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New and Extended Results On Fourth-Order Differential Subordinationfor Univalent Analytic Functions

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New and Extended Results On Fourth-Order Differential Subordination for Univalent Analytic Functions

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ABSTRACT

In this paper, we introduce new concept that is fourth-order differential subordination associated with linear operator $I_{s,\alpha,\mu}^{\lambda}$ for univalent analytic functions in the open unit disk. Here, we extended some lemmas. Also some interesting new results are obtained.

MSC: 30C45, 30C50

1. Introduction

Let $H(U)$ be the class of function which are analytic in the open unit disk

$$U = \{z: z \in C: |z| < 1\}.$$

For $n \in N = \{1,2,3,\dots\}$, and $a \in C$, let

$$H[a,n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

and also, let $H_0 = [0,1]$.

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Let Σ denote the subclass of H(U) consisting of all univalent and analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1.1}$$

Komatu [8] introduced and investigated a family of integral operator

$$J_{\mu}^{\lambda}: \Sigma \to \Sigma.$$

That is obtained as follows

$$J_{\mu}^{\lambda} f(z) = z + \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu+n-1}\right)^{\lambda} a_n z^n, (z \in U^*, n > 1, \lambda \geq 0). \tag{1.2}$$

The Hurwitz - Lerch Zeta function

$$\phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^k}{(k+a)^s}, a \in \mathbb{C}/z_0^{-}, s \in \mathbb{C}\ when\ 0 < |z| < 1.$$

Interims of Hadamard product (or convolution) where $G_{s,a(z)}$ is given by

$$G_{s,a(z)} = (1+a)^s [\phi(z,s,a) - a^{-s}], (\ z \in U).$$

Then a linear operator $I_{s,a,\mu}^{\lambda} f(z): \Sigma \to \Sigma\ (see[2])$ is defined

$$I_{s,a,\mu}^{\lambda} f(z) = G_{s,a(z)} * J_{\mu}^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s \left(\frac{\mu}{\mu+n-1}\right)^{\lambda} a_n z^n. \tag{1.3}$$

It is easily verified from (1.3) that

$$z\left(I_{s,a,\mu}^{\lambda+1} f(z)\right)' = \mu I_{s,a,\mu}^{\lambda} f(z) - (\mu - 1) I_{s,a,\mu}^{\lambda+1} f(z) \tag{1.4}$$

For several past years , there are many authors introduce and dealing with the theory of second-order differential subordination and superordination for example ([3,4,10]). Recently, many authors discussed the theory of third-order differential subordination and superordination for example ([5,6,7,11,12,13,14,15]). In the present paper, we investigated the extended theory of second-order differential subordination in the open unit disk introduced by Miller and Mocanu [9] to third-order case. Now, we extend this to fourth-order differential subordination and determined properties of functions p that satisfy the following fourth-order differential subordination:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\}.$$

To prove our main results, we need the basic concepts in theory of the fourth-order.

Definition(1.1): [9]. Let $f(z)$ and $F(z)$ be members of the analytic function class $H(U)$.The function $f(z)$ is said to be subordinate to $f(z)$ or $F(z)$ is superordinate to $f(z)$ if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, and such that $f(z) = F\big(w(z)\big)$. In such case, we write

$$f \prec F \text{ , } or \;\; f(z) \prec F(z).$$

If $F(z)$ is univalent in U, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)\ and\ f(U) \subset F(U)$.

Definition (1.2): [1]. Let $\mathbb{Q}$ be the set of all functions q that are analytic and univalent on $\overline{U}/E(q)$,where

$$E(q) = \left\{\xi : \xi \in \partial U : \lim_{z \to \xi}\{q(z)\} = \infty\right\},$$

and are such that $min|q'(\xi)| = p > 0\ for\ \xi \in \partial\, U/E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ with

$$\mathbb{Q}(0) = \mathbb{Q}_0 \text{ and } \mathbb{Q}(1) = \mathbb{Q}_1.$$

Definition (1.3): Let $\psi: \mathbb{C}^5 \times U \to \mathbb{C}$ and the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U and satisfies the following fourth-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \prec h(z), \tag{1.5}$$

then *p(z) i*s called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\breve{q}(z)$ that satisfies $\breve{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant.

Lemma (1.4): Let $z_0 \in U$ with $r_0 = |z_0|$. For $n \geq 1$. Let

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots$$

be continuous on $\overline{U_{r_0}}$ and analytic in $U_{r_0} \cup \{z_0\}$, with $f(z) \neq 0$. If

$$|f(z_0)| = \max\left\{|f(z)| : z \in \overline{U_{r_0}}\right\}, \tag{1.6}$$

then there exists $an, m \geq n$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \tag{1.7}$$

$$Re\left\{\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right\} \geq m, \tag{1.8}$$

and

$$Re\left\{\frac{z_0f'(z_0)+3z_0^2f''(z_0)+z_0^3f^{(3)}(z_0)}{z_0f'(z_0)}\right\}\geq m^2. \tag{1.9}$$

Then

$$Re\left\{\frac{z_0f'(z_0)+7z_0^2f''(z_0)+6z_0^{(3)}f^{(3)}(z_0)+z_0^{(4)}f^{(4)}(z_0)}{z_0f'(z_0)}\right\}\geq m^3. \tag{1.10}$$

Proof: The relations (1.7), (1.8) were proved in Lemma [9, Chap 2, page. 19] and (1.9) were proved in Lemma [9, Chap 6 ,page. 322]. We only need to prove (1.10).

If we $f(z)=R(r_0,\theta)e^{i\emptyset(r_0,\theta)}, for\ z=r_0e^{i\emptyset}$,

then

$$\frac{zf'(z)}{f(z)}=\frac{\partial\emptyset}{\partial\theta}-i\frac{1}{R}\frac{\partial R}{\partial\theta}. \tag{1.11}$$

Differentiating (1.11) with respect to θ, we obtain

$$i\frac{zf'(z)}{f(z)}\left[\frac{z\big(zf'(z)\big)'}{zf'(z)}-\frac{zf'(z)}{f(z)}\right]=\frac{\partial^2\emptyset}{\partial\theta^2}-i\left[\frac{1}{R}\frac{\partial^2R}{\partial\theta^2}-\left(\frac{1}{R}\frac{\partial R}{\partial\theta}\right)^2\right]. \tag{1.12}$$

Another differentiation with respect θ leads to

$$\frac{zf'(z)}{f(z)}\left[\frac{z\left(z\big(zf'(z)\big)'\right)'}{zf'(z)}-3\frac{z\big(zf'(z)\big)'}{zf'(z)}.\frac{zf'(z)}{f(z)}+2\left(\frac{zf'(z)}{f(z)}\right)^2\right]$$
$$=-\frac{\partial^3\emptyset}{\partial\theta^3}-i\frac{\partial}{\partial\theta}\left[\frac{1}{R}\frac{\partial^2R}{\partial\theta^2}-\left(\frac{1}{R}\frac{\partial R}{\partial\theta}\right)^2\right]. \tag{1.13}$$

Also differentiation (1.13) with respect to θ, we obtain

$$i\frac{zf'(z)}{f(z)}\left[\frac{z(z(z\big(zf'(z)\big)')')'}{zf'(z)}-4\frac{z\left(z\big(zf'(z)\big)'\right)'}{zf'}.\frac{zf'(z)}{f}-3\frac{z\big(zf'(z)\big)'}{zf'}.\frac{z(zf')'}{zf'}.\frac{zf'}{f}\right.$$
$$\left.+12\frac{z(zf')'}{zf'}.\left(\frac{zf'}{f}\right)^2-6\left(\frac{zf'}{f}\right)^3\right]=-\frac{\partial^4\emptyset}{\partial\theta^4}+i\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{R}\frac{\partial^2R}{\partial\theta^2}-\left(\frac{1}{R}\frac{\partial R}{\partial\theta}\right)^2\right].$$

Taking imaginary parts of this expression at z_0 and using (1.7) leads to

$$Re\left[\frac{z_0f'(z_0)+7{z_0}^2f''(z_0)+6{z_0}^3f'''(z_0)+z_0^4f^{(4)}(z_0)}{z_0f'(z_0)}\right]m-4m^2Re\left[\frac{z_0f'(z_0)+3{z_0}^2f''(z_0)+{z_0}^3f^{(3)}(z_0)}{z_0f'(z_0)}\right]-$$
$$3m^2Re\left[\frac{z_0f''(z_0)}{f'(z_0)}+1\right]^2+12m^2Re\left[\frac{z_0f''(z_0)}{f'(z_0)}+1\right]-6m^4=\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{R}\frac{\partial^2R}{\partial\theta^2}-\left(\frac{1}{R}\frac{\partial R}{\partial\theta}\right)^2\right].$$

Finally, by applying (1.8), (1.9), we obtain the desired result (1.10).

Lemma (1.5): Let $p \in H[a,n]$ and $q \in Q$ with $q(0)=a$ for $z \in \overline{U_{r_0}}$. Let

$$S = q^{-1}[p(z)] = f(z). \tag{1.14}$$

If there exists points $z_0 \in U$ and $s_0 \in \partial U / E(q)$ such that $p(z_0)=q(s_0)$ and $P\left(\overline{U_{r_0}}\right) \subset q(U)$,

$$Re\left\{\frac{s_0 q''(s_0)}{q'(s_0)}\right\} \geq 0 \quad , \quad \left|\frac{zp'(z)}{q'(s)}\right| \leq k \tag{1.15}$$

$$and \qquad Re\left\{\frac{s_0^2 q^{(3)}(s_0)}{q'(s_0)}\right\} \geq 0 \ , \quad \left|\frac{z^2 p''(z)}{q'(s)}\right| \leq k^2, \tag{1.16}$$

where $r_0 = |z_0|$. Then there exists an $m \geq n \geq 1$ such that

$$z_0 p'(z_0) = m s_0 q'(s_0) \tag{1.17}$$

$$Re\left\{\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right\} \geq m Re\left\{\frac{s_0 q''(s_0)}{q'(s_0)} + 1\right\} \tag{1.18}$$

and

$$Re\left\{\frac{z_0 p'(z_0) + 3z_0^2 p''(z_0) + z_0^3 p^{(3)}(z_0)}{z_0 p'(z_0)}\right\} \geq m^2 Re\left\{\frac{s_0 q'(s_0) + 3s_0^2 q''(s_0) + s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)}\right\}. \tag{1.19}$$

Then

$$Re\left\{\frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)}\right\}$$
$$\geq m^3 Re\left\{\frac{s_0 q'(s_0) + 7s_0^2 q''(s_0) + 6s_0^3 q^{(3)}(s_0) + s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)}\right\}, \tag{1.20}$$

$$or \quad Re\left\{\frac{z_0^3 p^{(4)}(z_0)}{p'(z_0)}\right\} \geq k^3 Re\left\{\frac{s_0^3 q^{(4)}(s_0)}{q'(s_0)}\right\}. \tag{1.21}$$

Proof: The relations (1.17), (1.18) proved in [9, Chap. 2, page 22] and (1.19) proved in [9, Chap. 6, pages 325 and 327].We only need to prove (1.20). Note that q is univalent at s_0 and hence $q'(s_0) \neq 0$ .Since p is analytic in U, the set $p(\overline{U_{r_0}})$ is a bounded set and $p(\overline{U_{r_0}}) \subset q(\overline{U})/E(q)$. From (1.14) we see that f is analytic in $(\overline{U_{r_0}})$ and satisfies $|f(z_0)| = |s_0| = 1, f(0) = 0 \ and \ |f(z)| \leq 1 \ for \ |z| \leq r_0$. A furlher calculation show that $f^{(k)}(0) = p^{(k)}(0) \ for \ k = 1,2,\dots,n-1$. Thus f satisfies the conditions of Lemma (1.4) and deduce that there exists an m,n such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \tag{1.22}$$

and

$$Re\left\{\frac{z_0 f''(z_0)}{f'(z_0)}+1\right\} \geq m. \tag{1.23}$$

Since from (1.14) we have $p(z) = q(s)$, with $s = f(z)$, we obtain

$$zp'(z) = q'(z)zf'(z). \tag{1.24}$$

Differentiating (1.20) leads to

$$\frac{z(zp'(z))'}{zp'(z)} = \frac{sq''(s)}{q'(s)}.\frac{zf'(z)}{f(z)} + \frac{z(zf'(z))'}{zf'(z)}, \tag{1.25}$$

Another differentiation leads to

$$\frac{z\left(z\left(zp'(z)\right)'\right)'}{zp'(z)} = \frac{s^3q^{(3)}(s)}{sq'(s)}\left(\frac{zf'(z)}{f(z)}\right)^2 + 3\frac{s^2q''(s)}{sq'(s)}.\frac{zf'(z)}{f(z)}.\frac{z\left(zf'(z)\right)'}{zf'(z)} + \frac{z\left(z\left(zf'(z)\right)'\right)'}{zf'(z)}. \tag{1.26}$$

While another differentiation leads to

$$\frac{z(z(z(zp'(z))')')'}{zp'(z)} = \frac{z\left(z\left(z\left(zf'(z)\right)'\right)'\right)'}{zf'(z)}$$

$$+4\frac{s^2q''(s)}{q'(s)}.\frac{z\left(z\left(zf'(z)\right)'\right)'}{zf'(z)}.\frac{zf'(z)}{f(z)} + 6\frac{s^3q^{(3)}(s)}{sq'(s)}.\frac{z\left(zf'(z)\right)'}{zf'(z)}.\left(\frac{zf'(z)}{f(z)}\right)^2$$

$$+3\frac{s^2q''(s)}{sq'(s)}.\left(\frac{z\left(zf'(z)\right)'}{zf'(z)}\right)^2.\frac{zf'(z)}{f(z)} + \frac{s^4q^{(4)}(s)}{sq'(s)}.\left(\frac{zf'(z)}{f(z)}\right)^3.$$

If we evaluate the real part of this expression at z_0 and use (1.23) and (1.25), we obtain

$$Re\left\{\frac{z_0p'(z_0)+7z_0^2p''(z_0)+6z_0^3p^{(3)}(z_0)+z_0{}^4p^{(4)}(z_0)}{z_0p'(z_0)}\right\} = Re\left\{\frac{z_0f'(z_0)+7z_0^2f''(z_0)+6z_0^3f^{(3)}(z_0)+z_0{}^4f^{(4)}(z_0)}{z_0f'(z_0)}\right\} +$$
$$4m\frac{s_0^2q''(s_0)}{s_0q'(s_0)}Re\left\{\frac{z_0f'(z_0)+3z_0^2f''(z_0)+z_0^3f^{(3)}(z_0)}{z_0f'(z_0)}\right\} + 3m\frac{s_0^2q''(s_0)}{s_0q'(s_0)}Re\left\{\left(\frac{z_0f''(z_0)}{f'(z_0)}+1\right)^2\right\} +$$
$$6m^2\frac{s_0^3q^{(3)}(s_0)}{s_0q'(s_0)}Re\left\{\frac{z_0f''(z_0)}{f'(z_0)}+1\right\} + \frac{m^3s_0^4q^{(4)}(s_0)}{s_0q'(s_0)}.$$

By use condition (1.9) and (1.10) in Lemma(1.4) and use (1.7), (1.8), we obtain

$$Re\left\{\frac{z_0p'(z_0)+7z_0^2p''(z_0)+6z_0^3p^{(3)}(z_0)+z_0{}^4p^{(4)}(z_0)}{z_0p'(z_0)}\right\}$$
$$\geq m^3 + 4m^3\frac{s_0^2q''(s_0)}{s_0q'(s_0)} + 3m^3\frac{s_0^2q''(s_0)}{s_0q'(s_0)} + 6m^3\frac{s_0^3q^{(3)}(s_0)}{s_0q'(s_0)} + m^3\frac{s_0^4q^{(4)}(s_0)}{s_0q'(s_0)},$$

hence

$$Re\left\{\frac{z_0p'(z_0)+7z_0^2p''(z_0)+6z_0^3p^{(3)}(z_0)+{z_0}^4p^{(4)}(z_0)}{z_0p'(z_0)}\right\}$$
$$\geq m^3\left[\frac{s_0q'(s_0)+7s_0^2q''(s_0)+6s_0^3q^{(3)}(s_0)+s_0^4q^{(4)}(s_0)}{s_0q'(s_0)}\right]$$

$$or\quad Re\left\{\frac{z_0^3p^{(4)}(z_0)}{p'(z_0)}\right\}\geq k^3Re\left\{\frac{s_0^3q^{(4)}(s_0)}{q'(s_0)}\right\}.$$

Definition(1.6): Let Ω be a set in $\mathbb{C}$, $q\in Q$ and $n\in N/\{2\}$. The class of admissible function $A_n[\Omega,q]$ consists of those function $\psi:\mathbb{C}^5\times U\longrightarrow\mathbb{C}$ that satisfies the following admissibility condition $\psi(r,s,t,w,b;z)\notin\Omega$, whenever

$$r=q(\xi),s=k\xi q'^{(\xi)},\Re\left(\frac{t}{s}+1\right)\geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right),\mathcal{R}\left(\frac{w}{s}\right)\geqq k^2\mathcal{R}\left(\frac{\xi^2q^{(3)}(\xi)}{q'(\xi)}\right)$$

$$\text{and}\quad \mathcal{R}\left(\frac{b}{s}\right)\geqq k^3\mathcal{R}\left(\frac{\xi^3q^{(4)}(\xi)}{q'(\xi)}\right),$$

where $z\in U$ $,\xi\in\partial U/E(q)$ and $k\geqq n$.

The next theorem is a foundation result in the theory of fourth-order differential subordinations. Its proof is very short because of we use Lemma (1.5) and the special conditions given in the definition of the class of admissible functions $A_n[\Omega,\text{q}]$.

Theorem (1.7): Let $p\in H[a,n]$ with $n\in N/\{2\}$. Also, let $q\in \text{Q}(a)$ and satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi^2q'''(\xi)}{q'(\xi)}\right)\geqq 0\qquad\qquad \left|\frac{z^2p''(z)}{q'(\xi)}\right|\leqq k^2, \tag{1.27}$$

where $z\in U$ $,\xi\in\partial U/E(q)$ and $k\geqq n$. If Ω a set in $\mathbb{C}$, $\psi\in A_n[\Omega,q]$ and

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p''''(z);z)\in\Omega, \tag{1.28}$$

then

$$p(z)\prec q(z)\qquad (z\in U).$$

Proof: If we assume that $p\nprec q$, then there exist point $z_0=r_oe^{i\theta_0}\in U$ and $s_o\in\partial U/E(q)$ such that $p(z_0)=q(s_0)$ and $p(\overline{U_{r_0}})\subset \text{q(U)}$. From (1.27), we see that the condition (1.16) of Lemma (1.5) is satisfied when $z\in\overline{U_{r_0}}$ and $s_o\in\partial U/E(q)$. Since all the conditions of that Lemma are satisfied, conclusions (1.17), (1.18), (1.19) and (1.21) follow. Using these last four results of Definition (1.6) leads to

$$\psi\big(p(z_0),zp'(z_0),z^2p''(z_0),z^3p^{(3)}(z_0),z^4p^{(4)}(z_0);z\big)\notin\Omega,$$

since this contradicts (1.28), we must have $p\prec q$.

2. Fourth-order differential subordination with $I_{s,a,\mu}^{\lambda}$

We first define the following class of admissible function, which are required in proving the differential subordination theorem involving the operator $I_{s,a,\mu}^{\lambda} f(z)$ defined by (1.3).

Definition (2.1): Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap \mathrm{H}_0$. The class function $B_I[\Omega, q]$ consists of those function $\phi: \mathbb{C}^5 \times U \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(\xi)\,, \qquad v = \frac{k\xi q'(\xi) + (\mu - 1)q(\xi)}{\mu}$$

$$\mathcal{R}\left(\frac{x\mu^2 - (\mu - 1)^2 u}{v\mu - u(\mu - 1)} + (2 - 2\mu)\right) \geqq \mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

$$\mathcal{R}\left(\frac{y\mu^3 - 3x\mu^3 + (2\mu + 1)(\mu - 1)^2}{v\mu - u(\mu - 1)} + (3\mu^2 - 1)\right) \geqq k^2 \mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right)$$

and

$$\mathcal{R}\left(\frac{g\mu^4 - (\mu - 1)^4 + (4\mu^4 - 2\mu^3)y + (\mu - 1)^3(4\mu + 2)u + (12\mu^2 + 6\mu)(x\mu^2 - (\mu - 1)^2 u}{v\mu - u(\mu - 1)} - (12\mu^3 + 6\mu^2 + 8\mu - 2)\right) \geqq k^3 \mathcal{R}\left(\frac{\xi^3 q''''(\xi)}{q'(\xi)}\right).$$

Theorem (2.2): Let $\phi \in B_I[\Omega, q]$.If the function $f(z) \in \Sigma$ and $q \in \mathbb{Q}_0$ and satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right) \geqq 0 \qquad \left|\frac{\mathrm{I}_{s,a,\mu}^{\lambda-1} f(z)}{q'(\xi)}\right| \leqq k^2, \tag{2.1}$$

and

$$\left\{\phi\left(I_{s,a,\mu}^{\lambda+1} f(z), I_{s,a,\mu}^{\lambda} f(z), I_{s,a,\mu}^{\lambda-1} f(z), I_{s,a,\mu}^{\lambda-2} f(z), I_{s,a,\mu}^{\lambda-3} f(z)\right); z \in \mathrm{U}\right\} \subset \Omega, \tag{2.2}$$

then

$$I_{s,a,\mu}^{\lambda+1} f(z) \prec q(z) \qquad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = I_{s,a,\mu}^{\lambda+1} f(z), \quad (z \in U). \tag{2.3}$$

Then, differentiating (2.3) with respect to z and using (1.4), we have

$$\boldsymbol{I}^{\lambda}_{s,a,\mu}f(z)=\frac{zp'(z)+(\mu-1)p(z)}{\mu}. \tag{2.4}$$

Further computations show that

$$\boldsymbol{I}^{\lambda-1}_{s,a,\mu}f(z)=\frac{z^2p''(z)+(2\mu-1)zp'(z)+(\mu-1)^2p(z)}{\mu^2} \tag{2.5}$$

$$\boldsymbol{I}^{\lambda-2}_{s,a,\mu}f(z)=\frac{z^3p'''(z)+3\mu z^2p''(z)+(3\mu^2-3\mu+1)zp'(z)+(\mu-1)^3p(z)}{\mu^3}. \tag{2.6}$$

and

$$\boldsymbol{I}^{\lambda-3}_{s,a,\mu}f=\frac{z^4p''''(z)+2(2\mu+1)z^3p'''(z)+(6\mu^2+1)z^2p''(z)+(4\mu^3-3\mu^2+\mu-1)zp'(z)+(\mu-1)^4p(z)}{\mu^4}. \tag{2.7}$$

We now define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$u(r,s,t,w,b)=r,\quad v(r,s,t,w,b)=\frac{s+(\mu-1)r}{\mu},$$

$$x(r,s,t,w,b)=\frac{t+(2\mu-1)s+(\mu-1)^2r}{\mu^2},$$

$$y(r,s,t,w,b)=\frac{w+3\mu t+(3\mu^2-3\mu+1)s+(\mu-1)^3r}{\mu^3},$$

$$g(r,s,t,w,b)=\frac{b+2(2\mu+1)w+(6\mu^2+1)t+(4\mu^3-3\mu^2+\mu-1)s+(\mu-1)^2r}{\mu^4}. \tag{2.8}$$

Let

$\psi(r,s,t,w,b;z)=\phi(u,v,x,y,g;z)=$

$$=\phi\left(\begin{array}{c}\frac{s+(\mu-1)r}{\mu},\frac{t+(2\mu-1)s+(\mu-1)^3r}{\mu^2},\frac{w+3\mu t+(3\mu^2-3\mu+1)s+(\mu-1)^3r}{\mu^3},\\ \frac{b+2(2\mu+1)w+(6\mu^2+1)t+(4\mu^3-3\mu^2+\mu-1)s+(\mu-1)^2r}{\mu^4}\end{array};z\right). \tag{2.9}$$

The proof will make use of Theorem(1.7). Using the equations (2.3) to (2.7), we have from (2.9) that

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p''''(z);z)=$$

$$\phi\left(\boldsymbol{I}^{\lambda+1}_{s,a,\mu}f(z),\boldsymbol{I}^{\lambda}_{s,a,\mu}f(z),\boldsymbol{I}^{\lambda-1}_{s,a,\mu}f(z),\boldsymbol{I}^{\lambda-2}_{s,a,\mu}f(z),\boldsymbol{I}^{\lambda-3}_{s,a,\mu}f(z)\right). \tag{2.10}$$

Hence, clearly, (2.2) becomes

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z),z^4p''''(z);z)\in\Omega,$$

we note that

$$\frac{t}{s}+1=\frac{x\mu^2-(\mu-1)^2u}{v\mu-u(\mu-1)}+(2-2\mu)$$

$$\frac{w}{s}=\frac{y\mu^3-3x\mu^3+(2\mu+1)(\mu-1)^2}{v\mu-u(\mu-1)}+(3\mu^2-1),$$

and

$$\frac{b}{s}=\frac{g\mu-(\mu-1)^4+(4\mu^4-2\mu^3)y+(\mu-1)^3(4\mu+2)u+(12\mu^2+6\mu)(x\mu^2-(\mu-1)^2u}{v\mu-u(\mu-1)}-(12\mu^3+6\mu^2+8\mu-2).$$

Therefore, the admissibility condition for $\phi \in B_I[\Omega, q]$ in Definition (2.1) is equivalent to admissibility condition for $\psi \in \mathrm{A}_3[\Omega, q]$ as given in Definition (1.6) with $n=3$.

Therefore, by using (2.1) and Theorem (1.7), we obtain

$$\mathrm{p(z)}=\mathrm{I}_{\mathrm{s,a,\mu}}^{\lambda+1}f(z)\prec q(z).$$

This completes the proof of Theorem (2.2).

Our next corollary is an extension of Theorem (2.2) to the case when the behavior of $q(z)$ *on* ∂U is not known.

Corollary (2.3): Let $\Omega \subset \mathbb{C}$ and let function $q(z)$ be univalent in U *with* $q(0)=0$. Let $\phi \in B_I[\Omega, q]$ for some $p \in (0,1)$, where $q_\rho(z)=q(\rho z)$ .If the function $f \in \Sigma$ and q_ρ satisfies the following conditions:

$$\mathcal{R}\left(\frac{\xi^2 q_\rho'''(\xi)}{q_\rho'(\xi)}\right)\geqq 0 \qquad \left|\frac{\boldsymbol{I}_{s,a,\mu}^{\lambda-1}f(z)}{q_\rho'(\xi)}\right|\leqq k^2, \quad (z\in U; k\geqq 2; \xi\in\partial U/E(q_\rho)), \tag{2.11}$$

and

$$\phi\left(\boldsymbol{I}_{s,a,\mu}^{\lambda+1}f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda}f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-1}f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-2}f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-3}f(z); z\right)\prec h(z), \tag{2.12}$$

then

$$\boldsymbol{I}_{s,a,\mu}^{\lambda+1}f(z)\prec q(z) \qquad (z\in U).$$

Proof: By using Theorem (2.2), yield

$$\boldsymbol{I}_{s,a,\mu}^{\lambda+1}f(z)\prec q_\rho(z) \qquad (z\in U),$$

then, we obtain the result from

$$q_\rho(z)\prec q(z) \qquad (z\in U).$$

This completes the proof of Corollary (1).

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ *of* U onto Ω .In this case , the class $B_I[\mathrm{h(U)}, q]$ is written as $B_I[\mathrm{h}, q]$. The following two results are immediate consequence of Theorem (2.2) and corollary (2.3).

Theorem (2.4): Let $\phi \in B_I[\mathrm{h}, q]$. If the function $f \in \Sigma$ and $q \in \mathbb{Q}_0$ satisfy the following conditions (2.1) and

$$\phi\left(\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-2} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-3} f(z); z\right) \prec h(z), \tag{2.13}$$

then

$$\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z) \prec q(z) \qquad (z \in U).$$

Corollary (2.5): Let $\Omega \subset \mathbb{C}$ and let function q be univalent in U *with* $q(0) = 0$. Also Let $\phi \in B_I\left[\mathrm{h}, q_p\right]$ for some $p \in (0,1)$, where $q_\rho(z) = q(\rho z)$ .If the function $f \in \Sigma$ and q_ρ satisfies the conditions (2.11), and

$$\phi\left(\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-2} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-3} f(z); z\right) \prec h(z), \tag{2.14}$$

then

$$\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z) \prec q(z) \qquad (z \in U).$$

The following result yield the best dominant of differential subordination (2.12).

Theorem (2.6): Let the function h be univalent in U. Also let $\phi: \mathbb{C}^5 \times U \to \mathbb{C}$ and Suppose that following differential equation:

$$\phi\begin{pmatrix} q(z), \dfrac{zq'(z) + (\mu-1)q(z)}{\mu}, \dfrac{z^2q''(z) + (2\mu-1)zq'(z) + (\mu-1)^2q(z)}{\mu^2}, \\ \dfrac{z^3q'''(z) + 3\mu z^2q''(z) + (3\mu^2-3\mu+1)zq'(z) + (\mu-1)^3q(z)}{\mu^3}, \\ \dfrac{z^4q''''(z) + 2(2\mu+1)z^3q'''(z) + (6\mu^2+1)z^2q''(z) + (4\mu^4-3\mu^2+\mu-1)zq'(z) + (\mu-1)^4q(z)}{\mu^4}; z \end{pmatrix} = h(z), \tag{2.15}$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies the condition (2.1). If $f \in \Sigma$ satisfies the condition (2.12) and if

$$\phi\left(\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-1} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-2} f(z), \boldsymbol{I}_{s,a,\mu}^{\lambda-3} f(z); z\right),$$

is analytic in U, then

$$\boldsymbol{I}_{s,a,\mu}^{\lambda+1} f(z) \prec \mathrm{q}(z) \qquad (z \in U)$$

and $q(z)$ is the best dominant.

Proof: Using Theorem(2.2), that $q(z)$ is a dominant of (2.12). Since $q(z)$ satisfies (2.14), it is also a solution of (2.12). Therefore, $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

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