Utilizing Parametric and Non-Parametric methods in Black-Scholes Process with Application to Finance

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Abstract:

In this article, we present that Black-Scholes process is a famous formula in financial mathematics. Our aim is to study the behavior of stochastic parameters in that model with application to Financial Time Stock Exchange FTSE100 Index. We use some parametric (Maximum likelihood, and Unbiased and Efficient) and nonparametric (Penalized least Squares, with different functions for the drift and diffusion coefficients and generally the Black-Scholes process) methods. Moreover, we study the change-point estimation for FTSE100 Index in order to determine these changes, and effects on the behavior of the Black-Scholes process.

Keywords: Black-Scholes Process, Parametric method, Non-parametric method, Change-point estimation, FTSE100 Index.

1. Introduction

Stochastic Differential Equations (SDE) has been studied in recent years particularly in statistics aspect. We can define Stochastic Differential Equation (SDE) that is a differential equation adding to white noise term. The white noise term is represented a standard Brownian motion in this article. We present a good example of Stochastic Differential Equations (SDE) that is called a Black-Scholes process. This process is introduced by Fisher Black and Myron Scholes [6] which is also called Geometric Brownian Motion [8, 9]. It has been suggested for European markets. We find that it is very interesting to study the Black-Scholes Process using real data.

Therefore, we motivate to study the behavior of the Black-Scholes process and it effect on FSTE100. Comte et al (2007) estimate drift and diffusion coefficients for some examples from Stochastic Differential Equations (SDE) using a penalized least squares approach [3]. Comte et al (2002) present penalized least squares using different functions of drift and integrated diffusion coefficients in a discrete time [4]. Kessler et al (2012) present current research trends and recent developments in statistical methods for Stochastic Differential Equations. Moreover, it presents a spectrum of estimation methods, including nonparametric estimation as well as parametric estimations based on likelihood methods, estimating functions, and simulation techniques for high-frequency data [10].

Kloke et al (2015) present traditional nonparametric methods and rank-based analyses, including estimation and inference for models ranging from simple location models to general linear and nonlinear models for uncorrelated and responses [12].

Iacus (2008) discusses the use of appropriate statistical techniques, with the choice of particularly financial models such as Black-Scholes process, starting from real financial data [9].

Iacus (2011) presents some elementary and advanced topics on modern option pricing, from basic models of the Black-Scholes theory to more sophisticated approaches [8].

The aim of this article is to estimate the parameters of Black-Scholes process, using parametric and nonparametric methods then determine the change points in our data by change-points estimation.

In Section 2, we present Stochastic Differential Equations (SDE), the properties of standard Brownian motion, and the Black-Scholes process is a famous example of Stochastic Differential Equations. In Section 3 and Section 4, we present some parametric and nonparametric methods of the Black-Scholes process to estimate those parameters.

In Section 5, the change-points estimation will be presented. Finally, In Section 6, we apply our methods to real data, for example FSTE100.

2. Stochastic Differential Equations

Stochastic Differential Equation (SDE) is a differential equation adding to Noise term. The general form of Stochastic Differential Equations (SDE) is as follows:

$$dX_t = a(X_t)\, dt + b(X_t) dW_t \qquad (1)$$

where

dX_t is the change of X_t in a continuous time t. $a(X_t)$ is the drift parameter. $b(X_t)$ is the volatility parameter, and dW_t is a standard Brownian motion.

A standard Brownian motion is a stochastic process (a continuous space and a continuous time) that describes the evolution of value of any random variable. It is sometimes called Wiener process that refers to Wiener (1923) [8]. The properties of standard Brownian motion are as follows:

1. It starts at zero. $W_0 = 0$.
2. Its sample path is everywhere continuous.
3. It is nowhere differentiable.
4. It has independent increments. This means that if (t_1,t_2) and (t_3,t_4) are disjoint intervals, then the increment or increase $W_{t4} - W_{t3}$ is independent of the increment $W_{t2} - W_{t1}$.
5. If $s < t$, then $W_t - W_s \sim N(0, t - s)$.

We will present a good example of Stochastic Differential Equations (SDE), which is called a Black-Scholes process.

2.1- *Black-Scholes Process*

Black-Scholes process (1973) is introduced by Fisher Black and Myron Scholes [6] which estimates the price S over time t. An option, from finance view, is contract to buy or sell an underlying asset at a specific price at time t, as shown in [13]. It is also called a Geometric Brownian motion [14]. This process is important mathematical model of a financial market. Mathematically, the Black- Scholes process can be written as follows:

$$dS_t = \theta_1 S_t\, dt + \theta_2 S_t\, dW_t \qquad (2)$$

where

S_t: represents the spot price of an underlying asset in time t.

θ_1: represents the drift parameter.

θ_2: represents the volatility parameter.

It is more convenient to work with $y_t = log(S_t/S_0)$, where (S_0) is an initial value of the spot price. Using the *Itô* Lemma [13], we can transform *(1)* into an equation for y_t.

$$dy_t = \left[\frac{\partial y_t}{\partial S_t}\theta_1 S_t + \frac{1}{2}\theta_2^2 S_t^2 \frac{\partial^2 y_t}{\partial S_t^2}\right] dt + \frac{\partial y_t}{\partial S_t}\theta_2 S_t dW_t \qquad (3)$$

where

$\frac{\partial y_t}{\partial S_t} = \frac{1}{S_t}, \frac{\partial^2 y_t}{\partial S_t^2} = -\frac{1}{S_t^2}$, and $\frac{\partial y_t}{\partial t} = 0$.

This leads to

$$\begin{aligned} dy_t &= \left[\frac{1}{S_t}\theta_1 S_t + 0 + \frac{1}{2}\theta_2^2 S_t^2 \left(-\frac{1}{S_t^2}\right)\right] dt + \frac{1}{S_t}\theta_2 S_t dW_t \\ &= (\theta_1 - \frac{1}{2}\theta_2^2)dt + \theta_2 dW_t. \end{aligned} \qquad (4)$$

Now, we will use the Euler scheme to transform the Black-Scholes [9] from continuous to discrete time. The aim of applying the Euler Scheme is that it is easier to deal with discrete rather continuous time.

The Euler Scheme of y_t is.

$$y_{t+\triangle} = y_t + \int_t^{t+\triangle} (\theta_1 - \frac{1}{2}\theta_2^2)ds + \int_t^{t+\triangle} \theta_2 dW_t, \qquad (5)$$

by integrated the above equation, we now let $\Delta \to 0$. When we move from continuous to discrete time, positivity of the diffusion parameter is unfortunately no longer guaranteed. Hence, we replace θ_2 with $|\theta_2|$

$$y_{t+\triangle} = y_t + (\theta_1 - \frac{1}{2}\theta_2^2) \triangle + |\theta_2|\sqrt{\triangle} Z_t,\ Z_t \sim N(0,1) \qquad (6)$$

since, $dW_t \sim N(0,\Delta)$ when $dW_t = W_{t+\Delta} - W_t$

We will estimate the drift and diffusion parameters, using parametric and nonparametric.

Moreover, we will determine the change points in our real data using change-points estimation. Then we can estimate our parameters before and after these points to choose the best estimate of the parameters.

3. *Parametric Methods*

We will use Unbiased and Efficient, and Maximum Likelihood methods to estimate the parameters for the Black-Scholes process.

3.1- *Unbiased and Efficient Estimation*

Using equation (6), we can consider y_t as random variables, taking from common distribution

$$N\left[(\theta_1 - \tfrac{1}{2}\theta_2^2) \triangle t, \theta_2\sqrt{\triangle t})\right]$$

Then, the parameters will be computed as the mean and the variance of a sample of i.i.d random variables for the Normal distribution with $\mu\Delta t\left(\theta_1 - \frac{1}{2}\theta_2^2\right)$ and $\sigma^2\Delta t = \theta_2$

From statistical view, these parameters are an Unbiased and Efficient estimations [8]. Mathematically, we can computed the mean and the variance as follows[9]:

$$\bar{y}_t = \theta_1 - \frac{1}{2}\theta_2^2 \qquad (7)$$

$$\sigma_{y_t}^2 = \theta_2^2. \qquad (8)$$

By solving the above equations, we can find the estimate of θ_1 and θ_2, where

$$\begin{aligned}\bar{y_t} &= \frac{\sum_{i=1}^{n} y_{ti}}{n \triangle t},\\ \sigma_{y_t}^2 &= \frac{\sum_{i=1}^{n}(y_{ti}-\bar{y_t})^2}{(n-1)\triangle t}\end{aligned}$$

3.2- Maximum Likelihood Estimation

We will write the likelihood function, assuming that the initial value y_0 is known. Mathematically, the likelihood function can be expressed as

follows :

$$\begin{aligned}L(y_t, y_{t-1}, y_{t-2}, \ldots, y_1|y_0, \theta_1, \theta_2) &= L(y_t|y_{t-1}, y_{t-2}, \ldots, y_1, y_0, \theta_1, \theta_2)\\ &\times L(y_{t-1}|y_{t-2}, y_{t-3}, \ldots, y_1, y_0, \theta_1, \theta_2)\\ &\times \cdots \times L(y_1|y_0, \theta_1, \theta_2).\end{aligned}$$

This process satisfies the Markov property. Therefore, we can rewrite the above equation as follows:

$$L(y_t, y_{t-1}, y_{t-2}, \ldots, y_1|y_0, \theta_1, \theta_2) = L(y_t|y_{t-1}, \theta_1, \theta_2) \times L(y_{t-1}|y_{t-2}, \theta_1, \theta_2) \times \cdots \times L(y_1|y_0, \theta_1, \theta_2).$$

As we mentioned before, this process is a normal distribution, so we use the likelihood function

to estimate the parameters θ_1 and θ_2.

We can also find the confidence intervals of our parameters in both methods as shown in [8, 9 and 10].

4- Non Parametric Method

We will use penalized least squares, with different functions for our parameters to estimate the functions of parameters for the Black-Scholes process. The initial aim of this method is to estimate the fitting function for our parameters and avoid excessive roughness, without loss of generality. Firstly, suppose it has been observed (x_i, y_i), $i = 1, 2, \ldots, n$

$$y_i = f(x_i) + noise.$$

where f is unknown. Secondly, we will try to make $\frac{\sum_{i=1}^{n}(y_i - f(x_i))^2}{n}$ in minimizing criterion.

Now, we will use this method to find the best function for the path of Black-Scholes process, and the function of drift and diffusion estimators. Using equation (6) to estimate the drift and diffusion function in discrete time and we will consider our process as the regression type equation [3 and 10] as follows:

$$\begin{aligned}y_{t+\triangle} &= \frac{S_{t+1}-S_t}{\triangle t}\\ &= \theta_1 S_{\triangle t} + Z_{\triangle t} + R_{\triangle t},\end{aligned}$$

where

$$\begin{aligned}Z_{\triangle t} &= \frac{1}{\triangle t}\int_t^{t+\triangle} \theta_2 S_{\triangle s} dW_s\\ R_{\triangle t} &= \frac{1}{\triangle t}\int_t^{t+\triangle} (\theta_1(S_s) - \theta_1(S_{\triangle s}))\, ds.\end{aligned}$$

The term $Z_{\Delta t}$ is a martingale increment [14] and the term $R_{\Delta t}$ belongs to the discretization case.

On the other hand, it can be estimated the function of diffusion coefficient which is faster than the function of drift coefficient. Moreover, the regression-type equation has to be more precise

for θ_2 than θ_1. We set

$$\begin{aligned}u_{t+\triangle} &= \frac{(y_{t+1}-y_t)^2}{\triangle t}\\ &= \theta_1 y_{\triangle t} + v_{\triangle t} + \tau_{\triangle t}, \qquad (10)\end{aligned}$$

by using Itô Lemma and Fubini formula [4], we can get the following relations, which is related the equation (10) as follows.

$$v_{\triangle t} = v_{\triangle t}^{(1)} + v_{\triangle t}^{(2)} + v_{\triangle t}^{(3)}$$

where

$$\begin{aligned}v_{\triangle t}^{(1)} &= \frac{1}{\triangle t}\left[(\int_t^{t+\triangle}\sqrt{\theta_2} S_s dW_s)^2 - \int_t^{t+1}\theta_2 S_t ds\right],\\ v_{\triangle t}^{(2)} &= \frac{1}{\triangle t}\int_t^{t+\triangle}((t+\triangle))\theta_2' S_s\sqrt{\theta_2 S_s}dW_s,\\ v_{\triangle t}^{(3)} &= 2\theta_1 y_{\triangle t}\int_t^{t+\triangle}\theta_2 S_s dW_s.\end{aligned}$$

In addition to, $\tau_{\triangle t} = \tau_{\triangle t}^{(1)} + \tau_{\triangle t}^{(2)} + \tau_{\triangle t}^{(3)}$, where

$$\begin{aligned}\tau_{\triangle t}^{(1)} &= \frac{1}{\triangle t}\left(\int_t^{t+\triangle}\theta_1 S_s ds\right)^2,\\ \tau_{\triangle t}^{(2)} &= \frac{2}{\triangle t}\int_t^{t+\triangle}(\theta_1 S_s - \theta_1 y_{\triangle t})\, ds\int_t^{t+\triangle}\theta_2 S_s dW_s,\\ \tau_{\triangle t}^{3} &= \frac{1}{\triangle t}\int_t^{t+\triangle} t\,\psi\, y_s ds,\end{aligned}$$

where

$$\psi = \frac{\theta_2^2}{2}(\theta_2^2)'' + \theta_1(\theta_2)' = L\theta_2$$

where

$$Lf = \frac{\theta_2^2}{2}f'' + \theta_1^2 f'.$$

The term $v_{\Delta t}$ refers to a sum of martingale increments whose variances have different orders

[10]. Moreover, the term $v^{1}_{\Delta t}$ is the main noise. The term $\tau_{\Delta t}$ refers to the discretization case. In two our parameters, we will estimate our process in min $\frac{1}{y}\sum[y_{\Delta t} - t(S_{\Delta t})]^2$

5- Change Points Estimation

In this article, the aim of the change points estimation is to determine a change case in our data of the Black-Scholes process. We will use the equations (2) and (6) in this method. Moreover, our aim of this method is to check how the effect on estimating the process, in particular the parameters before and after the change points happened. We will rewrite equation (6) using Euler scheme as follows:

$$y_{t+\triangle} = y_t + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\triangle + k\,\theta_2\,(W_{t+\triangle} - W_t), \tag{11}$$

where k is the parameter of our interest. Letting $k = k_1$ before the change point and $k = k_2$ after the change point. We will transform equation (11) to standardized residuals as follows:

$$\begin{aligned} Z_t &= \frac{1}{\sqrt{\triangle}\,\theta_2}\left[(y_{t+\triangle} - y_t) - \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\triangle\right], \\ &= k\left(\frac{W_{t+\triangle} - W_t}{\sqrt{\triangle}}\right), \end{aligned}$$

where $Z_t \sim N(0, 1)$.

Then, we can estimate the parameter before and after the change point k_1 and k_2 are

$$k_1 = \frac{\sum_{t=1}^{m} Z_t^2}{m} \tag{12}$$

$$k_2 = \frac{\sum_{t=1}^{n-m} Z_t^2}{n-m} \tag{13}$$

where n is the number of points in our data for the Black-Scholes process , m is a number of points before the change point happened, and n-m is a number of points after the change points happened.

In general, the parameter k can be estimated as.

$$k = \frac{\sum_{t=1}^{n} Z_t^2}{n}. \tag{14}$$

If θ_1 and θ_2 are unknown, we assume that θ_2 is constant and θ_1 can be estimated parametrically or non-parametrically as shown in Section 3.2 and Section 3.3.

6- Real Data

We apply our methods to real data, for example, Financial Time Stock Exchange (FTSE100). We use ***R*** software which is a free software using to analyses the data statistically.

6.1 Financial Times Stock Exchange100

Financial Times Stock Exchange100 (FTSE100) Index is organized as a joint venture between the Financial Times and the London Stock Exchange. When the market is open, this index is registered every 15 seconds. Essentially, this index is based on largest 100 companies in United Kingdom.

We apply the Black-Scholes process to the data shown in Figure 1. The number of observation is T = *7758.*

Figure 1: Graph of Financial Times Stock Exchange Index from 03/01/1984 to 10/08/2015.

6.2 Parametric Estimation

We begin by estimating the parameters $\theta = [\theta_1, \theta_2]$ of the Black-Scholes process as defined through (2) and (6) using Unbiased and Efficient, and Maximum Likelihood estimations as explained in Section 3.

Table 1 and Table 2 show the estimate of our parameters using Unbiased and Efficient and Maximum Likelihood estimations, Mean Square Error and Confidence Intervals for each parameter.

The estimation of the parameters is clearly good using our methods. Moreover, both methods have got the closer results. We can say that the estimate of maximum likelihood is better than the unbiased and efficient method because the mean square error is least value. Figure 2 presents the estimate of the profile likelihood for the parameters θ_1 and θ_2, which is clearly that the profile likelihood gives us a good estimation for our parameters.

Table 1: The results of Unbiased and Efficient Estimation

	Estimate of Parameters	Mean Squares Error	% 2.5	% 97.5
θ_1	0.07599	0.0025	0.014600	0.14013
θ_2	0.1746967	0.0097	0.17200	0.1789

Table 2: The results of Maximum Likelihood Estimation

	Estimate of Parameters	Mean Squares Error	% 2.5	% 97.5
θ_1	0.07597	0.0023	0.0145211	0.1374702
θ_2	0.17469020	0.0092	0.1719722	0.1775

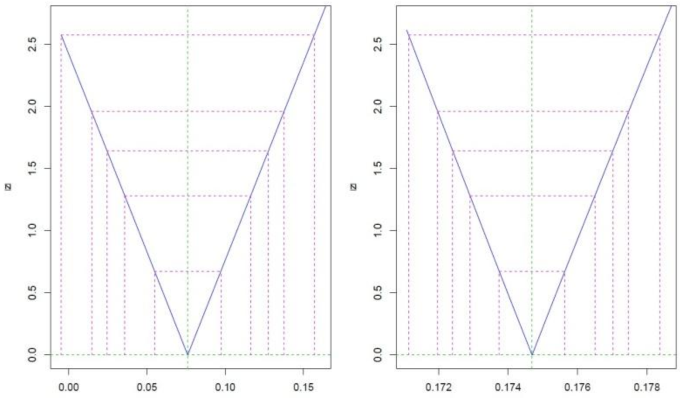

Figure 2: Graph of the profile likelihood for the θ_1 and θ_2.

6.3 Nonparametric Estimation

We estimate the parameters [θ_1,θ_2] and the general formula of the Black-Scholes process using nonparametric estimation as defined in equation (9) and (10). Figure 3 represents the estimate of drift coefficient (top graph), the estimate of diffusion coefficient (Middle graph) and the estimate of the general form of the Black-Scholes process (bottom graph). Clearly, the best estimation is for the diffusion coefficient because the red/broken line is very close to the black line whereas the red/broken line is away in some intervals for the drift coefficient and its away in a long interval for the general formula of the Black-Scholes process.

Figure 4 represents the estimate of drift coefficient (top graph), the estimate of diffusion coefficient (Middle graph) and the estimate of the general form of the Black-Scholes process (bottom graph). Clearly, the best estimation is for the drift and diffusion coefficients because the red/broken line is very close to the black line whereas the red/broken line is away in a long interval for the general formula of the Black-Scholes process.

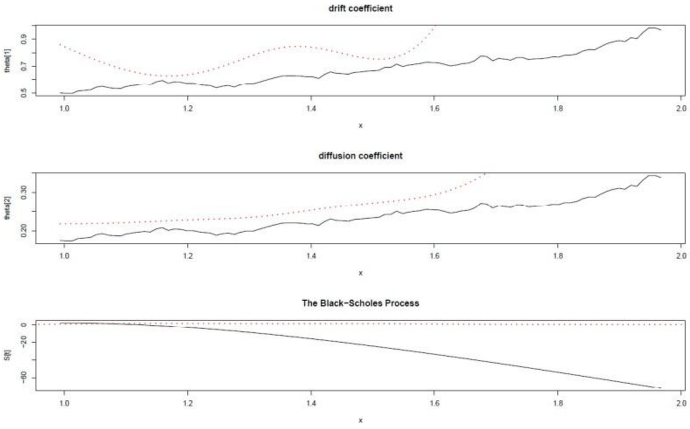

Figure 3: Nonparametric estimators of the drift and diffusion coefficients, and the general form for the Black-Scholes process. Assume that $\theta_1 = 0.5 \times S_t$ and $\theta_2 = 0.7 \times S_t$ are shown in the black line and the red/broken line represents the estimate of the drift and diffusion coefficient, and the estimate of general form for the Black-Scholes process.

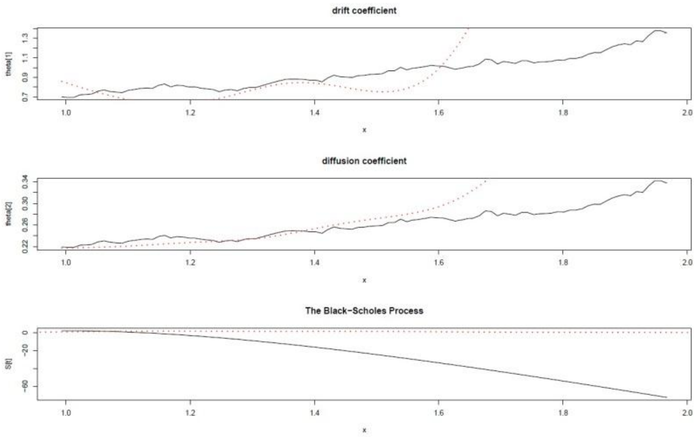

Figure 4: Nonparametric estimators of the drift and diffusion coefficients, and the general form for the Black-Scholes process. Assume that $\theta_1 = 0.75 * S_t$ and $\theta_2 = \sqrt{1 + S_t^2}$ are shown in the black line and the red/broken line represents the estimate of the drift and diffusion coefficient, and the estimate of general form for the Black-Scholes process.

6.4 Change Points Estimation

We use the change points estimation in our data and estimate the parameters before and after these points as shown in Section 4. Table 3 represents the change points and estimates the parameters k_1 and k_2 using maximum likelihood and nonparametric estimation. Figure 5 shows the change points shown as red/broken line in our data using maximum likelihood.

Table 4 represents the change points and estimates the parameters k_1 and k_2 using nonparametric method. Figure 6 shows the change points shown as red/broken line in our data using nonparametric method. Figure 7 represents the estimate of the θ_1 before and after the change points as shown in Table 4.

In general, the estimate of θ_1 after the change points is obviously better than before the change points because the estimated function of parameter (red/broken line) is so closer to the function of parameter (black line).

Table 3: The results of the Change points using Maximum Likelihood Estimation

Year of the change point	Estimate θ_1 before	Estimate θ_2 after	k_1	k_2
1988.276	0.2549453	0.18583	1.051124	0.7252491
1999.156	0.243	0.2275	0.8302445	1.143743
2008.576	0.2020	0.23520	1.031882	1.289755

Table 4: The results of the Change points using Nonparametric method

Year of the change point	k_1	k_2
1988.42	0.160314	0.1257458
1999.156	0.1262803	0.1968252
2008.576	0.1791424	0.2201445

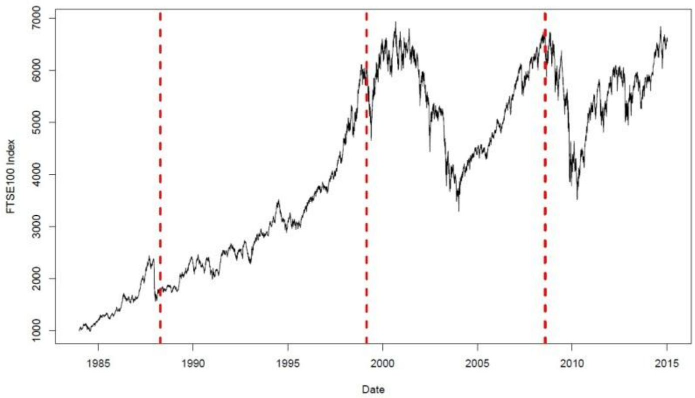

Figure 5: The Change points represent red/broken line in the FTSE100 using Maximum Likelihood method.

7- Conclusion

We have applied the Black-Scholes process to the real data, i.e., (FTSE100), using parametric and nonparametric estimation. We have performed inference for the parameters of the Black-Scholes process. We believe that the parametric performance is better than the nonparametric performance for the process because we have got the best estimated of the parameters. We also have applied the change points for (FTSE100) to show the change points happened in the real data from starting (FTSE100) until now, and estimated the parameters using parametric and nonparametric methods. We can conclude that the parametric estimation is better than nonparametric estimation particularly in our article.

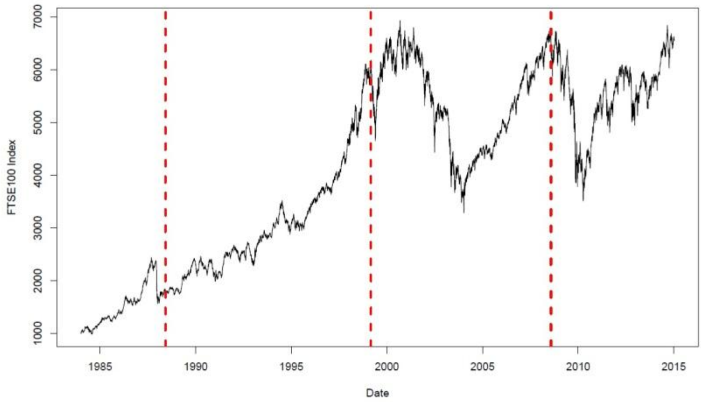

Figure 6: The Change points represent red/broken line in the FTSE100 using nonparametric method.

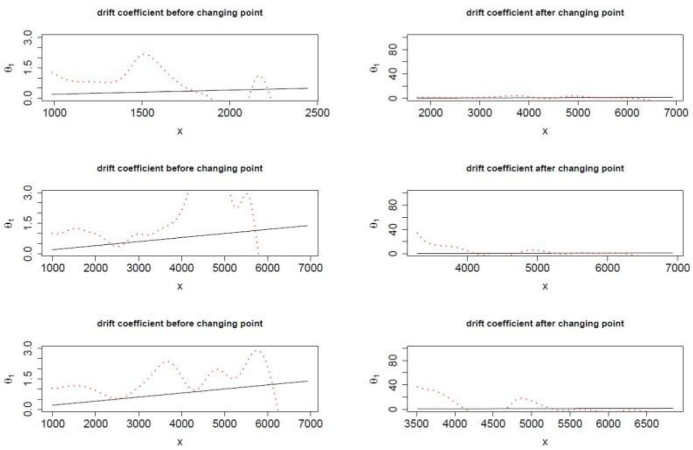

Figure 7: The estimated of θ_1 (red/broken line) and the function θ_1 (black line) of in the FTSE100 before and after the change points happened using nonparametric method.

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توظيف الطرائق المعلمي واللامعلمي في عملية بلاك-شوز مع تطبيقها في الجانب المالي

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في هذا البحث، سوف يتم التطرق الى عملية Black Scholes- والتي تعتبر من اشهر الصيغ في الرياضيات المالية. ان الهدف من هذا البحث هو دراسة المسار kBlac-Scholes لمعالم العشوائي من خلال تطبيقه على موشرFTSE١٠٠ . قد تم استخدم بعض الطرائق المعلمي(الامكان الاعظم وطرقة الكفاءة غير المتحيزة) واللامعلمي (penalized least squares) مع استخدم بعض الدوال المختلفة للمعالم driftوdiffusion وعملية Black Scholes يشكل عام، وتم ايضا دراسة تحليل نقاط التغيير في بياناتFTSE١٠٠ لتحديد هذه التغييرات وتاثيرها على مسار عملية Black Scholes

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