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Some Properties Related With $L^0(\Omega, F, \mu)$ Space

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ABSTRACT

In this paper we introduce the metric function on $L^0(\Omega, F, \mu)$ be the space of measurable functions defined on measure space (Ω, F, μ) .We prove the converge in this metric implies converge in measure ,but the converse not true only if μ is finite and prove $L^0(\Omega, F, \mu)$ with respect metric functions defined by $d(f, g) = \int_{\Omega} \frac{|f-g|}{1+|f-g|} d\mu$ for all $f, g \in L^0(\Omega, F, \mu)$ is complete metric space and a topological linear space and F-space .

1. Introduction

Metric space is the main topics in mathematical analysis , especially the topic of convergence and characteristics related to this topic and measure theory also from the main branches of mathematical analysis ,especially convergence in measurable functions and study of the relation between convergences and many sources touched on this topic and in particular Eugene in[3] (1975) where he defined a metric function on a space random variables defined on the probability space and showed that each sequence of random variables is converge to a random variable in relation to the . And Jordan was proved in [5] 2015 $L^0(\Omega, F, \mu)$ is a complete metric space . In this paper we proved $L^0(\Omega, F, \mu)$ is a complete metric space by using another metric function and proved $L^0(\Omega, F, \mu)$ is linear topological metric space and F-space .

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2.Fundamental concepts

In this section we introduce some basic concepts that we need such as measure, converges in measure ,types of convergence and relation between them.

Definition(2.1):[1] A nonempty family F of subsets of a set Ω is called a field (or algebra) on Ω if,
1.A$\in F$,then $A^C \in F$
2.$A_1 , A_2, \dots , A_n \in F$,then $\cup_{i=1}^{n} A_n \in F$
In other words,
A field is a nonempty family closed under the formation of complements and union .If (2) is replaced by the closed under countable union ,that is ,
3.If $A_n \in F$, n=1,2,…,then$\cup_{n=1}^{\infty} A_n \in F$
F is called a σ –field (σ-algebra) on a set Ω .i.e. a σ-field on a set Ω is afield which closed under countable union .A measurable space is a pair (Ω, F), where Ω is a set and F is σ-field on Ω. A subset A of Ω is called measurable (measurable with respect to the σ-field F) if A $\in$ F, i.e. any member of F is called a measurable set.

Definition(2.2):[6] Let (Ω, F) is measurable space .A set function $\mu: F \to R$ is said to be
1.σ- additive (sometimes called completely additive ,or Countable additive) if $\mu(\cup_{n=1}^{\infty} A_K) = \sum_{n=1}^{\infty} \mu(A_n)$,
Whenever $\{A_n\}$ is a sequence of disjoint sets in F .
2.Measure, if μ is a σ- additive and μ(A)≥ 0 for all $A \in F$.
A measure space is a triple (Ω, F, μ) where (Ω, F) is a measurable space ,and μ is measure on F .if μ is a probability measure on F ,(Ω, F, μ) is a probability space .

Definition(2.3):[9] Let $F(\Omega)$ be the family of all real valued functions defined on a set Ω .Let $f, f_n \in F(\Omega)$, $n \in N$ and $A \in \Omega$. We say that
1.$\{f_n\}$ converges pointwise to f on A ,if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ .We $\lim_{n\to\infty} f_n(x) = f(x)$ or $f_n \to f$ on A
2.$\{f_n\}$ uniformly convergent to f on A, if for every $\varepsilon > 0$ there is $k \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and all $x \in A$ .We write $f_n \overset{u}{\to} f$ on A .It is clear that every uniformly convergent sequence is pointwise convergent, but the converse is not true .

Definition (2.4):[3] Let (Ω, F, μ) be a measure space and $f, f_n \in \mu(\Omega)$, n$\in N$. We say that $\{f_n\}$
1.Converges almost everywhere to f ,denoted by $f_n \overset{a.e.}{\longrightarrow} f$,if there is a subset $A \in \Omega$ such that $\mu(A) = 0$ and $f_n \to f$ on A^c .
2. Converges almost uniformly to f ,denoted by ,$f_n \overset{a.u.}{\longrightarrow} f$,if for each $\varepsilon > 0$, there is a subset $A \subset \Omega$ such that $\mu(A) < \varepsilon$ and $f_n \overset{u}{\to} f$ on A^c
3.Converges in measure to f, denoted by $f_{n}\overset{\mu}{\to} f$, $\lim_{n\to\infty} \mu(\{X \in \Omega: |f_n(x)\_f(x)| \geq \varepsilon\}) = 0$, for each $\varepsilon > 0$.

Definition(2.5):[9] Let (Ω, F, μ) be a measure space and $f, f_n \in \mu(\Omega)$, $n \in N$. For $\delta > 0$, define $\beta_n(\delta) = \{x \in \Omega: |f_n(x) - f(x)| \geq \delta\}$, $n \in N$, and $\beta_\delta = \lim_{n\to\infty} sup\, \beta_n(\delta)$ since each $f_n - f$ is measurable function ,the sets $\beta_n(\delta)$ are measurable.

Theorem(2.6):[3] Let (Ω, F, μ) be a measure space $f, f_n \in \mu(\Omega)$, $n \in N$. Then
1.If $f_n \overset{a.u.}{\longrightarrow} f$,then $f_n \overset{\mu}{\to} f$ and $f_n \overset{a.e.}{\longrightarrow} f$

2.If μ is finite , then $f_n \xrightarrow{a.e.} f$ iff $\lim_{n\to\infty} \mu(\cup_{n=k}^{\infty} \beta_\delta)) = 0$ every $\delta > 0$

3.If μ is finite and $f_n \xrightarrow{a.e.} f$,then $f_n \xrightarrow{a.u.} f$

4.If μ is finite and $f_n \xrightarrow{a.e.} f$, then $f_n \xrightarrow{\mu} f$

3.Main result

In this section we prove that if a sequence in $L^0(\Omega, F, \mu)$ converge in metric implies to converge in measure ,but the converse is not true if and only if μ is finite and prove some properties of the metric space $L^0(\Omega, F, \mu)$ In this section we prove that if a sequence in $L^0(\Omega, F, \mu)$ converge in metric implies to converge in measure ,but the converse is not true only if μ is finite and prove some properties of the metric space $L^0(\Omega, F, \mu)$

Let $L^0(\Omega, F, \mu)$ be the space of measurable functions defined on measure space (Ω, F, μ) . Then $L^0(\Omega$,F,$\mu)$ is a linear space under the following addition and scalar multiplication

1.$(f + g)(x) = f(x) + g(x)$ for all $f, g \in L^0(\Omega, \mathcal{F}, \mu)$

2.$(\lambda f)(x) = \lambda(x)$ for all $f \in$ and $f \in L^0(\Omega, \mathcal{F}, \mu)$ or $\lambda \in R$.

Theorem (3.1)

Let (Ω, F, μ) be a finite measure space .$L^0(\Omega, F, \mu)$ be the collection of equivalent classes of measurable functions defined on (Ω, F, μ) where two f and g are equivalent when $\mu\{x \in \Omega : f(x) \neq g(x)\} = 0$.We define d:$L^0(\Omega, F, \mu) \times L^0(\Omega, F, \mu) \to \mathbb{R}$ by d(f ,g)= $\int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$ for all $f, g \in L^0(\Omega, F, \mu)$.Then d is a metric function on $L^0(\Omega, F, \mu)$. We call the topology induced by d the topology of convergence in measure .

Proof .1.Since $|f - g| \geq 0$ for all $f, g \in L^0(\Omega, F, \mu)$,then $\frac{|f-g|}{1+|f-g|} \geq 0$ for all $f, g \in L^0(\Omega, F, \mu)$
$\Rightarrow \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu \Rightarrow d(f, g) \geq 0$

2.Let $f, g \in L^0(\Omega, F, \mu)$,then $d(f, g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu = \int_\Omega \frac{|g-f|}{1+|g-f|} d\mu$

- If f and g are equivalent ,then $\mu\{x \in \Omega: f(x) \neq g(x)\} = 0$ .so that $f = g \ a.e., i.e. f - g = 0 \ a.e. \Longrightarrow \frac{|f-g|}{1+|f-g|} = 0 \ a.e. \Longrightarrow \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu = d(f, g) = 0$
- If $d(f , g) = 0$ then $\int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$, since $\frac{|f-g|}{1+|f-g|} \geq 0 \Longrightarrow \frac{|f-g|}{1+|f-g|} = 0 \ a.e. \Longrightarrow f - g = 0 \ a.e. \Longrightarrow f = g \ a.e.$

4.Let $f, g, h \in L^0(\Omega, F, \mu)$

Since $\frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|} \geq \frac{|f-g|}{1+|f-g|+|g-h|} + \frac{|g-h|}{1+|f-g|+|g-h|} = \frac{|f-g|+|g-h|}{1+|f-g|+|g-h|} \Longrightarrow \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|} \geq \frac{1}{\frac{1}{|f-g|}+1} \geq \frac{1}{\frac{1}{|f-h|}+1} = \frac{|f-g|}{1+|f-h|} \Longrightarrow \frac{|f-h|}{1+|f-h|} \leq \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|} \Longrightarrow \int_\Omega \frac{|f-h|}{1+|f-h|} d\mu \leq \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu + \int_\Omega \frac{|g-h|}{1+|g-h|} d\mu \Longrightarrow d(f , h) \leq d(f, g) + d(g - h)$.This show that $\int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$ is metric on $L^0(\Omega, F, \mu)$ .We now prove that if a sequence in $L^0(\Omega, F, \mu)$ converges in metric then it converges in measure and convers is true if and only if μ is finite.

Theorem(3.2) : Let $\{f_n\}$ be a sequence in $L^0(\Omega, F, \mu)$,

1.if d$(f_n, f) \to 0$ then $f_n \xrightarrow{\mu} f$.

2.If μ is finite and $f_n \overset{\mu}{\to} f$, then $d(f_n, f) \to 0$

Proof 1.Let $\varepsilon > 0$.For each n, let $A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\}$. To prove $\lim_{n\to\infty} \mu(A) = 0$ $A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon\} = \left\{x \in \Omega : \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} \geq \frac{\varepsilon}{1+\varepsilon}\right\}$, since $\frac{\varepsilon}{1+\varepsilon} I_{A_n} \leq \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}$,we have $\int_\Omega \frac{\varepsilon}{1+\varepsilon} I_{A_n} d\mu \leq \int_\Omega \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} d\mu$
$i.e.\ \frac{\varepsilon}{1+\varepsilon}\mu(A_n) \leq d(f_n, f) \Rightarrow \mu(A_n) \leq \frac{\mathbf{1+\varepsilon}}{\boldsymbol{\varepsilon}} d(f_n, f)$ Since $d(f_n, f) \to 0$, then $\mu(A_n) \to 0$, i.e. $\lim_{n\to\infty} \mu(A_n) = 0$
2.Let $\varepsilon > 0$ since μ is finite ,then $\mu(\Omega) < \infty$.Take $\delta = \frac{\varepsilon}{1+\mu(\Omega)}$, then For each n ,let $A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \delta\}$. $A_n = \{x \in \Omega : |f_n(x) - f(x)| \geq \delta\} = \left\{x \in \Omega : \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} \geq \frac{\delta}{1+\delta}\right\}$ .Since $\mu(A_n) \to 0$ as $n \to \infty$,there exists $k \in \mathbb{Z}^+$ such that $\mu(A_n) < \delta$ for all $n \geq k$,d$(f_n, f) = \int_\Omega \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} d\mu = \int_{A_n} \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} d\mu + \int_{A_n^c} \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} d\mu \Rightarrow$d$(f_n, f) \leq \int_\Omega d\mu + \int_{A_n^c} \frac{\delta}{1+\delta} d\mu = \mu(A_n) + \frac{\delta}{1+\delta}\mu(A_n^c) = \frac{1}{1+\delta}(\mu(A_n) + \delta(\mu(A_n) + (\mu(A_n^c))) = \frac{1}{1+\delta}\Big(\mu(A_n) + \delta\big(\mu(A_n \cup A_n^c)\big)\Big) = \frac{1}{1+\delta}(\mu(A_n) + \delta\mu(\Omega)$ Since $\mu(A_n) < \delta$ for all $n \geq k$, we have $d(f_n, f) < \frac{1}{1+\delta}\big(\delta + \delta\mu(\Omega)\big) = \frac{\delta}{1+\delta}\big(1 + \mu(\Omega)\big) < \big(1 + \mu(\Omega)\big)\delta = \varepsilon$. Hence d$(f_n, f) \to 0$.

Corollary (3.3) :[5] Let $\{f_n\}$ be a sequence in $L^0(\Omega, F, \mu)$. If μ is finite and $f_n \overset{a.e.}{\longrightarrow} f$, then $d(f_n, f) \to 0$

Proof: Since μ is finite and $f_n \overset{a.e.}{\longrightarrow} f$,then by Egro$v$'es theorem we have $f_n \overset{\mu}{\to} f$,by theorem (3.2) ,we have $d(f_n, f) \to 0$. We now to prove the metric space $L^0(\Omega, \mathrm{F}, \mu)$ is complete.

Theorem(3.4) : The metric space $L^0(\Omega, F, \mu)$ is complete .
Proof: Suppose that f_n is a Cauchy sequence in $L^0(\Omega, F, \mu)$.
Let $\varepsilon > 0$,for each n, let $A_n = \{x \in \Omega : f_n(x) - f(x) \geq \varepsilon\}$ to prove $\lim_{n\to\infty} \mu(A_n) = 0$,$A_{n,m} = \{x \in \Omega : |f_n(x) - f_m(x)| = A_n = \{x \in \Omega : \frac{|f_n(x)-f_m(x)|}{1+|f_n(x)-f_m(x)|} \geq \frac{\varepsilon}{1+\varepsilon}\}$
Since $\frac{\varepsilon}{1+\varepsilon} I_{A_{n,m}} \leq \frac{|f_n(x)-f_m(x)|}{1+|f_n(x)-f_m(x)|}$,we have $\int_\Omega \frac{\varepsilon}{1+\varepsilon} I_{A_{n,m}} \leq \int_\Omega \frac{|f_n(x)-f_m(x)|}{1+|f_n(x)-f_m(x)|} d\mu$
i.e. $\frac{\varepsilon}{1+\varepsilon}\mu\big(A_{n,m}\big) \leq d(f_n, f_m) \Rightarrow \mu\big(A_{n,m}\big) \leq \frac{1+\varepsilon}{\varepsilon} d(f_n, f_m)$. Since d$(f_n, f_m) \to 0$, then $\mu(A_{n,m}) \to 0$ as $n, m \to \infty$, i.e. $\lim_{n\to\infty} \mu\big(A_{n,m}\big) = 0$ .Hence $\{f_n\}$ is a Cauchy in measure , so that there is $f, f_n \in L^0(\Omega, F, \mu)$ such that $f_n \overset{\mu}{\to} f$ since μ is finite ,then $d(f_n, f) \to 0$ as n$\to \infty$ .then $L^0(\Omega, F, \mu)$ is complete . We now to prove the metric space $L^0(\Omega, F, \mu)$ is a topological linear space.

Theorem (3.5) :$L^0(\Omega, F, \mu)$ is a topological linear space
Proof: let $X = L_0(\Omega, F, \mu)$ is a metric space ,then X is topological space . Let $f, g \in X$, then f and g are measurable function for all $\alpha, \beta \in \mathbb{R}$,hence X is a linear space over $\mathbb{R}$.
Define u:$X \times X \to X$ by $u(f, g)(x) = f(x) + g(x)$ .To prove u is continuous .
Let $f_n, g_n, f, g \in X$ such that $d(f_n, f) \to 0$ as $n \to \infty$ and $d(g_n, g) \to 0$ and $n \to \infty$ .Then $f_n \to f$ and $g_n \to g$,so that $f_n + g_n \to f + g$ .Since μ is finite by (1) of theorem (3.2) , then $d(f_n + g_n, f + g) \to 0\ \ as\ n \to \infty$,i.e.$u(f_n, g_n) \to u(f\ , g)$.
Define v:$\mathbb{R} \times X \to$ by $v(\lambda f)(x) = \lambda f(x)$ .To prove v is continuous .

Let f , $f_n \in L^0(\Omega, F, \mu)$ and $\lambda_n, \lambda \in \mathbb{R}$ such that $d(f_n, f) \to 0$ $as\ n \to \infty$ and $|\lambda_n - \lambda| \to 0$ $as\ n \to \infty$ .Then $\lambda_n f_n \overset{\mu}{\to} \lambda f$,since μ is finite ,then $d(\lambda_n f_n, \lambda f) \to 0$ $as\ n \to \infty$,i.e. $v(f_n, \lambda_n) \to v(f, \lambda)$. we now to prove the metric space $L^0(\Omega, F, \mu)$ is F-space.

Theorem(3.6) : $L^0(\Omega, F, \mu)$ is F-space.

Proof: Since $L^0(\Omega, F, \mu)$ is complete metric space we need to show that d is invariant .

Let $f, g, h \in L_0(\Omega, F, \mu)$ $d(f + h, g + h) =$
$\int_\Omega \frac{|f+h-(g+h)|}{1+|f+h-(h+g)|} d\mu = \int_\Omega \frac{|f+h-g-h|}{1+|f+h-g-h|} d\mu = \int_\Omega \frac{|f-g|}{1+|f-g|} = d(f, g)$ .then $L^0(\Omega, F, \mu)$ is F-space.

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