap Y = 0$. Assume that $a(\neq 0) \in K \cap Y$, so $a \in Y$ and $a.1 \neq 0$ but $X \cap Y \leq_{sl} Y$, there exists $r \in R$ such that $ar(\neq 0) \in X \cap Y$, then $ar \in X \subseteq N$ but $ar \in K$, hence $ar(\neq 0) \in N \cap K$ which is a contradiction. So $K \cap Y = 0$, but M satisfy C_3, then $K \oplus Y$ is a direct summand of M. Now, we claim that $K \oplus Y = K \oplus \pi(Y)$. Let $x = k + y \in K \oplus Y$ where $k \in K$ and $y \in Y$. Put $y = n + k_1$ where $n \in N$ and $k_1 \in K$. Then $x = k + k_1 + \pi(y) \in K \oplus \pi(Y)$, and hence $K \oplus Y \subseteq K \oplus \pi(Y)$.Conversely, let $b \in K \oplus \pi(Y)$, $b = k + \pi(y)$ where $k \in K$ and $y \in Y$. Put $y = n + k_1$ where $n \in N$, $k_1 \in K$. So $b = k - k_1 + y \in K \oplus Y$. Therefore, $K \oplus Y = K \oplus \pi(Y)$. Thus $\pi(Y)$ is a direct summand of M, but $\pi(Y) \leq N$, so $\pi(Y)$ is a direct summand of N. Next to prove that $X \rho \pi(Y)$. For any $n \in \pi(Y)$, $n = \pi(y)$ for some $y \in Y$, and let $r \in R$ such that $nr \neq 0$. Hence $\pi(yr) = \pi(y)r \neq 0$, thus $yr \neq 0$ but $X \cap Y \leq_{sl} Y$, then there exists $s \in R$ such that $ys \in X \cap Y$ and $yrs \neq 0$, so $ys \in X$ and $ys \in Y$. Then $nrs = \pi(y)rs = \pi(yrs) \neq 0$. Since $ys \in X \subseteq N$, then $\pi(ys) = ys$. Hence $ns = \pi(y)s = \pi(ys) = ys \in X \cap \pi(Y)$.Thus $X \cap \pi(Y) \leq_{sl} \pi(Y)$. It is clear $\pi(Y) = N \cap (K \oplus \pi(Y))$, to see this; let $x = n = k + \pi(y) \in N \cap (K \oplus \pi(Y))$, where $n \in N$, $k \in K$ and $y \in Y$. Thus $k = n - \pi(y) \in N \cap K = 0$, and so $k = 0$, then $x = \pi(y) \in \pi(Y)$. So $N \cap (K \oplus \pi(Y)) \subseteq \pi(Y)$, and hence $\pi(Y) = N \cap \big(K \oplus \pi(Y)\big) = N \cap (K \oplus Y)$. Thus $X \cap \pi(Y) = X \cap (N \cap (K \oplus Y))$=$X \cap (K \oplus Y) \leq_{sl} X$, therefore $X \cap \pi(Y) \leq_{sl} X$. Thus N is strongly $\mathcal{G}$-extending, and hence M is a strongly $\mathcal{G}^+$-extending module. The converse is clear. ∎

Corollary 3.28. Let M be a module satisfying C_2. Then M is strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}^+$- extending.

Proof. Since C_2 implies C_3, hence the result is obtained. ∎

Now, we will investigate the behavior of strongly $\mathcal{G}$-extending modules. We need the next Lemma.

Lemma 3.29. Let M be an R-module satisfies condition (*) and $N, L \leq M$. If $S^{-1}N \leq_{sl} S^{-1}L$ in $S^{-1}M$ as $S^{-1}R$-module, then $N \leq_{sl} L$ in M as R-module, where S is a multiplicative closed subset of R.

Proof. Let $l \in L$ and $r \in R$ such that $lr \neq 0$. For $s \in S$, $l/s \in S^{-1}L$ and $r/s \in S^{-1}R$ such that $(l/s)(r/s) \neq 0$ (because, if $(l/s)(r/s) = 0$, so there exists $(0 \neq)t \in R$ such that lrt=0; that is $rt \in r_R(l)$, but $r \notin r_R(l)$ and M satisfies condition (*) this implies $t = 0$ which is a contradiction). Since $S^{-1}N \leq_{sl} S^{-1}L$, then there exists $r_1/s_1 \in S^{-1}R$ such that $(l/s)(r_1/s_1) \in S^{-1}N$ and $(l/s)(r_1/s_1)(r/s) \neq 0$, this implies $lr_1 \in N$ and $lr_1 r \neq 0$ for some $r_1 \in R$ (because if $lr_1 r = 0$, $(lr_1 r/s_1 s^2) = 0$, so $(l/s)(r_1/s_1)(r/s) = 0$ which is a contradiction.) Thus the result is obtained. ∎

Proposition 3.30. Let M be an R-module satisfies condition (*) and S be a multiplicative closed subset of R. Then M is a strongly $\mathcal{G}$-extending as R-module if and only if $S^{-1}M$ is a strongly $\mathcal{G}$-extending as $S^{-1}R$-module, provided $S^{-1}A = S^{-1}B$ iff $A = B$.

Proof. Suppose that M is a strongly $\mathcal{G}$-extending R-module. Let $S^{-1}A \leq S^{-1}M$, so $A \leq M$, then there exists a submodule B of M such that $A\rho B$ (i.e. $A \cap B \leq_{sl} A$ and $A \cap B \leq_{sl} B$), $B \oplus C = M$ for some $C \leq M$. It is clear that $S^{-1}B \leq^{\oplus} S^{-1}M$. Since a module M satisfies the condition (*), then $(S^{-1}A) \cap (S^{-1}B) = S^{-1}(A \cap B) \leq_{sl} S^{-1}A$ and $(S^{-1}A) \cap (S^{-1}B) = S^{-1}(A \cap B) \leq_{sl} S^{-1}B$, by [7, Lemma 2.1.47], that is; $(S^{-1}A)\rho(S^{-1}B)$. Therefore $S^{-1}M$ is a strongly $\mathcal{G}$-extending $S^{-1}R$-module.

Conversely, assume $X \leq M$, so $S^{-1}X \leq S^{-1}M$, there exists a submodule $S^{-1}Y$ of $S^{-1}M$ such that $(S^{-1}X)\rho(S^{-1}Y)$, and $(S^{-1}Y)\oplus(S^{-1}K) = S^{-1}M$ for some $S^{-1}K \leq S^{-1}M$. But $S^{-1}(Y \oplus K) = (S^{-1}Y)\oplus(S^{-1}K) = S^{-1}M$, then by assumption $Y \oplus K = M$, that is; Y is a direct summand of M. Since $(S^{-1}X) \cap (S^{-1}Y) \leq_{sl} S^{-1}X$ and $(S^{-1}X) \cap (S^{-1}Y) \leq_{sl} S^{-1}Y$, then $S^{-1}(X \cap Y) \leq_{sl} S^{-1}X$ and $S^{-1}(X \cap Y) \leq_{sl} S^{-1}Y$, so by previous Lemma, $X \cap Y \leq_{sl} X$ and $X \cap Y \leq_{sl} Y$, that is; $X\rho Y$. Thus M is a strongly $\mathcal{G}$-extending as R-module. ■

Corollary 3.31. Let M be an R-module satisfies condition (*). Then M is a strongly $\mathcal{G}$-extending R-module if and only if M_p is a strongly $\mathcal{G}$-extending R_p-module, for all maximal ideals P of R.

The following Proposition is appeared in [16, Prop. 2.25].

Proposition 3.32. Let M be an R-module with the condition (*), $m \in M$ and I be a right ideal of R. Then

(i) If I is a strongly large in R, then mI is a strongly large in mR.

(ii) If $r_R(m) \leq M$ and mI is a strongly large in mR, then I is a strongly large in R.

Next, we will consider multiplication modules with the strongly $\mathcal{G}$-extending.

Proposition 3.33. Let M be a faithful finitely generated multiplication R-module satisfies (*). Then M is strongly $\mathcal{G}$-extending if and only if R is strongly $\mathcal{G}$-extending.

Proof. Suppose that M is a strongly $\mathcal{G}$-extending R-module. Let A be an ideal of R, so MA is a submodule of M, then there exists a direct summand $K = MB$ such that $(MA)\rho K$, for some ideal B of R, and $K \oplus L = M$ where $L = MC \leq M$ for some ideal C of R. Then $M(B \oplus C) = MB \oplus MC = M$, but M is faithful multiplication, implies $B \oplus C = R$, that is; $B \leq^{\oplus} R$. On the other hand, $r_R(M) \leq A \cap B$, $M(A \cap B) \leq_{sl} MA$ and $M(A \cap B) \leq_{sl} MB$, so by previous Proposition (ii), $A \cap B \leq_{sl} A$ and $A \cap B \leq_{sl} B$, this mean $A\rho B$. Thus R is strongly $\mathcal{G}$-extending.

Conversely, assume that $N \leq M$. Since M is multiplication, so $N = MI$ for some ideal I of R. But R is strongly $\mathcal{G}$-extending, so there exists an ideal J of R such that $I\rho J$, and $J \oplus P = R$ for some ideal P of R. Thus $I \cap J \leq_{sl} I$ and $I \cap J \leq_{sl} J$, but M satisfies the condition (*), so by previous Proposition(i), $N \cap MJ \leq_{sl} N$ and $N \cap MJ \leq_{sl} MJ$. Since $J \leq^{\oplus} R$ this implies $MJ \leq^{\oplus} M$. Hence M is strongly $\mathcal{G}$-extending. ■

However, we shall gave the following Corollary. But, the next Lemma we needed which appeared in [16, Prop. 2.21].

Lemma 3.34. Every torsion free module satisfies condition (*).

By comparing the above Lemma and Proposition 3.33, we have the following Corollary.

Corollary 3.35. Let M be a torsion free finitely generated multiplication R-module. Then M is strongly $\mathcal{G}$-extending if and only if R is strongly $\mathcal{G}$-extending.

Recall that an R-module M is called a scalar module if, for each $\varphi \in End(M)$, there exists an $a \in R$ such that $\varphi(m) = ma$ for all $m \in M$ [10].

Now, we finish this section by the following Proposition.

Proposition 3.36. Let M be a faithful scalar R-module. Then R is strongly $\mathcal{G}$-extending if and only if $S = End(M)$ is strongly $\mathcal{G}$-extending.

Proof. Since M is a scalar R-module, so by [10, Lemma 6.2] $S = End(M) \cong R/r_R(M)$. But M is faithful, so $S = End(M) \cong R$. Thus the result is obtained. ■

4 Direct Sums Of Strongly $\mathcal{G}$-extending Modules

In this section, we investigate some various conditions for a direct sum of strongly $\mathcal{G}$-extending modules to be strongly $\mathcal{G}$-extending.

We begin with the following example.

Example 4.1. It is well known that Z is an integral domain, so clear that the polynomial ring of Z is also an integral domain (i.e. $Z[X]$ is an integral domain), then by Rem.and.Ex. 3.2(v), $Z[X]$ is strongly $\mathcal{G}$-extending. But $Z[X] \oplus Z[X]$ is not strongly $\mathcal{G}$-extending. In fact, $Z[X] \oplus Z[X]$ is not $\mathcal{G}$-extending, see [2]. This example show that the class of strongly $\mathcal{G}$-extending modules is not closed under direct sums.

Following K.R. Goodearl in [8], if $\{A_\alpha\}$ is an independent family of submodules of M and $A_\alpha \leq_e B_\alpha$ for each α, then $\{B_\alpha\}$ is an independent family of submodules and $\oplus A_\alpha \leq_e \oplus B_\alpha$. But "strongly large" version of this statement is an open question.

However, we introduce the following definition.

Definition 4.2. Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be collections of submodules of a module M. The module M is called SL-direct sum if for each α, $A_\alpha \leq_{sl} B_\alpha$ implies $\oplus A_\alpha \leq_{sl} \oplus B_\alpha$.

Lemma 4.3. Let $\{ M_\alpha : \alpha \in \Lambda \}$ be a family of SL-direct sum modules. If $A_\alpha \, \rho \, B_\alpha$ of M_α for all $\alpha \in \Lambda$, then $(\oplus A_\alpha)\rho(\oplus B_\alpha)$.

Proof. Assume that $A_\alpha \, \rho \, B_\alpha$ of a module M_α, for each $\alpha \in \Lambda$. Then $(A_\alpha \cap B_\alpha) \leq_{sl} A_\alpha$ and $(A_\alpha \cap B_\alpha) \leq_{sl} B_\alpha$ for each $\alpha \in \Lambda$. But M_α is SL-direct sum, then $\oplus (A_\alpha \cap B_\alpha) \leq_{sl} \oplus A_\alpha$ and $\oplus (A_\alpha \cap B_\alpha) \leq_{sl} \oplus B_\alpha$, thus $(\oplus A_\alpha) \cap (\oplus B_\alpha) \leq_{sl} \oplus A_\alpha$ and $(\oplus A_\alpha) \cap (\oplus B_\alpha) \leq_{sl} \oplus B_\alpha$. Thus $(\oplus A_\alpha) \, \rho \, (\oplus B_\alpha)$. ■

Proposition 4.4. Let M_1 and M_2 be R-modules such that $M = M_1 \oplus M_2$ be a duo SL-direct sum R-module. Then M_1, M_2 are strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}$-extending.

Proof. Suppose M_1, M_2 are strongly $\mathcal{G}$-extending R-modules, and $X \leq M$. Since $M = M_1 \oplus M_2$ is a duo module, $X = (X \cap M_1) \oplus (X \cap M_2)$. On the other hand, for (i = 1,2), $X \cap M_i \leq M_i$ and M_i is strongly $\mathcal{G}$-extending, so there exists a direct summand D_i of M_i such that $(X \cap M_i) \, \rho \, D_i$. By Lemma 4.3, $X = (X \cap M_1) \oplus (X \cap M_2) \, \rho \, (D_1 \oplus D_2)$, since M is a SL-direct sum module. Notice that $D_1 \oplus D_2$ is a direct summand of M. Thus

M is a strongly $\mathcal{G}$-extending R-module. The converse, follows directly by Corollary 3.19. ■

Proposition 4.5. Let $M = M_1 \oplus M_2$ be a distributive SL-direct sum R-module. Then M_1, M_2 are strongly $\mathcal{G}$-extending if and only if M is strongly $\mathcal{G}$-extending.

Proof. Assume M_1, M_2 are strongly $\mathcal{G}$-extending R-modules. Let $X \leq M$, then $X = X \cap M = X \cap (M_1 \oplus M_2) = (X \cap M_1) \oplus (X \cap M_2)$,since M is a distributive R-module. By same argument of Proposition 4.4, M is strongly $\mathcal{G}$-extending module. The converse, follows from Corollary 3.19. ■

Proposition 4.6. Let $M = M_1 \oplus M_2$ be a SL-direct sum R-module, $r_R(M_1) + r_R(M_2) = R$. If M_1 and M_2 are strongly $\mathcal{G}$-extending, then M is strongly $\mathcal{G}$-extending.

Proof. Let X be a nonzero submodule of M. Since $r_R(M_1) + r_R(M_2) = R$, then by the same way of the proof of [1, Prop. 1.2.4] $X = A \oplus B$, where $A \leq M_1$ and $B \leq M_2$. Since $X \neq 0$, then we have three cases: **Case 1,** if $A \neq 0$ and $B = 0$, then $X = A$ is a submodule of M_1, but M_1 is strongly $\mathcal{G}$-extending, then there exists a direct summand D of M_1 such that $X\rho D$. It is clear that D is a direct summand of M. **Case 2,** if $A = 0$ and $B \neq 0$, then by a similar way, we get a direct summand C of M_2 (also in M) such that $X\rho C$. **Case 3,** if A and B are both nonzero submodules, so there exists direct summands D_1, D_2 of M_1 and M_2 respectively, such that $A\rho D_1$ and $B\rho D_2$. Since M is SL-direct sum, then by Lemma 4.3, $X = (A \oplus B)\ \rho\ (D_1 \oplus D_2)$ and $D_1 \oplus D_2$ is a direct summand of M. From above cases, we get the result. ■

Proposition 4.7. The following statements are equivalent for a PID R.

(i) $\bigoplus_{i\in I} R$ is strongly $\mathcal{G}$-extending, for all index set I ;

(ii) Every projective R-module is strongly $\mathcal{G}$-extending.

Proof. $(i) \Rightarrow (ii)$ Assume that P is a projective R-module. Choose a free R-module F and an epimorphism $\varphi\colon F \to R$. Since F is free, so by [9, Lemma 4.4.1] $F \cong \bigoplus_{i\in I} R$ for some index set I. Consider the short exact sequence $0 \to Ker\varphi \overset{i}{\to} \bigoplus_{i\in I} R \overset{\varphi}{\to} P \to 0$ where *i* is the inclusion map. Since P is projective, so the sequence is splits. Thus $\bigoplus_{i\in I} R = Ker\varphi \oplus P$. Since $\bigoplus_{i\in I} R$ is free strongly $\mathcal{G}$-extending over a PID R, so by Remarks 3.26 (*iv*), $\bigoplus_{i\in I} R$ is strongly $\mathcal{G}^+$-extending, but P is a direct summand of $\bigoplus_{i\in I} R$, therefore P is strongly $\mathcal{G}$-extending.

$(ii) \Rightarrow (i)$ Obvious. ■

References

[1] Abbas, M.S., 1990, On Fully Stable Modules, Ph.D. Thesis, Univ. Of Baghdad, Iraq.

[2] Akalan, E., Birkenmeier, G.F., Tercan, A., 2009, Goldie Extending Modules, Comm. Algebra 37, PP. 663-683.

[3] Al-Bahraany, B.H., 2000, Modules With The Pure Intersection Property, Ph.D. Thesis, Univ. Of Baghdad, Iraq.

[4] Barnard, A., 1981, Multiplication Modules, J. Algebra 71, PP. 174-178.

[5] Dung, N.V., Huyn, D.V., Smith, P.F. and Wisbauer, R., 1994, Extending Modules, Pitman Research notes in math. Series, Longman Harlow.

[6] Erdogdu, V., 1987, Distributive Modules, Canad. mat. Bull. 302, PP. 248-254.

[7] Ghawi, Th.Y., 2015, Modules With Closed Intersection (Sum) Property, Ph.D. Thesis, Univ. Of Al-Mustansiriyah, Iraq.

[8] Goodearl, K.R., 1976, Ring Theory, Nonsingular Rings And Modules, Dekker, Newyork.

[9] Kasch, F., 1982, Modules And Rings, Academic press, London.

[10] Mohamed-Ali, E.A., 2006, On Ikeda-Nakayama Modules, Ph.D. Thesis, Univ. Of Baghdad, Iraq.

[11] Ozcan, A.C., Harmanci, A. and Smith, P.F., 2006, Duo Modules, Glasgow math. J, PP. 533-545.

[12] Sharpe, D.W. and Vamos, P., 1972, Injective Modules, Lectures in pure math., Cambridge Univ. Press.

[13] Smith, P.F. and Tercan, A., 1992, Continuous And Quasi-continuous Modules, Houston J. math, PP. 339-348.

[14] Smith, P.F. and Tercan, A., 1993, Generalizations Of CS-Modules, Comm. Algebra, PP. 1809-1847.

[15] Ungor, B. and Halicioglu, S., 2013, Strongly Extending Modules, Hacettepe J. math. And Statistic, PP. 465-478.

[16] Ungor, B. and Halicioglu, S., Kamal, M.A. and Harmanci, A., 2013, Strongly Large Module extensions, An.Stiint. Univ. Al.I.Cuza Iasi. Math. (S.N.) 59, PP. 431-452.

[17] Wilson, G.V., 1986, Modules With The Summand Intersection Property, Comm. In Algebra, PP. 21-38.

[18] Wisbauer, R., 1991, Foundations Of Module And Ring Theory, reading, Gordon and Breach Science Publishers.

مقاسات التوسع القوي من النمط غولدي

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المستخلص

المقاس الجزئي N من المقاس الايمن M على الحلقة R يدعى جوهري قوي في حالة كون كل عنصر $m \in M$ و $r \in R$ مع $mr \neq 0$ فانه يوجد عنصر $r \in R$ بحيث ان $mr \in N$ و $mrs \neq 0$. في هذا البحث قدمنا مقاسات التوسع القوي من النمط-$\mathcal{G}$ التي هي حالة خاصة من مقاسات التوسع من النمط-$\mathcal{G}$ وتحققنا من خواصها وشواخصها. المقاس M على الحلقة R يسمى مقاس توسع قوي من النمط-$\mathcal{G}$ اذا كان لكل مقاس جزئي X من M، توجد مركبة جمع مباشر D من M بحيث ان $X \cap D$ مقاس جزئي جوهري قوي في كل من X و D. بعض الشروط الضرورية التي تجعل المجموع المباشر لمقاسات التوسع القوي من النمط-$\mathcal{G}$ هو صفة مغلقة تم الحصول عليها. امثلة توضح هذا النوع من المقاسات قد اعطيت.