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Separation Axioms by Using *gh*- Closed Sets

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Abstract

Gh- separation axioms is introduced in our work via h-open sets, where we discuss some properties and several characterizations of this class. Additionally, we discuss the relationship between the *gh*-separation axioms and provide numerous examples. On the other hand, we have get that separation axioms give *gh*- separation axioms, we demonstrate that the contrary may not be true by using examples. Also, the notion of *gh*- Regular space, *gh*-Normal space, *gh*-Completely Regular space, *gh*-Completely Normal space and *gh*- Perfectly Normal space is introduced and some properties are proved. Finely, we defined T_{gh}- space and we study the relation between T_{gh} and $T_{1/2}, T_h$. .

Introduction:

The first step of generalized closed sets introduced by Levine [1]. After that there is a vast progress occurred in the field of generalized open sets (compliment of respective closed sets) which became the base for separation axioms in the respective context. Using the idea of pre-open sets, Fatima, M. Mohammad proposed pre-Techonov and Pre-Hausdorff Separation Axioms in Intuitionistic Fuzzy Special Topological Spaces [2] in 2006. In 2012 Sabih W. Askandar [3] defined another type of separation axioms depend on an *i* - open sets. In 2020 F. Abbas [4] used *h* - open sets to produce the concept of *h*-separation axioms. In this paper, we introduce the generalized forms of *h*- separation axioms using the concepts of generalized *h*-open sets called *gh* - T_k (briefly denoted by T_{kgh}) spaces, k= 0, 1, 2, 3, 4, 5, 6. Among other things, the concern basic properties and relative preservation properties of these spaces are projected under *gh*- irresolute and *gh*-continuous mapping [5]. We present our work in three sections. In section 1, the gh- separation axioms and lots of examples have been described and provided, and the relationship with other classes of separation axioms have been investigated. In section 2, in order to discuss the gh- separation axioms' property, established a number of significant theorems have been established. Finally, in the third section, some fundamental properties of *gh*-Regular space, *gh*-Normal space, *gh*-Completely Regular space, *gh*-Completely Normal space and *gh*- Perfectly Normal space have been presented. χ and γ, respectively, have been used to indicate the topological spaces (χ, τ) and (γ, σ). Topological spaces by TS, (os), (cs), and open sets (vs. closed sets), the following definitions and notations have been recalled. Throughout this paper (χ, τ^{gh}) a topological space is always, (where τ^{gh} is family of all *gh*-open set) [5] of χ.

Definition1.1. A subset A of a spaces (χ,τ) is defined

1. (h-os)[4] stands for (h-open set),if for every non empty set U in χ , $U \neq \chi$ and $U \in \tau$, s.t. $A \subseteq int(A \cup U)$. The complement of (*h- os*) is said *h*-closed set denoted by (*h-cs*).
2. generalized h-closed [5] (briefly, *gh*-closed) denoted by (*gh-cs*) set, if $CL_h(A) \subseteq U$ whenever $A \subseteq U$ and U is (*os*) in (χ,τ). The complement of *gh*- closed set is said *gh*-open set denoted by (*gh-os*).
3. $ho(\chi)$, $hc(\chi)$ and $ghc(\chi)$ are family of *h*-open, *h*-closed and g*h*-closed sets respectively.

Definition1.2. Let χ be a *TS* and let A subset of χ. The intersection of all (*gh-cs*) containing A is named *gh*-closure of *A*, denoted by $CL_{gh}(A)$.[5]

Definition1.3. A mapping $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ is named

1. Continuous designated by *(contm)*[1], if $f^{-1}(F)$ is (*cs*) in χ. $\forall F \in (cs)$ in γ.
2. *h*-continuous designated by (*h-contm*) [4], if $f^{-1}(F)$ is (*h*-cs) in χ.$\forall F \in (cs)$ in γ.
3. *gh*-continuous designated by (*gh-contm*) [5], if $f^{-1}(F)$ is (*gh*-cs) in χ.$\forall F \in (cs)$ in γ.
4. *gh*-irresolute designated by (*gh-irrem*)[5],if $f^{-1}(F)$is (*gh*-cs) set in χ. $\forall F \in (gh-cs)$ in γ.

Theorem 1.4.

1. *Each (cs) in TS is (g-cs) [1].*
2. *Each (cs) in TS is (h-cs) [4].*
3. *Each (cs) in TS is (gh-cs)[5].*

Definition1.5. A *TS* (χ,τ) is said to be:

(1) $T_{0gh}-$space: if a,b are two distinct points in χ, $\exists\, U \in$ (*gh-os*) s.t. either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$.[5]
(2) $T_{1gh}-$space: if $a,b \in \chi$ and $a \neq b$, $\exists\, U,V \in$ (*gh-os*) containing a,b respectively, s.t. $b \notin U$, and $a \notin V$.[5]
(3) $T_{2gh}-$space: if $a,b \in \chi$ and $a \neq b$, $\exists\, disjoint\ U,V \in$ (*gh-os*) containing a,b respectively. [5]

2. Some Separation Axioms by Using *gh-Closed Set*.

In this chapter, we introduce and study the notion of *(gh-cs)* in *TS* and obtain some of its basic properties.

Definition2.1. "A *TS* (χ,τ) is said to be:

1- *gh*-Regular space(shortly R_{gh}-space) if for every *(gh- cs) F* and each point *x* of χ which is not in *F*, there exists disjoint *(gh-os) U* and *V* s.t. $x \in U$, $F \subset V$.
2- g*h*-Normal space (shortly N_{gh}-space) if for every pair of disjoint (*gh-cs*) F_1 and F_2 in χ,there exists disjoint *(gh-os) U* and *V* s.t. $F_1 \subset U$, $F_2 \subset V$.
3- g*h*- CN_{gh}-space, also known as completely normal space, if it satisfies *gh* -Titus axiom: If $A_1 \subseteq \chi, A_2 \subseteq \chi, A_1 \cap A_2 = \emptyset\ \exists\, I_1, I_2 \subseteq \chi\ such\ that\ A_1 \subseteq I_1, A_2 \subseteq I_2\ where\ I_1 \cap I_2 = \emptyset$, I_1, I_2 are (*gh-os*).
4- g*h*- Completely Regular space (shortly CR_{gh}-space) if the following axiom is true: If F is (*gh-cs*) in χ and $x \in \chi$, $x \notin F$ there exists an (*gh-contm*) [5] $f\colon \chi \longrightarrow [0,1]$ s.t. $f(F) = 1$, $f(x) = 0$.

5- gh- Perfectly Normal space (shortly PN_{gh}-space) if it satisfies the following axiom: If C_1 *and* C_2 are disjoint (*gh-cs*), there exists (*gh-contm*) [5] $f: \chi \longrightarrow [0,1]$ s.t. $f^{-1}(\{0\}) = C_1$and $f^{-1}(\{1\}) = C_2$.

6- $T_{1/2}$-space [6] if each (*g-cs*) in it is (*cs*)

Definition2.2. A $T_{1gh} -$ space is named $T_{3gh} -$ space if its R_{gh} space.

Definition2.3. A $T_{1gh} -$ space is named $T_{4gh} -$ space if its N_{gh} space.

Definition2.4. A $T_{1gh} -$ space is named $T_{5gh} -$ space if its CN_{gh} space.

Definition2.5. A $T_{1gh} -$ space is named $T_{6gh} -$ space if its PN_{gh} space.

Definition2.6. A $T_{1gh} -$ space is named $T_{(3\frac{1}{2})gh} -$ space if its CR_{gh} space.

Example2.6. Let χ ={1, 2} and $\tau = \{\emptyset, \chi, \{1\}, \{2\}\}$, $ghc(\chi) = \tau$ then

$ho(\chi) = \{\emptyset, \chi, \{1\}, \{2\}\} = hc(\chi) = ghc(\chi)$

1. $1, 2 \in \chi (1 \neq 2) \exists \{1\}, \{2\} \in \tau^{gh}$s.t. 1$\in \{1\}, 2 \in \{2\}$.Therefore; *$(\chi, \tau)$* is T_{1gh}.
2. $1, 2 \in \chi\ (1 \neq 2) \exists \{1\}, \{2\} \in \tau^{gh}$ s.t. 1$\in \{1\}, 2 \in \{2\}$, $\{1\} \cap \{2\} = \emptyset$. Therefore; *(χ, τ)* is T_{2gh}.
3. $\{2\}$ is (*gh*-cs) and 1$\notin \{2\}$ there is two (*gh*-os){1},{2} s.t. $1 \in \{1\}, \{2\} \subseteq \{2\}$. Therefore; *$(\chi, \tau)$* is R_{gh} space.
4. By (1) and (3) we have : *(χ, τ)* is T_{3gh}
5. {1}, {2} are (*gh-cs*) there are two (*gh*-os) {1}, {2} s.t. $\{1\} \subseteq \{1\}, \{2\} \subseteq \{2\}, \{1\} \cap \{2\} = \emptyset$. Therefore; *$(\chi, \tau)$* is N_{gh} space.
6. By (1) and (5) we have : *(χ, τ)* is T_{4gh}
7. $\{1\}, \{2\} \subseteq \chi$, there are two (*gh*-os){1}, {2} s.t. $\{1\} \subseteq \{1\}, \{2\} \subseteq \{2\}\, where \{1\} \cap \{2\} = \emptyset$. Therefore; *$(\chi, \tau)$* is CN_{gh} space.
8. By (1) and (7) we have : *(χ, τ)* is T_{5gh}
9. Let $f: \chi \longrightarrow [0,1]$ be (*gh-contm*) and {1}, {2} are disjoint (*gh-cs*) s.t. $f^{-1}(\{0\}) = \{1\}$, $f^{-1}(\{1\}) = \{2\}$. Therefore; *(χ, τ)* is PN_{gh} space.
10. By (1) and (9) we have : *(χ, τ)* is T_{6gh}

Theorem 2.7. *A TS (χ, τ) is T_{0gh} $-$space iff for each pair of distinct points x, y of χ, $cl_{gh}(\{x\}) \neq cl_{gh}(\{y\})$.*

Proof: Suppose χ be a T_{0gh} $-$space, and $x, y \in \chi$ s.t. $x \neq y$, then there exists (*gh-os*) U containing one of the points but not the other, then $x \in U$ and $y \notin U$. Then $\chi \backslash U$ is $(gh-$cs) containing y but not x. But $cl_{gh}(\{y\})$ is the smallest ($gh-$cs) containing y. Therefore;$cl_{gh}(\{y\}) \subset \chi \backslash U$ and hence $x \notin cl_{gh}(\{y\})$. Thus $cl_{gh}(\{y\}) \neq cl_{gh}(\{x\})$.

Conversely, suppose $x, y \in \chi$, $x \neq y$ and $cl_{gh}(\{y\}) \neq cl_{gh}(\{x\})$. Let $z \in \chi$ s.t. $z \in cl_{gh}(\{x\})$ but $z \notin cl_{gh}(\{y\})$. If $x \in cl_{gh}(\{y\})$ then $cl_{gh}(\{x\}) \subset cl_{gh}(\{y\})$ and hence $z \in cl_{gh}(\{y\})$. This is a contradiction. Therefore; $x \notin cl_{gh}(\{y\})$. That is $x \in \chi \backslash cl_{gh}(\{y\})$. Therefore; $X \backslash cl_{gh}(\{y\})$ is ($gh -$ os) containing x but not y. Hence (χ, τ) is T_{0gh} $-$space.■

Theorem 2.8. *A TS (χ,τ) is T_{1gh} $-$space iff for every $k \in \chi$ singleton $\{k\}$ is $(gh-cs)$ in χ.*

Proof: Assume that (χ,τ) be T_{1gh} $-$space and let $k \in \chi$, to prove that $\{k\}$ is ($gh-$cs), we will prove $\chi\backslash\{k\}$ is ($gh-$os) in χ. Let $y \in \chi\backslash\{k\}$, implies $k \neq y$ and since χ is T_{1gh} $-$space then, their exists two (g$h-$ os) U,V s.t. $k \notin U$, $y \in V \subset \chi\backslash\{k\}$. Since $y \in V \subset \chi\backslash\{k\}$ then $\chi\backslash\{k\}$ is ($gh-$os). Hence $\{k\}$ is ($gh-$cs).

Conversely, let $k \neq y \in \chi$, then $\{k\},\{y\}$ are ($gh-$cs). That is $\chi\backslash\{k\}$ is ($gh-$os), clearly, $k \notin \chi\backslash\{k\}$ and $y \in \chi\backslash\{k\}$. Similarly, $\chi\backslash\{y\}$ is ($gh-$os), $y \notin \chi\backslash\{y\}$ and $k \in \chi\backslash\{y\}$. Hence (χ,τ) is T_{1gh} $-$space. ■

Theorem 2.9.*Aspace (χ,τ) is T_{2gh} $-$space iff (χ,τ^{gh}) is Hausdorff -space.*

Proof: Assumes n, $m\in \chi$ with $n\neq m$. Since χ is T_{2gh} $-$space, there exists disjoint (*gh-os*) H and K in χ s.t. $n \in H$ and $m \in K$, $H\cap K=\emptyset$. Here H, $K\in \tau^{gh}$, so, obviously (χ,τ^{gh}) ceases to be a T_{2gh} $-$space i.e. a Hausdorff space.

Conversely, whenever (χ,τ^{gh}) is a T_{2gh} $-$space , there exists a pair of members of τ^{gh}, say, P & Q for a pair of distinct points p & q of χ such that $p \in P$ & $q \in Q$ & $P \cap Q = \emptyset$. But $gho(\chi,\tau)= \tau^{gh}$. Combing all these facts (χ, τ) is T_{2gh} $-$space. ■

Theorem 2.10. *Each open subspace of a T_{2gh} $-$space is T_{2gh} .*

Proof: Suppose U be an open subspace of a T_{2gh} $-$space (χ,τ). Let k and p represent any two separate points on U. Since χ is T_{2gh} $-$space and $U \subset \chi$, there exists two disjoint (*gh-os*) G and H in χ such that $k \in G$ & $p \in H$. Let $A = U \cap G$ & $B = U \cap H$. Then A & B are (*gh-os*) in U containing k and p. Also, $A \cap B = \emptyset$. Hence (U,T_u) is T_{2gh}. ■

Theorem 2.11. *Each subspace of* $R_{gh}-$ space *is* R_{gh} .

Proof: let (χ,τ) be an $R_{gh}-$ space and A a subset, $k \in A$ and C is (cs) in A, "now k is a point in χ, D is an (g$h-$ cs) subset of χ s.t. $D \cap A = C$. Such D exists by the way that the subspace topology is defined. Clearly, whatever D is picked up for the purpose, k cannot lie in D because the only points in $D \cap A$ are in a set not containing k. Since χ is R_{gh} , we can find (*gh-os*) U and V in χ s.t. $k \in U$, $C \subseteq V$ and U and V are disjoint. Now, $U \cap A$ and $V \cap A$ are disjoint ($gh-$os) subsets of A, with $k \in U \cap A$ and $C \subseteq V \cap A$". ■

Theorem 2.12.*Let (χ,τ) be TS then the following statements are equivalent.*

1- *χ is $T_{2gh} - space$.*
2- *Let $k \in \chi$.For each $k\neq p$, there exists(gh-os) U containing k s.t. $p\notin cl_{gh}(U)$.*
3- *For each $k \in \chi$ $\cap\{cl_{gh}(U)$: $U\in \tau^{gh}$ & $k \in U\}=\{k\}$*

Proof:(1)→(2): Assumes (χ,τ) is $T_{2gh}-$ space, there exists disjoint (*gh-os*) U and G containing k and p respectively. So, $U \subset \chi\backslash G$. Therefore; $cl_{gh}(U) \subset \chi\backslash G$. So $p\notin cl_{gh}(U)$.

(2) →(3): If possible for some $k\neq p$, we have $p\in cl_{gh}(U)$ for every (g*h-os*) *U containing k*, which is contradiction (2).

(3) →(1): Suppose *k*, $p \in \chi$ **&** $k\neq p$. Then there exists *(gh-os) U containing k s.t.* $p \notin cl_{gh}(U)$. Let $V= \chi\backslash cl_{gh}(U)$, then $p\in V$ and $k\in U$ and also $U \cap V=\emptyset$.■

3. The Relation Ships among *gh*-Separation Axioms.

In this part, we compared among T_{0gh}, T_{1gh}, T_{2gh}, T_{3gh}, $T_{(3\frac{1}{2})gh}$, T_{4gh}, T_{5gh} and T_{6gh}- space. Also we obtain some of its basic properties.

Theorem 3.1. Each $T_{2gh} -$ space is T_{1gh} and also is T_{0gh}.[5]

Theorem 3.2. *Each Regular space is* R_{gh} .

Proof: Assume that (χ, τ) be a regular space. Let $k \in \chi$ and F be any (cs) on χ, s.t. $k \notin F$, and let U, V be any (os) in χ , s.t. $U \cap V = \emptyset$. Form Theorem (1.4) (3) each (cs) is (gh-cs). Then F is (gh-cs) & $k \notin F$. Each *(os)* being (gh-os), U and V are then (gh-os). Hence (χ, τ) is R_{gh} . ■

"The converse of theorem 3.2 is not true in general as shown in the following example".

If $\chi = \{1, 2, 3\}$, $\tau = \{\emptyset, \chi, \{1\}, \{1, 3\}$.

Then χ is R_{gh} space but not Regular space.

Theorem 3.3. *Each Normal space is* $N_{gh} -$ *space* .

Proof: Suppose that (χ, τ) be Normal space and let F_1, F_2 be disjoint (cs) in χ and U , V are disjoint (os) in χ , s.t. $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \emptyset$. Form Theorem (1.4) each (cs) is (gh-cs) and every (os) is (gh-os) in χ. Then F_1, F_2 are (gh-cs) and U, V are (gh-os). Hence (χ, τ) is N_{gh} . ■

The converse of theorem 3.3 is not true in general as shown in the following example.

If $\chi = \{a, b, c\}$, $\tau = \{\emptyset, \chi, \{b\}, \{a, b\}, \{b, c\}\}$.

Then χ is N_{gh} space but not Normal space.

Theorem 3.4. Each T_{3gh}- space is T_{2gh}-space.

Proof: Suppose (χ, τ) be T_{3gh}- space and let n, $m \in \chi$ s.t. $n \neq m$. Since χ be T_{1gh}-space, then $\{n\}$ is (gh-cs), since $n \neq m$, then $m \notin \{n\}$. Now, since χ be gh- Regular space, there exists disjoint *(gh-os)* U and V s.t. $m \in U$, $\{n\} \subset V$ and $U \cap V = \emptyset$. Therefore; (χ, τ) is T_{2gh}-space. ■

"The converse of theorem 3.4 is not true in general as shown in the following example".

Example 3.5. If $\chi = \mathbb{R}$, $\boldsymbol{\tau = \{(-P, P) \backslash \left\{\frac{1}{n} : n \epsilon N\right\} : P > 0\}}$

Then (χ, τ) is $T_{2gh} -$ space but not T_{3gh}.

Theorem 3.6. *Each* $T_{(3\frac{1}{2})gh}$ *space is* T_{3gh}.

Theorem 3.7. *Each* T_{4gh}- *space is* $T_{(3\frac{1}{2})gh}$.

Proof: A TS (χ, τ) satisfies 2.3 of T_{4gh}- space, This results in 2.6 of $T_{(3\frac{1}{2})gh}$- space and by 2.1(2), we get that the proof is complete. ■

Theorem 3.8. Each T_{4gh}- space is T_{3gh}.

Proof: A TS (χ, τ) satisfies 2.3 of T_{4gh}- space, This results in 2.2 of T_{3gh}- space and since every two discrete *(gh-cs)* are separated, we get that the proof is complete. ■

Theorem 3.9. Each T_{5gh}- space is T_{4gh}.

Proof: A TS (χ, τ) satisfies 2.4 of T_{5gh}- space, This results in 2.3 of T_{4gh}- space and since every two discrete *(gh-cs)* are separated, We realize the proof is clear. ■

Corollaries 3.10.

1. Each T_{3gh} (respect. $T_{(3\frac{1}{2})gh}$,T_{4gh}, T_{5gh} and T_{6gh}- space) is T_{1gh} but the converse is not true because T_{1gh} -space is not necessary to be R_{gh} space (respect. CR_{gh}, N_{gh}, CN_{gh} and PN_{gh} space).
2. Each T_{1gh} is T_{0gh}-space but the converse is not true.[5]
3. From above we have the following diagram:

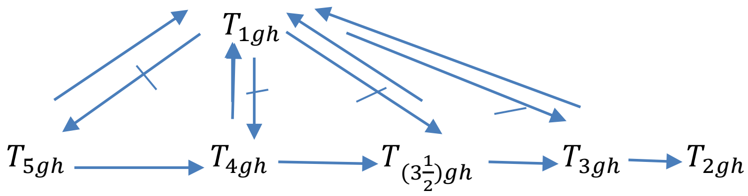

Fig. 1

4. Property of *gh*- Separation Axioms

In this section, we discuss the properties of *gh*-separation axioms.

Theorem 4.1. *Let $f:(\chi,\tau)\to(\gamma,\sigma)$ be bijection, (gh-contm) and γ is $T_0-space$, then χ is $T_{0gh}-space$.*

Proof: Let $f:(\chi,\tau)\to(\gamma,\sigma)$ be bijection, (*gh-contm*) and γ is" T_0- space". *Demonstrating that χ is a* "$T_{0gh}-$ space" . Let $x_1,\ x_2\in\chi$ with $x_1\neq x_2$. Since f is a bijection, there exists $y_1,\ y_2\in\gamma$with $y_1\neq y_2$ s.t. $f(x_1)=y_1$ and $f(x_2)=y_2$, then $x_1=f^{-1}(y_1)$ and $x_2=f^{-1}(y_2)$. Since γ is" T_0- space", there exists an (*os*)U in χ s.t. $y_1\in U$ & $y_2\notin U$. Since f is (*gh- contm*), $f^{-1}(U)$ is a (*gh-os*) in γ. We now have, $y_1\in U$ then $f^{-1}(y_1)\subset f^{-1}(U)$ then $x_1\in f^{-1}(U)$ and $x_2\notin f^{-1}$(U). Henceforth, for any two separate points $y_1,\ y_2$ in γ, there is (g*h-os*) $f^{-1}(U)$ in γ s.t. $x_1\in f^{-1}(U)$ & $x_2\notin f^{-1}(U)$. This shows that X is a T_{0gh}. ■

Theorem 4.2. *If $f:(\chi,\tau)\to(\chi,\tau)$ be an 1-1, (gh-irrem) and γ is $T_{0gh}-space$, then χ is "$T_{0gh}-space$".*

Proof: Assumes that *n, m* $\in\chi$ with *n≠ m.* Since *f* is 1-1 and γ is "$T_{0gh}-$ space" there exists (*gh-os*) U in γ s.t. *f(n)* ∈U and *f(m)* ∉U or there exists (*gh- os*) G in γ s.t. *f(m)* ∈G and *f(n)* ∉G with *f(n)* ≠ *f(m)*. By (*gh-irrem*) of *f*, $f^{-1}(U)$ is (*gh-os*) in χ such that $n\in f^{-1}(U)$ and $m\notin f^{-1}(U)$ *or* $f^{-1}(G)$ is (*gh-os*) in χ such that $m\in f^{-1}(G)$ and $n\notin f^{-1}(G)$. This demonstrates χ is "$T_{0gh}-$ space". ■

Theorem 4.3. *If $f:(\chi,\tau)\to(\chi,\tau)$ be an 1-1, "(gh-irrem)" and γ is "$T_{1gh}-space$", then χ is $T_{1gh}-space$.*

Proof: "Theorem 4.2's" mention of the argument is accurate, with the necessary modifications.

Theorem 4.4. *Let (χ,τ) be TS and γ is $T_1-space$. If $f:(\chi,\tau)\to(\gamma,\sigma)$ is 1-1 and (gh- contm) , then χ is $T_{1gh}-space$.*

Proof: Suppose *n, m* $\in\chi$. s.t. *n≠ m*. Since f is 1-1 then f *(n)≠* f *(m)*. Since γ *is"* $T_{1gh}-$ space*"* then there are two (*gh- os*) *H* and *K* in γ s.t. f *(n)* ∈*H* , f *(m)*∉*H* and f *(m)* ∈*K*, f *(n)* ∉*K*. Since f is (*gh- contm*) then $f^{-1}(H)$, $f^{-1}(K)$ are two (*gh-os*) in χ , *n*∈ $f^{-1}(H)$, *m*∉ $f^{-1}(H)$ and *m*∈ $f^{-1}(K)$, *n*∉ $f^{-1}(K)$. This shows that χ is T_{1gh}. ■

Theorem 4.5. *Let (χ,τ) be TS and γ is $T_{2gh}-space$. If $f:(\chi,\tau)\to(\gamma,\sigma)$ is 1-1 and (gh- irrem) , then χ is $T_{2gh}-space$.*

Proof: Suppose *n, m* $\in\chi$. *s.t. n≠ m.* Since " f *is 1-1* " then f *(n)≠* f *(m)*. Since γ *is"* $T_{2gh}-$ space*"* then there are two (*gh- os*) *H* and K in γ s.t. f *(n)* ∈*H* , f *(m)* ∈*K* and *H*∩*K*=∅, since f is (*gh- irrem)* then $f^{-1}(H)$, $f^{-1}(K)$ are two (*gh-os*) in χ, *n*∈ $f^{-1}(H)$, *m*∈ $f^{-1}(K)$ and $f^{-1}(K)\cap f^{-1}(H)=\emptyset$. This shows that χ is T_{2gh}. ■

Theorem 4.6. *Let* (χ,τ) *be TS and* γ *is* $T_2 - space$*. If* $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ *is 1-1 and* (*gh- contm*) *, then* χ *is* $T_{2gh} - space$*.*

Proof: Suppose *n, m* $\in \chi$*. s.t.* n $\neq$ *m*. Since " f *is 1-1"* then $f(n)\neq f(m)$. Since γ *is "*$T_2 -$ space*"* then there are two (*os*) *H* and *K* in γ s.t. $f(n) \in H$, $f(m) \in K$ and $H\cap K=\emptyset$, since f is (*gh- contm*) then $f^{-1}(H)$, $f^{-1}(K)$ are two (*gh-os*) in χ, *n*$\in$ $f^{-1}(H)$, *m*$\in f^{-1}(K)$ and $f^{-1}(H) \cap f^{-1}(K)=\emptyset$. This shows that χ is a T_{2gh}. ■

Theorem 4.7. If *f is a (gh- irrem) and gh-closed map injection of a TS* (χ,τ) *in to* $R_{gh} -$ *space* γ*, then* χ *is* R_{gh}*.*

Proof: *Suppose* $k \in \chi$ and *A* be a (g*h-cs*) in χ not containing k . Since f is (*gh-cs*), $f(A)$ is a (*gh-cs*) in γ not containing $f(k)$. Since γ is R_{gh}, there exists disjoint (g*h-os*) *U* and *V* in γ s.t. $f(k) \in U$ and $f(A) \subset V$. Since f is *(gh- irrem),* $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint (g*h-os*) in χ containing *k* and *A* respectively. Hence (χ,τ) *is* $R_{gh} - space$*.* ■

Remark 4.8. Let (X^*,τ^*) be a partial topological space of *TS* (χ,τ) and let $F \subseteq X^*$ then $\tau^* \subseteq \tau \subseteq \tau^{gh}$ if and only if $X^* \in \tau$.

Theorem 4.9. *A TS* (χ,τ) *is* CN_{gh} *iff each partial TS of it is* N_{gh}*.*

Proof: 1. Assumes that (χ,τ) is CN_{gh} and let (X^*,τ^*) is a partial space of X . Let F_1 and F_2 be two discrete (*gh-cs*) in χ, then:

$$F_1 \cap CL_{gh}(F_2) = CL^*_{gh}(F_1) \cap CL_{gh}(F_2) = X^* \cap CL_{gh}(F_1) \cap CL_{gh}(F_2)$$
$$= CL^*_{gh}(F_1) \cap CL^*_{gh}(F_2) = F_1 \cap F_2 = \emptyset$$

Then F_1 and F_2 are separated sets in χ. By definition of CN_{gh} there exists two (*gh-os*) I_1 *and* I_2 s.t. $F_1 \subseteq I_1$ *and* $F_2 \subseteq I_2$ then $X^* \cap I_1, X^* \cap I_2$ are discrete *(gh-os*) in X^*. Where $F_1 \subseteq X^* \cap I_1, F_2 \subseteq X^* \cap I_2$. Hence, (X^*,τ^*) is N_{gh}.

2. On the other hand assume that each partial *TS* of (χ,τ) is N_{gh} and we must prove that χ is CN_{gh}. Let A_1 *and* A_2 be separated sets in χ and assumes that (*gh-os*) $\left[CL_{gh}(A) \cap CL_{gh}(B)\right]^C = X^*$ be partial space of χ. This space is N_{gh} (by suppose) and $X^* \cap CL_{gh}(A), X^* \cap CL_{gh}(B)$ are two discrete *(gh-cs*) in X^*. Then there exists two discrete *(gh-os)* G_A *and* G_B in X^*, s.t. $X^* \cap CL_{gh}(A) \subseteq G_A, X^* \cap CL_{gh}(B) \subseteq G_B$. Since $X^*$is *(gh-os*) in χ, then G_A *and* G_B are *(gh-os*) in χ too (4.8). Then A $\subseteq X^* \cap CL_{gh}(A) \subseteq G_A$, $B \subseteq X^* \cap CL_{gh}(B) \subseteq G_B$. Therefore; (X,τ) is CN_{gh} ■

Definition4.10. A mapping $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ is said to be point *gh*-closure 1-1 iff $n,m \in \chi$ such that $CL_{gh}\{n\} \neq CL_{gh}\{m\}$ then $CL_{gh}\{f(n)\} \neq CL_{gh}\{f(m)\}$

Theorem 4.11. *If* $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ *is point gh-closure 1-1 and* χ *is* $T_{0gh} - space$*, then f is 1-1.*

Proof: suppose $n,m \in \chi$ with $n \neq m$. Since χ is $T_{0gh} -$ space, then $CL_{gh}\{n\} \neq CL_{gh}\{m\}$ by theorem 2.7. But *f* is point *gh*-closure 1-1 implies that $CL_{gh}\{f(n)\} \neq CL_{gh}\{f(m)\}$. Hence $f(n) \neq f(m)$ Thus, f is 1-1. ■

Theorem 4.12. *A point gh-closure 1-1mapping* $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ *from* T_{0gh}*-space* χ *in to* $T_{0gh} -$ *spac* γ *exists iff f is 1-1.*

Proof: The necessity follows from the fact mentioned in theorem 4.2

For sufficiency, let $f:(\chi,\tau)\rightarrow(\gamma,\sigma)$ from T_{0gh}-space χ in to $T_{0gh} -$ spac γ be an 1-1 mapping. Now for every pair of distinct points *n & m*$\in \chi$, $CL_{gh}\{n\} \neq CL_{gh}\{m\}$ *as* χ is $T_{0gh} -$ space. Since, *f* is 1-1 mapping $f(CL_{gh}\{n\}) \neq f(CL_{gh}\{m\})$.i.e., $CL_{gh}\{f(n)\} \neq CL_{gh}\{f(m)\}$. Consequently, f is point gh-closure 1-1mapping. ■

Definition4.13. A mapping $f : (\chi, \tau) \rightarrow (\gamma, \sigma)$ is called gh-closed denoted by (*gh-cm*) if the image *f(F)* of each (*cs*) *F* in (χ, τ) is (*gh-cs*) in (γ, σ).

Theorem4.14. *Let (χ, τ) and (γ, σ) be TS and γ is R_{gh} space. If $f : (\chi, \tau) \rightarrow (\gamma, \sigma)$ is (gh-cm), (gh-irrem) and 1-1, then χ is R_{gh} .*

Proof: Assumes that H be (*cs*) in χ , $n \notin H$. Let f (*gh-cm*), then *f(H)* is (*gh-cs*) in γ. *f(n)*= $m \notin f(H)$. But γ is R_{gh} space, then there are two (*gh-os*) I_1 and I_2 in Y such that $f(H) \subseteq I_2, m \in I_1$ *and* $I_1 \cap I_2 = \emptyset$. Since *f* is (*gh-irrem*) and one to one , so $f^{-1}(I_1), f^{-1}(I_2)$ are two (*gh-os*) in χ and $n \in f^{-1}(I_1), H \subseteq f^{-1}(I_2), f^{-1}(I_1) \cap f^{-1}(I_2) = \emptyset$. Hence (χ, τ) is R_{gh} space. ■

Definition4.15. A space *(χ, τ)* is called a

1. T_h- space if each *(h-cs*) in it is (*cs*).
2. T_{gh}- space if each (*gh-cs*) in it is (*cs*).

Example4.16. If χ ={1, 2} and $\tau = \{\emptyset, \chi, \{1\}\}$, $ho\,(\chi) = \tau$ then

$ghc\,(\chi) = \{\emptyset, \chi, \{2\}\} = C(\tau)$

$C(\tau) = \{\emptyset, \chi, \{2\}\} = hc\,(\chi)$

This shows that χ is T_{gh}-space. Also χ is T_h.

Example4.17. If χ ={1, 2,3} and $\tau = \{\emptyset, \chi, \{1\}, \{2\}, \{1,2\}\}$, $ho\,(\chi) = \tau$ then

$ghc\,(\chi) = \{\emptyset, \chi, \{3\}, \{1,3\}, \{2,3\}\} = C(\tau)$, $C(\tau) = \{\emptyset, \chi, \{3\}, \{1,3\}, \{2,3\}\} = hc\,(\chi)$

This shows that χ is T_{gh}-space. Also χ is T_h.

Theorem 4.18. *If (χ, τ) is T_{gh}-space then, for each $k \in \chi$, {k} is (gh-cs) or open.*

Proof: Let *TS (χ, τ)* be T_{gh}-space. Let us Suppose that for some $k \in \chi$, {k} is not (*gh- cs*) in χ. By theorem 1.4(3) *{k}* is not (*cs*) in χ. So $\chi\backslash\{k\}$ is not (*os*) in χ and χ is the only (*os*) containing $\chi\backslash\{k\}$. So $\chi\backslash\{k\}$ is *(gh-cs*) in χ, by (suppose), $\chi\backslash\{k\}$ is (*cs*) in χ, it means *{k}* is (*os*) in χ. ■

Theorem4.19. *Each T_{gh}-space is T_h-space.*

Proof: Suppose *(χ, τ)* be T_{gh}-space and let *k* be *(h-cs*) in χ. Since each *(h-cs*) is (*gh-cs*), therefore; *k* is (*gh-cs*) in χ, by (suppose), *k* is (*cs*) in χ. This shows that χ is T_h. ■

"The converse of theorem 4.19 is not true in general as shown in the following example".

Example 4.20. If χ ={1, 2} and $\tau = \{\emptyset, \chi\}$ then

$C(\tau) = \tau = hc\,(\chi) = ho\,(\chi) = \{\emptyset, \chi\}$. Hence (χ, τ) is T_h- space. but (χ, τ) is not T_{gh}, because {1}is (*gh-cs*) in (χ, τ) but {1} is not (*cs*) in (χ, τ).

Example 4.21. If χ ={1, 2,3} and $\tau = \{\emptyset, \chi, \{1\}, \{2,3\}\}$ then

$C(\tau) = \tau = hc\,(\chi) = ho\,(\chi) = \{\emptyset, \chi\{1\}, \{2,3\}\}$. Hence (χ, τ) is T_h- space. but (χ, τ) is not T_{gh}, because {2}is (*gh-cs*) in (χ, τ) but it is not (*cs*) in (χ, τ).

Theorem4.22. *Each T_{gh}-space is $T_{1/2}$-space .*

Proof: Suppose (χ, τ) be T_{gh}-space and let n be (*g-cs*) in χ. Since each (*g-cs*) is (*gh-cs*), therefore; n is (*gh-cs*) in χ, by (suppose), n is (*cs*) in χ. This shows that χ is $T_{1/2}$. ■
The converse of theorem 4.22 is not true in general as shown in the following example.

Example4.23. If χ ={5, 4, 7} and $\tau = \{\emptyset, \chi, \{4\}, \{5, 4\}, \{4, 7\}\}$ then

$g -$ closed sets are $= \{\emptyset, \chi, \{5, 7\}, \{7\}, \{5\}\} = c(\tau)$. Hence (χ, τ) is $T_{1/2}$- space. but (χ, τ) is not T_{gh}, because {4, 7}is (*gh-cs*) in (χ, τ) but {4, 7} is not (*cs*) in (χ, τ).
Remark 4.24. *From above we have the following diagram:*

$$T_h \underset{\longleftarrow}{\not\longrightarrow} \quad T_{gh} \underset{\not\longleftarrow}{\longrightarrow} T_{1/2}$$

Fig. 2

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بديهيات الفصل باستخدام المجاميع المغلقة المعممة من النمط-h

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الخلاصة:

في البحث الحالي درسنا صنف جديد من بديهيات الفصل سمي بديهيات الفصل المعممة من النمط-h باستخدام المجاميع المفتوحة المعممة من النمط-h . حيث ناقشنا الخصائص مع عدد من المميزات لهذا الصنف. ايضا شرحنا العلاقة بين بديهيات الفصل المعممة من النمط-h مع اعطاء عدة امثلة عن ذلك. من ناحية اخرى برهننا بان بديهيات الفصل تؤدي الى بديهيات الفصل المعممة من النمط-h واعطينا امثلة تبين بان العكس غير صحيح. ايضا قدمنا تعريف الفضاء المنتظم المعمم من النمط-h ، الفضاء السوي المعمم من النمط-h، الفضاء المنتظم الكامل المعمم من النمط-h، الفضاء السوي الكامل المعمم من النمط-h والفضاء السوي التام المعمم من النمط-h وقمنا ببرهان بعض خصائصها. اخيرا عرفنا الفضاء T_{gh} واوجدنا العلاقة بينه وبين الفضاءات $T_{1/2}$ وT_h.