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2-quasi-prime submodules

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Abstract

Let R be a commutative ring with identity, and W be an R-module. In This paper introduces the notion of 2-quasi prime submodules as a generalization of quasi-prime submodules, where a proper submodule K of module W over a ring R is said to be 2-quasi-prime if whenever $a, b \in R, x \in W$ and $abx \in K$ then either $a^2x \in K$ or $b^2x \in K$. We prove many properties for this kind of submodules such as if K is a submodule of module W over a ring R, then K is 2-quasi prime submodule if and only if $[K_{\dot{R}}(x)]$ is 2-prime ideal for all $x \in W$

1. Introduction:

Let W be an R-module over a ring R with unity, an ideal J of a ring R is prime if for all elements n, m∈R, nm∈J implies that either n∈J or m∈J [1], Definition (2.8), p4], as a generalization of the prime ideal [2] introduced a prime submodule, where a proper submodule K of module W is said to be prime if rx∈K, for r∈R, x∈W, then either x∈K or r∈[K: W].

In [3] the concept of 2-prime ideals was introduced by W.Messirdi, where a proper ideal J of a ring R is 2-prime ideal if for all x, y ∈R suchthat $xy \in J$, either x^2or y^2 lies in J. And each prime ideal is 2-prime ideal [3] but the converse is not true. In [4] we introduced the concept of 2-prime submodule as a generalization of 2-prime ideal, where a proper submodule K of W if whenever r∈R, w∈W, rw ∈K implies w∈K or r^2 ∈[K: W] where [K: W] ={r∈R, rW⊆K}.

As a generalization of prime submodule [5] introduced the conception of quasi-prime submodule, where a proper submodule K of W is said to be quasi-prime submodule if whenever $abx \in K$ for $a, b \in R$ and $x \in W$, then either $ax \in K$ or $bx \in K$. We generalized this concept and introduced 2-quasi-prime submodule, where a proper submodule K of W is called 2-quasi-prime submodule if whenever, $a, b \in R, x \in W$ and $abx \in K$, then either $a^2x \in K$ or $b^2x \in K$. We prove many properties for this kind of submodules. Every 2-prime submodule is quasi-prime submodule but the converse is not true, and K be a submodule of an R-module W. Then, K is a 2-quasi-prime submodule of W if and only if $[K_{\dot{W}}J]$ is a 2-quasi-prime submodule of W for every J of R.

2. Basic properties of 2-quasi prime submodules.

In this section, we introduce the notion of 2-quasi prime submodule as a generalization of quasi-prime submodule and we study some properties of this kind of submodules.

Definition (2.1): Let W be an R-module and K is a proper submodule of W. K is called 2-quasi prime submodule, whenever $a, b \in R$, $x \in W$ and $abx \in K$ then either $a^2x \in K$ or $b^2x \in K$. and an ideal J of ring R is called 2-quasi-prime ideal if it 2-quasi-prime R-submodule of R.

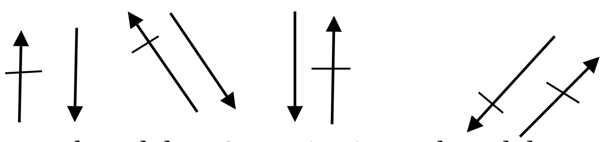
Examples and remarks (2.2).

1. Each quasi-prime submodule is 2-quasi-prime submodule.
   <u>Proof</u>: Let K be a quasi-prime submodule of an R-module W and suppose that $abx \in K$, where $a, b \in R$, $x \in W$. Since K is quasi-prime then either $ax \in K$ or$bx \in K$. Thus either $a^2x \in K$ or $b^2x \in K$. Therefore K is 2-quasi-prime submodule.
2. The converse of (1) is false see the next example: consider the Z-module Z_8 and let$K = \langle\bar{4}\rangle$. Notice that $2 \cdot 2 \cdot \bar{1} = \bar{4} \in K$, but $2 \cdot \bar{1} = \bar{2} \notin K$, so K is not a quasi-prime submodule, but it is clear that K is a 2-quasi-prime submodule.
3. Every prime submodule is 2-quasi-prime submodule.
   Proof: Since every prime submodule is quasi-prime submodule [5] and by (1) get the result.

4. The converse of (3) is false in general in the example; Let $W = Z \oplus Z$ as Z-module and $K = 2Z \oplus (0)$, K is a quasi-prime submodule by[5 Examples and remarks 2.1.2,p38] and by (1) K is 2-quasi-prime submodule. But K is not a prime submodule since $2(3,0) = (6,0) \in K$ and $(3,0) \notin K$ and $2 \notin [2Z\oplus(0): Z\oplus Z] = 0$
5. Every 2-prime submodule is a 2-quasi-prime submodule.
   Proof: Let $abx \in K$ and we have K is 2-prime submodule then either $ax \in K$, where $x \in W$ or $b^2 \in [K: W]$ then $a^2x \in K$ or $b^2x \in K$.
6. The converse of (5) is not true $\langle\bar{8}\rangle$ of the Z-module Z_{16} is 2-quasi-primesubmodule but not 2-prime submodule since $2 \cdot \bar{4} \in \langle\bar{8}\rangle$, $\langle\bar{4}\rangle \notin \langle 8\rangle$, and $2^2 = 4 \notin [\langle\bar{8}\rangle :_Z Z_{16}] = 8Z$.
   2-quasi-prime submodule$\nrightarrow$2-prime submodule

7. Each quasi-prime submodule is semi-prime [5]. This is not true for 2-quasi-prime submodule. Where K proper submodule of W is "semi-prime submodule" when $r^2w \in K$ for $r \in R$ and $w \in W$, then $rw \in K$. [6]
   In fact there is no relationship between 2-quasi-prime and semi-prime submodule $\langle\bar{4}\rangle$ of the Z-module $Z_8$is 2-quasi-prime submodule but it is not semi-prime since $2^2 \cdot 1 \in \langle\bar{4}\rangle$ but $2 \cdot 1 \notin \langle\bar{4}\rangle$.
   Also, Z-module Z is not 2-quasi-prime but it is semi-prime submodule.

Prime submodule → quasi-prime submodule →semi-prime submodule



2-prime submodule →2-quasi-prime submodule

8. Let Q as Z-module and let L=Z is a submodule of Q which is not 2-quasi-prime submodule since $4 \cdot 3 \cdot \frac{1}{2} \in Z$ but $4^2 \cdot \frac{1}{12} \notin Z$ and $3^2 \cdot \frac{1}{12} \notin Z$.But it is 2-prime submodule since if $rx \in Z, x \in Q, r \in Z$. Then either $x \in Z$ or $r^2 \in [Z\colon Q] = 0$
9. Each maximal submodule of module is a 2-quasi-primesubmodule.
   Proof: since, every maximal submodule is 2-prime submodule, see [4] and by (5) where submodule is maximal, then the result follows.
10. The converse of (9) is not true (0) is 2quasi-prime submodule of the Z-module Z. but it is not maximal.

Proposition (2.3). The proper submodule K of W is 2-quasi-prime submodule if and only if $[K\colon (w)]$ be 2-prime ideal. For all $w \in W, w \notin K$.

Proof: (⇒) suppose K is 2-quasi-prime submodule and let $ab \in [K\colon (w)]$, where $w \in W, w \notin K$ and $a, b \in R$. Thus $ab(w) \subseteq K \Rightarrow abrw \in K$. But K is 2-quasi-prime submodule then either $a^2w \in K$ or $b^2r^2w \in K$ (since K is 2-quasi-prime)
$a^2 \in [K\colon (w)]$ or $b^2 \in [K\colon (w)]$]. Thus $[K\colon (w)]$ is 2-prime ideal.
(⇐) If $[K_{\dot{R}}(w)]$ is a 2-prime ideal where $w \in W$ and $w \notin K$. to prove that K is 2-quasi-prime.
Let $abw \in K$, Thus $ab \in [K_{\dot{R}}(w)]$ is 2-prime ideal then either $a^2 \in [K_{\dot{R}}(w)]$ or $b^2 \in [K_{\dot{R}}(w)]$ so $a^2w \in K$ or $b^2w \in K$ then K is 2-quasi-prime submodule.

We needed the following lemma is to prove the next proposition (2.5)

Lemma (2.4). Let U and V are two submodules of R-module W, if for every $x \in V$. $[U_{\dot{R}}(x)]$ Which is 2-prime ideal, so $[U_{\dot{R}}V]$ be 2-prime ideal.

Proof: let y, t be in R such that $yt \in [U_{\dot{R}}V]$, so $ytx \in U$, for every $x \in V, yt \in [U_{\dot{R}}(x)] \cdots (*)$.
But $[U_{\dot{R}}(x)]$ is 2-prime ideal. So either $y^2 \in [U_{\dot{R}}(x)]$ or $t^2 \in [U_{\dot{R}}(x)]$. Thus for any $x \in V$ either $y^2x \in U$ or $t^2x \in U$, suppose that $y^2 \notin [U_{\dot{R}}(x)]$ and $t^2 \notin [U_{\dot{R}}(x)]$, so there exists x_1, x_2 in V such that $y^2{x_1}^2 \notin V$ and $t^2x_2 \notin V$. Hence $y^2 \notin [U_{\dot{R}}(x_1)]$ and $t^2 \in [U_{\dot{R}}(x_2)]$ by (*) $yt \in [U\colon (x_1)]$ which is 2-prime ideal, hence $t^2 \in [U_{\dot{R}}(x_1)]$. Thus $t^2x_1 \in U$, similarity $yt \in [U\colon (x_2)]$ implies that $t^2x_2 \in U$.
On the other hand by (*), $yt \in [U_{\dot{R}}(x_1 + x_2)]$, so either $y^2 \in [U_{\dot{R}}(x_1 + x_2)]$ or $t^2 \in [U_{\dot{R}}(x_1 + x_2)]$. Hence, either $y^2(x_1 + x_2) \in U$ or $t^2(x_1 + x_2) \in U$. Which means that either $y^2x_1 + y^2x_2 = u_1 \in U$ or $t^2(x_1) + t^2(x_2) = u_2 \in U$. Therefore either $y^2x_1 = u_1 - y^2x_2 \in U$ or $t^2x_2 = u_2 - t^2x_1 \in U$, which is contradiction. Thus either $y^2 \in [U_{\dot{R}}V]$ or $t^2 \in [U_{\dot{R}}V]$. Hence $[U\colon V]$ is 2-prime ideal.

Theorem (2.5). Let W be module over ring R and K is proper submodule of W then K is 2-quasi-prime submodule of W if and only if $[K_{\dot{R}}L]$ is a 2-prime ideal of R for each submodule L of W where $[K_{\dot{R}}L] = \{r\colon r \in R, rL \subseteq K\}$

Proof: Let K be a 2-quasi-primesubmodule of W. Then $[K\colon (x)]$ is 2-prime ideal of R, for each $x \in W$. By proposition (2.3) so $[K_{\dot{R}}(x)]$ is 2-prime ideal for each $x \in L$. So by lemma (2.4) we have $[K_{\dot{R}}L]$ is a 2-prime ideal of R.
Now, for the converse Let $abx \in K$ $a, b \in R$ and $x \in W$ i.e. $ab \in [K_{\dot{R}}(x)]$ and $x \in L \subseteq W$ by (2) $[K_{\dot{R}}(x)]$ is 2-prime ideal then either $a^2 \in [K_{\dot{R}}(x)]$ or $b^2 \in [K_{\dot{R}}(x)]$ that is mean $a^2x \in K$ or $b^2x \in K$. Thus K is 2-quasi-prime submodule.

Corollary (2.6). Let W a module over R and K a proper submodule of W. If K is 2-quasi-prime submodule of W then $[K_R^{\cdot}W]$ is a 2-prime ideal of R.

Note (2.7). In corollary (2.6) converse is false for example; the Z-module $Q + Z$, and a submodule $K = Z \oplus Z$. Then $[K_Z^{\cdot}W] = [Z \oplus Z_Z^{\cdot} Q \oplus Z] = (0)$ which is a 2-prime ideal of Z. However, K is not 2-quasi-prime Z-submodule since $\left[K_Z^{\cdot}(\frac{1}{6}, 0)\right] = 6Z$ is not a 2-prime ideal of Z.

Remark (2.8). Let W be a module over PID, R. And K is 2-quasi-prime submodule of W if $[K_R^{\cdot}W]$ is non-trivial 2-prime ideal of R.

Remark (2.9). The intersection of any two 2-quasi-prime submodules of an R-module need not be 2-quasi-prime submodule for example;$K_1 = (\bar{2}) = \{\bar{0}, \bar{2}, \bar{4}\}, K_2 = \{\bar{0}, \bar{3}\}$, $K_1 \cap K_2 = (0)$ is not 2-quasi-prime submodule of Z_6 since $2 \cdot 3 \cdot \bar{1} = 6 \in (0), 2^2 \cdot 1 = 4 \notin (0)$ and $3^2 \cdot 1 \notin (0)$.

Now, Let W be an R-module and K be submodule of W, S is multiplicative set of R, Then, $K(S) = \{x \in W: \exists t \in S$ such that$tx \in K\}$[7].

The following proposition gives a partial converse of Corollary (2.6)

Proposition (2.10). Let W be R-module and K be a proper submodule of W. íf $P = [K_R^{\cdot}W]$ be a 2-prime ideal of R and $K(S) = K$ where (S=R-P), then K is a 2-quasi-prime submodule.

Proof: R-P is multiplicatively closed by [1]. If P is a 2-prime ideal.
Let $a, b \in R, x \in W$ such that $abx \in K$, suppose $a^2x \notin K$ so $a \notin [K_R^{\cdot}W] = P$, then $a \in S$ implies that $bx \in K(S)$. But $K(S) = K$ so $b^2x \in K$ which means K is a 2-quasi-prime submodule.
We give converse of proposition (2.10) which is false. As the upcoming example show.

Example (2.11): Let $W = Z \oplus Z$ as Z-module and $K = 2Z \oplus 0$ is a 2-quasi-prime submodule of W. Then $[K_R^{\cdot}W] = (0)$ is a 2-prime ideal of module Z. But S=Z-(0). So that $K(S) = \{(x, y) \in W: \exists t \in S, t(x, y) \in K\}$. $K(S) = Z + (0) \neq K$.

3. Properties of 2-quasi-prime submodule.

We end this section by study the relation between maximal submodule and 2-quasi-prime submodules, and we will give the homomorphic and inverse image.

The upcoming proposition gives another characterization of 2-quasi-prime submodule in class of multiplication modules.

Recall "W is called multiplication module for every submodule L of W, there exists an ideal J of R such that JW=L" [8]

Proposition (3.1). Let W is a multiplícation R-module and K is a proper submodule of W. Then the following statements are equivalent.

1. K be 2-quasi-prime submodule.
2. $[K_R^{\cdot}W]$ be 2-prime ideal
3. K is 2-prime submodule.

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ let K is submodule of W then, let $x, y \in [K_R^{\cdot}W] \Rightarrow xyw \in K \ \forall w \in W$ since K is 2-quasi-prime submodule of W then either $x^2 \in [K: (w)]$ or $y^2 \in [K: (w)]$ then by lemma(2.4) ∴ $[K_R^{\cdot}W]$ is 2-prime ideal of R.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ $[K_{\dot{R}}W]$ be 2-prime ideal by [4] then K is 2-prime submodule.
$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ K is 2-prime submodule then K be 2-quasi-prime submodule by (remarks and examples (2.2) (5)).

(Remark 3.2). The intersection of any collection of 2-Quãsi-prime submodules of R-module is not 2-quasi-prime submodules. For example; Z-module Z_6 has two 2-quasi-prime submodules.
$K_1 = (\bar{2})$ and $K_2 = (\bar{3})$ but $K_1 \cap K_2 = \{0\}$ is not 2-quasi-prime submodule of Z_6 since $[(0)_{\dot{R}}Z_6] = 6Z$ is not 2-prime ideal of Z.

Proposition (3.3). Let K and L are two submodules of R-module W where K is a 2-quasi-prime submodule of W and L is not contained in K. So that, $L \cap K$ is a 2-quasi-prime submodule of L.

Proof: since $L \not\subset L \cap K$ is proper submodule of L. Now, let $a, b \in R$, and $y \in L$ such that $aby \in L \cap K$, so $aby \in L$ and $aby \in K$. But, K is a 2-quasi-prime submodule of W so either $a^2y \in K$ or $b^2y \in K$ by definition (2.1) since $y \in L$ and $a^2y \in K$ implies that $a^2y \in L \cap K$ or$b^2y \in L \cap K$.

Propositíon (3.4). Let W and $\acute{W}$ be two R-module and let $f: W \longrightarrow \acute{W}$ epimorphsim. If K is 2-quasi-primesubmodule of $\acute{W}$, then $f^{-1}(K)$ is also 2-quasi-primesubmodule of W.

Proof: To prove $f^{-1}(K)$ is 2-quasi-primesubmodule of W. We want to prove $[f^{-1}(K)_{\dot{R}}K]$ is a 2-prime ideal. $\forall K \leq \acute{W}$ suchthat $f^{-1}(K) \subsetneq L$ let $ab \in [f^{-1}(K)_{\dot{R}}L]$ and so $abL \subseteq f^{-1}(K)$. Thus $f(abL) \subseteq f(f^{-1}(K))$, so $ab(f(L) \subseteq K$. Therefore $ab \in [K: f(L)]$. But K is 2-quasi-prime submodule of $\acute{W}$. Then either $a^2 \in [K: f(L)]$ or $b^2 \in [K: f(L)]$. Thus either $a^2f(L) \subseteq K$ or $b^2f(L) \subseteq K$, i.e. either $a^2L \subseteq f^{-1}(K)$ or $b^2L \subseteq f^{-1}(K)$. Therefore either $a^2 \in [f^{-1}(K): L]$ or $b^2 \in [f^{-1}(K): L]$ i.e. $[f^{-1}(K): L]$ is 2-prime ideal. So $f^{-1}(K)$ is 2-quasi-prime submodule of W.

Proposition (3.5). If $f: W \rightarrow \acute{W}$ be an epimorphsim such that $Kerf \subseteq L$ where L is a 2-quasi-prime submodule of W. Then $f(L)$ is a 2-quasi-primesubmodule of $\acute{W}$.

Proof: To prove that $f(L)$ is a 2-quasi-prime submodule of $\acute{W}$, we prove that $\left[f(L)_{\dot{R}}\acute{K}\right]$ is a 2prime ideal of R, for all $\acute{K} \leq \acute{W}$ and $\acute{K} \nsubseteq f(L)$. Since f is epimorphsim, then $ff^{-1}(\acute{K}) = \acute{K}$
Let $f(K) = \acute{K}$. It follow that $f(K) \supsetneq f(L)$. Now to prove that $[f(L)_{\dot{R}}f(L)]$ is 2-prime ideal of R. Let $a, b \in R$ such that $ab \in [f(L)_{\dot{R}}f(K)]$, so $abf(K) \subseteq f(L)$. Thus for each $x \in K$, $abf(x) \in f(L)$ so $f(abx) = f(l)$, for some $l \in L$. Then $abx - l \in kerf \subseteq L$ and hence $abx \in L$, for each $x \in K$. Hence $ab \in [L_{\dot{R}}K]$ But $[L_{\dot{R}}K]$ is 2-prime ideal, so either $a^2 \in [L: K]$ or $b^2 \in [L: K]$. Thus either $a^2K \subseteq L$ or $b^2K \subseteq L$ so either $a^2f(K) \subseteq f(L)$ or $b^2f(K) \subseteq f(L)$. Therefore either $a^2 \in [f(L): f(K)]$ or $b^2 \in [f(L): f(K)]$ is 2-prime ideal of R and hence $f(L)$ is a 2-quasi-prime submodule of $\acute{W}$.

Corollary (3.6). Let K and L be two submodule of R-module W and $K \subseteq L$. Then $\frac{L}{K}$ is a 2-quasi-prime of $\frac{W}{K}$ if and only if L is a 2-quasi-primeRsubmodule of W.

Proof: Let $f: W \rightarrow \frac{W}{K}$ be the natural mapping, then the result follows by proposition (3.5).

Proposition (3.7). Let L be a 2-quasi-prime submodule of an R-module W, such that L_s is a proper of W_s, then L_s is 2-quasi-prime submodule of W_s.

Proof: let $\frac{r_1}{t_1}, \frac{r_2}{t_2} \in R_s$ and $\frac{w}{t} \in W_s$. Suppose that $\frac{r_1}{t_1} \cdot \frac{r_2}{t_2} \cdot \frac{w}{t} \in L_s$, thus there exists $l \in L, s \in S$ such that $\frac{r_1}{t_1} \cdot \frac{r_2}{t_2} \cdot \frac{w}{t} = \frac{l}{s}$. Such that $(r_1r_2ws - t_1t_2tl)w = 0$ which implies that $r_1r_2ws \in L$ But L is a 2-

quasi-prime submodule of W, Thus either ${r_1}^2ws \in L$ or ${r_2}^2ws \in L$. Hence either $\frac{{r_1}^2ws}{{t_1}^2ts} \in l_s$ or $\frac{{r_2}^2ws}{{t_1}^2ts} \in l_s$. This means that either $\frac{{r_1}^2w}{{r_1}^2t} \in l_s$ or $\frac{{r_2}^2w}{{t_2}^2t} \in l_s$ i.e. l_s is 2-quasi-primesubmodule of W_s

Proposition (3.8). Let W_1 and W_2be two modules and let $W = W_1 \oplus W_2$. If $K = K_1 \oplus K_2$ is a 2-quasi-prime R-submodule such that K_1 and K_2 are two proper submodule of W_1 and W_2 respectively, then $K_1$and K_2 are 2-quasi-prime submodule of W_1 and W_2 respectively.

Proof: To show that $K_1$is a 2-quasi-prime R-submodule of W_1. Let $r_1, r_2 \in R$ and $a \in W_1$, such that $r_1 r_2 a \in K_1$, thus $r_1 r_2 \in [K_1 : (a)]$ and then $r_1 r_2 \in [K : (a, 0)]$. But K is 2-quasi-prime submodule so $[K \dot{_R} (a, 0)]$ is 2-prime ideal so either ${r_1}^2 \in [K \dot{_R} (a, 0)]$ or ${r_2}^2 \in [K \dot{_R} (a, 0)]$ and hence either ${r_1}^2 (a, 0) \in K_1 \oplus K_2$ or ${r_2}^2 (a, 0) \in K_1 \oplus K_2$ which implies that either ${r_1}^2 \in [K_1 : (a)]$ or ${r_2}^2 \in [K_1 : (a, 0)]$ and hence K_1 is 2-quasi-prime submodule of W.

Remark (3.9). Let $W = W_1 \oplus W_2$ and let W_1 and W_2 be two R-modules. If K_1 and K_2 are 2-quasi-prime submodules of W_1 and W_2 respectively, then $K = K_1 \oplus K_2$ is not necessarily 2-quasi-prime submodule. As the following example shows

Example (3.10). Consider the Z-module $W = Z \oplus Z$ and let $K = 2Z \oplus 3Z$, 2Z and 3Z are 2-quasi-prime submodule of Z. But $[K \dot{_R} W] = 6Z$ which is not 2-prime ideal of Z. thus by proposition (2.3) K is not a 2-quasi-prime submodule of W.

Proposition (3.11). Let W be an R-module. If K is 2-quasi-primesubmodule of W, then $K^2 = K \oplus K$ which is 2-quasi-príme submodule of $W^2 = W \oplus W$.

Proof: Let $r_1, r_2 \in R$ and $w = (x, y) \in W^2$ such that $r_1 r_2 w \in K$. Thus $r_1 r_2 x \in K$ and $r_1 r_2 y \in K$. This implies that (${r_1}^2 x \in K$ or ${r_2}^2 x \in K$) and (${r_1}^2 y \in K$ or ${r_2}^2 y \in K$) so either ${r_1}^2 w \in K^2$ or ${r_2}^2 w \in K^2$. Therefore K^2 is a 2-quasi-prime R-submodule.

Now, we have the upcoming result.

Proposition (3.12). Let $W = W_1 \oplus W_2$ direct sum of two R-modules W_1 and W_2. If K_1 is 2-quasi-prime of W_1. Then, $K_1 \oplus W_2$ is a 2-quasi-prime submodule of W.

Proof: let $xy \in [K_1 \oplus W_2 \dot{_R} (w)]$ where $x, y \in R, w \in W$. But $W = W_1 \oplus W_2$ and $w \in W$, so $w_1 + w_2 = w$, $w_1 \in W_1$ and $w_2 \in W_2$. Thus $xy(w) = xy(w_1 + w_2) \in K_1 \oplus W_2$ and hence $xyw_1 + xyw_2 = t + a$ for $t \in K_1$ and $a \in W$. Hence $xyw_1 - t = a - xywz \in W_1 \cap W_2 = 0$, so $xyw_1 = t \in K_1$, but K_1 is 2-prime submodule of W_1 then either $x^2 w_1 \in K_1$ or $y^2 w_1 \in K_1$. This implies that $x^2 w = x^2(w_1 + w_2) \in K_1 \oplus W_2$ or $y^2(w_1 \oplus w_2) \in K_1 \oplus W_2$, therefore $K_1 \oplus W_2$ is 2-quasi-prime submodule of W.

Recall $Hom_R(W_1, W_2)$ is the set of all R-homomorphsims from W_1 to W_2 [9]

The upcoming proposition describes the behaviour of 2-quasi-prime submodules in $Hom_R(W_1, W_2)$.

Proposition (3.13). Let W_1 and W_2 be two R-modules and let K be a 2-quasi-prime submodule of W_2 such that $Hom(W, K)$ is a proper submodule of $Hom_R(W_1, W_2)$, then $Hom(W_1, K)$ is a 2-quasi-prime submodule of $Hom_R(W_1, W_2)$.

Proof: To show that $Hom_R(W_1, K)$ is a 2-quasi-prime submodule of $Hom_R(W_1, W_2)$

Let $r_1 r_2 f \in Hom_R(W_1, K)$. Then for each $x \in W_1, r_1 r_2 f(x) \in K$. But K is a 2-quasi-prime submodule of W_2, then either ${r_1}^2 f(x) \in K$ or ${r_2}^2 f(x) \in K$. This implies that either ${r_1}^2 f \in Hom_R(W_1, K)$ or ${r_2}^2 f \in Hom(W_1, K)$.

Proposition (3.14). Let K be a submodule of an R-module W. Then, K is a 2-quasi-prime submodule of W if and only if $[K \dot{_W} J]$ is a 2-quasi-prime submodule of W for every J of R.

Proof: Suppose K is a 2-quasi-prime submodule of W, let $r_1, r_2 \in R$ and $x \in W$ such that $r_1 r_2 x \in [K_{W}\dot{:}J]$. Hence $a r_1 r_2 w \in K \ \forall a \in J$. But K is a 2-quasi-prime submodule of W, thus either $a{r_1}^2 x \in K$ or $a{r_2}^2 x \in K \ \forall a \in J$. Then either ${r_1}^2 x \in [K_{W}\dot{:}J]$ or ${r_2}^2 x \in [K_{W}\dot{:}J]$ that is mean $[K_{W}\dot{:}J]$ is 2-quasi-prime submodule of W.

Conclusion

1. Every 2-prime submodule is 2-quasi-prime submodule. [Examples and remarks (2.2)]
2. A proper submodule K of W is 2-quasi-prime submodule if and only if $[K:(w)]$ is 2-prime ideal. For all $w \in W, w \notin K$.
3. Let K be a proper submodule of an R-module W. Then the following are equivalent.
   - K is 2-quasi-prime submodule of W.
   - $[K_{R}\dot{:}L]$ is a 2-prime ideal of R for each submodule L of W where $[K_{R}\dot{:}L] = \{r: r \in R, rL \subseteq K\}$
   - $[K:(rx)] = [K:(x)]$ For each $x \in W, r \in R$, $r^2 \notin [K_{R}\dot{:}(x)]$. Then $[K_{R}\dot{:}(x)]$ is semi-prime ideal.
4. Let K be a submodule of an R-module W. If K is 2-quasi-prime submodule of W, then $[K_{R}\dot{:}W]$ is a 2-prime ideal of R.

   J is 2-prime ideal of R and M be cyclic such that $ann_R M \subseteq J$. Then JW is a 2-quasi-prime

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المقاسات الجزئية الاولية الظاهرية من النمط-2-

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المقاس الجزئي الاولي الظاهري

معلومات المؤلف

الايميل:
الموبايل:

الخلاصة:

لتكن R حلقه ابداليه ذات عنصر محايد وليكن W مقاسا على R , في هذا البحث عرفنا مفهوم مقاسات الجزئية الاولية الظاهرية من النمط 2 كتعميم للمقاسات الجزئية الأولية الظاهرية يقال ان المقاس الجزئي الفعلي K من المقاسW على الحلقة R انه مقاس جزئي اولي ظاهري من النمط2 متى ما كان $a, b \in R, x \in W$ و $abx \in K$فأنه اما $a^2x \in K$ او $b^2x \in K$ لقد برهنا خصائص عديدة لهذا النوع من المقاسات الجزئية حيث اذا كان K مقاس جزئي من المقاس W فأن K يكون مقاس جزئي اولي ظاهر من النمط 2 اذا وفقط اذا $[K_{\dot{R}}(x)]$ مثالي اولي من النمط 2 لكل $x \in W$