On the Stabilizability of Nonlinear Control Systems

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Abstract:

In this paper a stablizability of nonlinear control system $\dot{x}(t) = f(x) + g(x)u(t)$ has been studied by using a suitable form for control function $u(t)$. We used some conditions on this system to find the positive definite function which has negative definite derivative which led the system to be asymptotically stabilizable. Then We used some hypothesis and some conditions to get a system with difference types of stabilizability.

حول قابلية الأستقرارية لأنظمة السيطرة غير الخطية

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المستخلص

في هذا البحث قمنا بدراسة قابلية الأستقرارية لنظام السيطرة غير الخطي $\dot{x}(t) = f(x) + g(x)u(t)$ من خلال ايجاد الصيغة المناسبة لدالة السيطرة ومن خلال وضع شروط معينة على هذا النظام قمنا بايجاد دالة(positive definite) والتي تكون مشتققتها (negative definite) والتي تجعل النظام (asymptotically stabilizable). وبالاعتماد على هذه الدالة وضعنا مجموعة من الفرضيات وبشروط معينه مما قاد النظام أن يمتلك قابلية الأستقرارية وبأنواع مختلفه

1. Introduction:

A fundamental feedback control problem is that of obtaining some prespecifued desired behavior from a system about which there is uncertain or incomplete information. In recent years, much effort has focused on utilizing Lyapunov theory to obtain controllers which yields desirable behavior from systems whose uncertainties are characterized deterministically, rather than stochastically [4, 5].

Stability of dynamical control systems is very important, because of its wide applications in different fields. Since the process of finding the general solution to dynamical control system is almost impossible except for linear systems with constant coefficients. Therefore, theorems for determining the stability of the solution of a particular system without actually solving the system are established [13]. Most physical systems encountered in engineering applications are inherently nonlinear. Thus, control of nonlinear systems is a subject of active research and increasing interest. However, most controller design techniques for nonlinear systems are not systematic and/or apply only to very specific case [11].

One of the most basic issues in system theory is the stability of dynamical systems. The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov. Lyapunov's results, along with the Krasovskii-Lasalle invariance principle, provide a powerful framework for analyzing the stability of nonlinear dynamical systems. Lyapunov methods have also been used by control system designers to obtain stabilizing controllers for nonlinear systems. IN particular, for smooth feedback, Lyapunov-based methods were inspired by Jurdjevic and Quinn [7] who gives sufficient conditions for smooth stabilization based on the ability of constructing a Lyapunov function for the closed-loop system. More recently, Art stein [1] introduced the notion of a control Lyapunov function whose existence guarantees a feedback control law which globally stabilizes a nonlinear dynamical system. In general the feedback control law is not necessarily smooth, but can be guaranteed to be at least continuous at the origin in addition to being smooth everywhere else. Even though for certain classes of nonlinear dynamical systems a universal construction of feedback stabilizer can be obtained using control Lyapunov functions [18, 20], there does not exist a unified procedure for finding a Lyapunov function candidate that will stabilize the closed-loop system for general nonlinear systems [17].

Stabilization problem of nonlinear control system has been for many years a subject of great interest to researchers in dynamical system theory [2, 6, 12, 15, 16, 19, 22]. This problem regarding as an extension of the classical Kalman result [8] is to find a control such that the corresponding solution of the system has desired properties. Depending on the properties involved one defines various qualitative problems

2.Preliminaries:

The following notation will be used this paper:

R^n is the n-dimensional Euclidean vector space; R^+ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in R^n$.

Consider the nonlinear system:

$$\begin{aligned} &\dot{x}(t) = F(x(t),t)\,;\, t \geq 0 \\ &x(t_0) = x_0\,;\, t_0 \geq 0 \end{aligned} \qquad (2.1)$$

where $x(t) \in R^n$, $F(x,t): R^n \times R^+ \to R^n$ is a given nonlinear function satisfying $F(0,t) = 0$ for all $t \in R^+$. We shall assume that conditions are imposed on system (2.1) such that the existence of its solutions is guaranteed.

Definition(2.1): [10]

The zero solution of system(2.1) is exponentially stable if any solution $x(x_0,t)$ of (2.1) satisfies:

$$\|x(x_0,t)\| \le \beta(\|x_0\|,t_0)e^{-\delta(t-t_0)} \ ,\forall\ t \ge t_0$$

where $\beta(h,t): R^+ \times R^+ \to R^+$ is a non-negative function increasing in $h \in R^+$ and δ is a positive constant.

If the function $\beta(.)$ in the above definition does not depend on t_0, the zero solution is called uniformly exponentially stable.From now on ,yo shorten expressions,instead of saying the zero solution is stable ,We say that the system is stable.

Associated with system (2.1) We consider a nonlinear control system:

$$\dot{x}(t) = F(x(t),u(t),t) \quad , \quad t \ge 0 \qquad (2.2)$$

where $x \in R^n, u \in R^m, F(x,u,t): R^n \times R^m \times R^+ \to R^n$.

Definition(2.2): [10]

Control system (2.2) is exponentially stabilizable by the feedback control $u(t) = h(x(t))$

Where $h(x): R^n \to R^m$, if the closed-lop system:

$$\dot{x}(t) = F(x(t),h(x(t)),t)$$

is exponentially stable.

Lemma(2.3):[3]

Let P be a symmetric matrix and let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ be the smallest and largest eigenvalues of P, respectively,then:

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\|x\|^2 \quad , \quad \forall x \in R^n$$

where $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$, x_i is the i-th component of x.

Theorem(2.4):[14]

If there exists a scalar function $V(x,t)$ with continuous first partial derivative satisfying the following conditions:

a. $V(x,t) > 0$ for all $x \ne 0$ in Ω and all t.

$V(0,t) = 0$ for all t.

b. $\dot{V}(x,t) < 0$ for all $x \ne 0$ in Ω and all t.

$\dot{V}(0,t) = 0$ for all t.

Ω is the region(can be the entire state space),which includes the origin.Then the origin of the system $\dot{x}(t) = F(x(t),t)$ is uniformly asymptotically stable.

Remark(2.5):[21]

Consider the nonlinear control system (2.2), assuming that $F(0,0,t)=0$, for all $t \ge 0$.
We recall that system (2.2) is asymptotically stabilizable by a feedback control $u(t)=h(x(t))$, where $h(x): R^n \to R^m$, $h(0)=0$, if the zero solution of the system without control:

$$\dot{x}(t)=F(x(t),h(x),t)\ ,\ t \ge 0$$

$$x(t_0)=x_0 \quad , \quad t_0 \ge 0$$

is asymptotically stable in the Lyapunov sence.

Definition(2.6):[9]

Let the origin be a solution of :

$$\dot{x}(t)=F(x(t))$$

it is said to be globally asymptotically stable if there exists a function β such that for each $x \in R^n$,all the solutions $x(x_0,t)$ are defined on $[0,\infty)$ and satisfy:

$$\|x(x_0,t)\| \le \beta(\|x_0,t_0)\| \quad , \quad \forall t \ge 0$$

3.Main result:

Consider the nonliear contror system described by:

$$\dot{x}(t)= f(x)+g(x)u(t) \tag{3.1}$$

where $f(x): R^n \to R^n$, $g(x): R^n \to R^n$, $u(t): R^+ \to R^m$ and $g(x) \ne 0$ for all x. In theorem below , We give sufficient conditions for the stabilizability of the system(3.1)

Theorem(3.1):

The system (3.1)is asymptotically stabilizable if:

a. $\|f(x)\| \le \eta\|x\| \quad ; \quad \eta > 0$

b. $u(t)=-\dfrac{x^T Pf(x)+f^T(x)Px+\alpha\|x\|}{\|g(x)\|}$**,** where P is a positive definite symmetric matrix and $\alpha > \eta$

Proof:

Let $V(x)=x^T Px$ then from lemma (2.3) ,we have :

$$\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2 \tag{3.1.1}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of P respectively.

$\dot{V}(x)=x^T P\dot{x}+\dot{x}^T Px$

$$= x^T P\left[f(x) + g(x)u\right] + \left[f(x) + g(x)u\right]^T Px$$
$$= x^T Pf(x) + f^T(x)Px + x^T Pg(x)u + u^T g^T(x)Px$$
$$= x^T Pf(x) + f^T(x)Px - x^T Pg(x)\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|} - \left[\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right]^T g^T(x)Px$$
$$\leq \left\|x^T Pf(x)\right\| + \left\|f^T(x)Px\right\| - \left\|x^T Pg(x)\right\|\left\|\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right\| - \left\|\left[\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right]^T\right\|\left\|g^T(x)Px\right\|$$
$$\leq \left\|x^T\right\|\|P\|\|f(x)\| + \left\|f^T(x)\right\|\|P\|\|x\| - \left\|x^T\right\|\|P\|\|g(x)\|\left\|\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right\| - \left\|\left[\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right]^T\right\|\left\|g^T(x)\right\|\|P\|\|x\|$$

Since for any vector A we have $\|A\| = \left\|A^T\right\|$ then:

$$\dot{V}(x) \leq 2\|x\|\|P\|\|f(x)\| - 2\|x\|\|P\|\|g(x)\|\left\|\frac{x^T Pf(x) + f^T(x)Px + \alpha\|x\|}{\|g(x)\|}\right\|$$
$$\leq 2\|x\|\|P\|\|f(x)\| - 2\|x\|\|P\|\|g(x)\|\frac{\left\|x^T\right\|\|P\|\|f(x)\| + \left\|f^T(x)\right\|\|P\|\|x\| + \alpha\|x\|}{\|g(x)\|}$$
$$\leq 2\|x\|\|P\|\|f(x)\| - 2\|x\|\|P\|\left[2\|x\|\|P\|\|f(x)\| + \alpha\|x\|\right]$$

From the condition (a) We obtain:

$$\dot{V}(x) \leq 2\eta\|P\|\|x\|^2 - 2\|P\|\|x\|\left[2\eta\|P\|\|x\|^2 + \alpha\|x\|\right]$$
$$\leq 2\eta\|P\|\|x\|^2 - 4\eta\|P\|^2\|x\|^3 - 2\alpha\|P\|\|x\|^2$$

Since $4\eta\|P\|^2\|x\|^3 \geq 0$ then:

$$\dot{V}(x) \leq 2\eta\|P\|\|x\|^2 - 2\alpha\|P\|\|x\|^2$$
$$\leq 2(\eta - \alpha)\|P\|\|x\|^2$$

Since $\alpha > \eta$, $\|P\| > 0$, $\|x\|^2 \geq 0$ then :

$$\dot{V}(x) \leq 0$$

By theorem (2.4) and remark (2.5) the system (3.1) is asymptotically stabilizable.

<u>Lemma(3.2)</u>

The system (3.1) which is satisfies the conditions of theorem (3.1) is exponentially stabilizable.

<u>Proof:</u>

From theorem (3.1),we have:

$V(x) = x^T Px$ such that $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$ and $\dot{V}(x) \leq 2(\eta - \alpha)\|P\|\|x\|^2$

Let: $Q(x,t) = V(x)e^{mt}$, where $m = \dfrac{2(\alpha - \eta)\|P\|}{\lambda_{\max}(P)} > 0$

$$\dot{Q}(x,t) = \dot{V}(x)e^{mt} + mV(x)e^{mt}$$

$$\leq 2(\eta-\alpha)\|P\|\|x\|^2 e^{mt} + m\lambda_{\max}(P)\|x\|^2 e^{mt}$$
$$\leq \left[2(\eta-\alpha)\|P\| + m\lambda_{\max}(P)\right]\|x\|^2 e^{mt}$$
$$Q(x,t) - Q(x_0,t_0) \leq \left[\frac{2(\eta-\alpha)\|P\| + m\lambda_{\max}(P)}{m}\right]\|x\|^2 \left[e^{mt} - e^{mt_0}\right]$$

Since: $m = \dfrac{2(\alpha-\eta)\|P\|}{\lambda_{\max}(P)}$, then:

$$Q(x,t) - Q(x_0,t_0) \leq 0$$
$$Q(x,t) \leq Q(x_0,t_0)$$
$$Q(x,t) \leq V(x_0)e^{mt_0} \leq \lambda_{\max}(P)\|x_0\|^2 e^{mt_0}$$

Setting: $\beta(\|x_0\|,t_0) = \lambda_{\max}(P)\|x_0\|^2 e^{mt_0}$, hence:

$$Q(x,t) \leq \beta(\|x_0\|,t_0)$$

From the left hand of (3.1.1), we have:

$$\|x\| \leq \left[\frac{V(x)}{\lambda_{\min}(P)}\right]^{1/2}$$
$$\leq \left[\frac{Q(x,t)e^{-mt}}{\lambda_{\min}(P)}\right]^{1/2} \leq \left[\frac{\beta(\|x_0\|,t_0)}{\lambda_{\min}(P)}\right]^{1/2} e^{\frac{-mt}{2}}$$

Therefore the system(3.1) is exponentially stabilizable.

Corollary(3.3):

In lemma (3.2) if $Q(x,t) = V(x)e^{m(t-t_0)}$, then the system (3.1) is uniformly exponentially stabilizable.

Proof:

$$Q(x,t) = V(x)e^{m(t-t_0)}$$
$$\dot{Q}(x,t) = \dot{V}(x)e^{m(t-t_0)} + mV(x)e^{m(t-t_0)}$$
$$\leq \left[2(\eta-\alpha)\|P\| + m\lambda_{\max}(P)\right]\|x\|^2 e^{m(t-t_0)}$$
$$Q(x,t) - Q(x_0,t_0) \leq \left[\frac{2(\eta-\alpha)\|P\| + m\lambda_{\max}(P)}{m}\right]\|x\|^2 \left[e^{m(t-t_0)} - 1\right]$$

Since: $m = \dfrac{2(\alpha-\eta)\|P\|}{\lambda_{\max}(P)}$, then:

$$Q(x,t) - Q(x_0,t_0) \leq 0$$
$$Q(x,t) \leq Q(x_0,t_0) \leq V(x_0) \leq \lambda_{\max}(P)\|x_0\|^2$$

Setting: $\beta(\|x_0\|) = \lambda_{\max}(P)\|x_0\|^2$, hence: $Q(x,t) \leq \beta(\|x_0\|)$

From the left hand of(3.1.1),we have:

$$\|x\| \leq \left[\frac{V(x)}{\lambda_{\min}(P)}\right]^{1/2}$$

$$\leq \left[\frac{Q(x,t)e^{-m(t-t_0)}}{\lambda_{\min}(P)}\right]^{1/2} \leq \left[\frac{\beta(\|x_0\|)}{\lambda_{\min}(P)}\right]^{1/2} e^{\frac{-m}{2}(t-t_0)}$$

Therefore the system (3.1) is uniformly exponentially stabilizable.

Corollary(3.4):

In lemma (3.2) , if $Q(x,t) = V(x) + m\lambda_{\max}(P)\|x\|^2 t$, then the system(3.1) is a globally asymptotically stabilizable.

Proof:

$$Q(x,t) = V(x) + m\lambda_{\max}(P)\|x\|^2 t$$

$$\dot{Q}(x,t) = \dot{V}(x) + m\lambda_{\max}(P)\|x\|^2$$

$$\leq 2(\eta-\alpha)\|P\|\|x\|^2 + m\lambda_{\max}(P)\|x\|^2$$

$$\leq \left[2(\eta-\alpha)\|P\| + m\lambda_{\max}(P)\right]\|x\|^2$$

Since: $m = \dfrac{2(\alpha-\eta)\|P\|}{\lambda_{\max}(P)}$, then: $\dot{Q}(x,t) \leq 0$,hence:

$$Q(x,t) - Q(x_0,t_0) \leq 0$$

$$Q(x,t) \leq Q(x_0,t_0) \leq V(x_0) + m\lambda_{\max}(P)\|x_0\|^2 t_0$$

From the right hand of (3.1.1),we obtain:

$$Q(x,t) \leq \lambda_{\max}(P)\|x_0\|^2 + m\lambda_{\max}(P)\|x_0\|^2 t_0$$

$$\leq (mt_0+1)\lambda_{\max}(P)\|x_0\|^2$$

Setting: $\beta(\|x_0\|,t_0) = (mt_0+1)\lambda_{\max}(P)\|x_0\|^2$, hence: $Q(x,t) \leq \beta(\|x_0\|,t_0)$

From the left hand of (3.1.1) we have:

$$\|x\| \leq \left[\frac{V(x)}{\lambda_{\min}(P)}\right]^{1/2} \leq \left[\frac{Q(x,t) - m\lambda_{\max}(P)\|x\|^2 t}{\lambda_{\min}(P)}\right]^{1/2}$$

Since: $m\lambda_{\max}(P)\|x\|^2 t \geq 0$, then:

$$\|x\| \leq \left[\frac{Q(x,t)}{\lambda_{\min}(P)}\right]^{1/2}$$

$$\leq \left[\frac{\beta(\|x_0\|,t_0)}{\lambda_{\min}(P)}\right]^{1/2}$$

Therefore the system (3.1) is a globally asymptotically stabilizable.

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