Al-Qadisiyah Journal of Pure Science

Volume 25 | Number 2 Article 9

4-7-2020

Numerical Methods for Solving Optimal Control ProblemUsing Scaling Boubaker Function

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Recommended Citation

Ouda, Eman Hassan and Ahmed, Imad Noah (2020) "Numerical Methods for Solving Optimal Control ProblemUsing Scaling Boubaker Function," *Al-Qadisiyah Journal of Pure Science*: Vol. 25: No. 2, Article 9.
DOI: 10.29350/2411-3514.1198
Available at: https://qjps.researchcommons.org/home/vol25/iss2/9

Al-Qadisiyah Journal of Pure Science
ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514
DOI: /10.29350/jops.
http://qu.edu.iq/journalsc/index.php/JOPS

Numerical Methods for Solving Optimal Control Problem Using Scaling Boubaker Function

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Article History
Received on: 15/04/2020
Revised on: 22/04/2020
Accepted on: 06/05/2020

Keywords:
nonlinear optimal control
Scaling Boubaker function
state parameterization
control parameterization

DOI: doi.org/10.29350/jops.2020.25. 2.1120

ABSTRACT

In this paper, the approximation methods are used to solve optimal control problem (OCP), two techniques for state parameterization and control parameterization have been considered with the aid of Scaling Polynomials (SBP) represent a new important technique for solving (OCP's). The algorithms were illustrated by several numerical examples using Matlab program. The results were evaluated and graphed to show the accuracy of the methods.

MSC:

1. Introduction

The main idea of the solution process for an optimal control problems is finding a control $u^*(t)$ which minimizes a given performance index while satisfying the required constraints. In general, this is done by approximating state or control functions. The finite terms of scaling functions whose unknown coefficients values are sought giving an approximate optimal solution.

Optimal control is a wide field where a lot of researchers used different proceedings for solving its problems. Kafash B. et al. presented new efficient algorithms for solving OCPs and the controlled Duffing oscillator [7]. Lotfi A. Yousefi S. A. and Dehghan M. suggested a numerical direct method for solving a general class of fractional optimal control problems (FOCPs)[10]. Jaddu H.and Majdalawi A. proposed a computational algorithm for solving a class of nonlinear optimal control problems (NOCPs) [6]. Naseif J. and Imad N., used functional analysis technique for studying and deducing sufficient conditions for the controllability of nonlinear boundary value control systems in Banach spaces [12]. Ramezani M. gave a new numerical method for solving (OPC) based on state parameterization and new second kind Chebyshev wavelet [13].

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Kafash B. and Delavarkhalafi A. introduced an efficient algorithm based on state parameterization solving (OPCs) and Van Der Pol oscillator [8].

The general nonlinear optimal control (NOC) is stated as follows:

$$min\ J = \int_{t_0}^{t_f} F\ (t, x(t), u(t))dt \qquad \dots (1)$$

Subject to the system constraints:

$$\dot{x}\ (t) = f\big(t, x(t), u(t)\big), \quad t_0 \leq t \leq t_f \qquad \dots (2)$$

with conditions $x(t_0) = x_0$ ... (3)

Where $x(t)(\ the\ state\ vector)$ is n × 1, and u (t) (the input vector) is m × 1.

The linear quadratic optimal control problem (LQOCP) is a particular form of the general nonlinear (NOC) is as follows:

$$J = \int_{t_0}^{t_f} \big(x^T Qx(t) + u^T Ru(t)\big)dt \qquad \dots (4a)$$

Subject to the linear

$$\dot{x}\ (t) = Ax + Bu \text{ and } x(t_0) = x_0 \qquad \dots (4b)$$

where Q (n x n)and R (m x m) are positive definite matrices. A (n x n), and B (m x m) are constant matrices.

Boubaker polynomials were firstly used as a tool for solving heat equation, then these polynomials were utilized for solving optimal control problems (OCPs) with different proceedings. Kafash B. et al. used Boubaker polynomials with expansion scheme for solving OCPs [9]. Ahmed I., et al utilizing indirect method with Boubaker polynomial for solving an optimal control problem [1].

Goy T. evaluated several families of Toeplitz - Hessenberg determinants whose entries are Boubaker polynomials [4].

Scaling Boubaker polynomials (SBP) and their properties provide powerful mathematical technique for solving some problems in pure and applied mathematics. Scaling functions are widely used in the last decades for dealing with different applied fields in science and engineering. They represent a dilation equation which is a functional of the form

$$f(t) = \sum_{k=0}^{N} C_k f(2t - K) \qquad \dots (5)$$

with a non – zero solution. Colella D. and Heil Ch. gave a characterization of all dilation equations with continuous and compactly supported solutions [2]. Gradimir V. and Joksimovic D. added some new properties of Boubaker polynomials and applications for achieving an approximate analytical solution of Love's integral equation [5]. Yousfi S. A. presented a numerical solution for the generalized Emden – Fowler equations using Legendre Scaling function [14] .Elaydi H., Jaddu H. and Wadi M. used a wavelet approach where the basis orthogonal function were Legendre scaling functions for solving nonlinear optimal control problems[3]. Malmir I. presented an arbitrary scaling parameter for operational matrices of Legendre wavelets [11].

In this paper, two techniques state parameterization and control parameterization have been considered with the aid of scaling Boubaker polynomials (SBP) which represent a new important technique for solving OCP's, the results were evaluated and graphed .

This paper is arranged as follows, in section2, the Boubaker polynomials are introduced, in section3 the Scaling Boubaker polynomials are presented, then in section 4 and 5 the state Parameterization technique and control parameterization technique are explained, in section 6 the convergence of two previous proceedings is studied, then some numerical examples were applied for the proposed method, in section7 comparing the results with exact solutions. Finally the conclusion.

2. Boubaker polynomials

Boubaker polynomials *Bo* have been appeared by Boubaker et al, for solving different equations in many applications ...etc. see [5].

$$B_m(t) = \sum_{s=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{(m-4s)}{(m-s)} \binom{m-s}{s} (-1)^s t^{m-2s}, \qquad m = 0,1,2,\ldots \qquad \ldots (6)$$

The first terms of Boubaker polynomials are

$B_0(t) = 1$, $B_1(t) = t$, $B_2(t) = t^2 + 2$, ...

and the recurrence relation $B_m(t) = tB_{m-1}(t) - B_{m-2}(t)$, $m > 2$

3. Scaling Boubaker polynomials (*SBP*)

The Scaling Boubaker polynomials (*SBP*), can be defined as follows:

$$SB_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} B_m(2^{k+1}t - 2n - 1) & \frac{2n-1}{2^{k+1}} \le t \le \frac{2n}{2^{k+1}} \\ 0 & otherwise \end{cases} \qquad \ldots (7)$$

The arguments of scaling (*k*, *n*, *m*, *t*), *k* is positive integer, *n* = 0, 1, 2, 3,..., 2*k*, *m* is degree of Boubaker polynomials and *t* is the time.

Choosing *k*=1 and *m*=5. The first five terms Scaling Boubaker $SB_m(t)$ were found by using Eq.7 to be:

$SB_0 = \sqrt{2}$,

$SB_1 = \sqrt{2}(4t - 1)$,

$SB_2 = \sqrt{2}(16t^2 - 8t + 3)$,

$SB_3 = \sqrt{2}\,(64t^3 - 48t^2 + 16t - 2)$,

$SB_4 = \sqrt{2}\,(256t^4 - 256t^3 + 96t^2 - 16t - 1)$.

The direct methods can be applied by using the parameterization techniques for state variables or control variables or both of them.

4. The State Parameterization Technique

The state parameterization is based on approximating state variables by using Scaling Boubaker polynomials (SBP) with unknown coefficients as follows,

$$x(t) = \sum_{i=0}^{m} a_i\, SB_i(t) = a^T SB\,(t) \qquad t_0 \le t \le t_f \qquad \dots (8)$$

where a_i are unknown coefficients of state and *SB* are the Scaling Boubaker polynomials

-Expanding $x\,(t)$ using SBP into five order ($m = 5$) in Eq.7 we get,

$x(t) = a_0SB_0 + a_1SB_1 + a_2SB_2 + a_3SB_3 + a_4SB_4$, where a_i the state coefficients must be found

with initial condition $x(t_0) = \sum_{i=0}^{4} a_i\, SB_i(t_0) = x_0$ ... (9)

-Finding $\dot{x}\,(t) = \sum_{i=0}^{4} a_i\, \dot{SB_i}(t) \qquad t_0 \le t \le t_f$

where the differential of Scaling Boubaker polynomials ($\dot{SB_i}$) is given as follows

$\dot{SB_0}(t) = 0$

$\dot{SB_1}(t) = 4\sqrt{2}$

$\dot{SB_2}(t) = \sqrt{2}\,(32t - 8)$

$\dot{SB_3}(t) = \sqrt{2}\,(192t^2 - 96t + 16)$

$\dot{SB_4}(t) = \sqrt{2}\,(1024t^3 - 768t^2 + 192t - 16)$.

Substituting in state equation (4*b*) defined by

$$u(t) = SB(t, x(t), \dot{x}(t)) \qquad \dots (10)$$

We get $u\,(t)$ which is the control variable

-Deducing the optimal control $u(t)$ with the unknown coefficients of a_i we can evaluate J using Eq.1 as in follows

$$J \cong \int_{t_0}^{t_f} (a^T SB^T QaSB + a^T SB^T RaSB)dt \quad , t_0 \leq t \leq t_f$$

The performance index can written as follows

$$J = \frac{1}{2} a^T H a$$

Where H is Hessian matrix with the unknown parameters $a^T = [\ a_0\ a_1\ a_2\ \ a_3\ a_4\ ]^T$ with initial condition using Eq.9 we get the system of equations.
-Solving the system to find coefficients of state.
-Deducing the control, then evaluating the performance function J by Eq.4.

5. The Control Parameterization Technique

In this section, the following steps were used to evaluate the control and state by using control parameterization,

- Approximating control variable, as follows

$u(t) = \sum_{i=0}^{4} b_i\, SB_i\ ,\ t_0 \leq t \leq t_f$

where b_i are the unknown coefficients of control and SB are Scaling Boubaker polynomials of order 5 and

- By integrating the state equation system with initial condition we get

$$\int_{t_0}^{t_f} u(t)dt = \int_{t_0}^{t_f} SB(t, x(t), \dot{x}(t))dt \qquad \dots (11)$$

- Obtaining the solution at the points (t_n, t_{n+1}), the fundamental theorem of calculus can be used, and then Eq.(11) is integrated over $[t_n, t_{n+1}]$
- We get the linear algebraic equations which can be solved by Gauss elimination method.
- The coefficients of control b_i can be found.
- The state variables can be deduced from the control coefficients by solving state equation.
- Now, substituting $u(t)$ and $x(t)$ to evaluate the performance function J by using Eq.4.

6. The Convergence Test:

6.1. The State Technique:

$x_m(t) = \sum_{i=0}^{\infty} a_i\, SB_i(t) \qquad t_0 \leq t \leq t_f$

The truncated series would be

$$x_m(t) = \sum_{i=0}^{m} a_i\, SB_i(t)$$

There using the convergence test for the state parameterization technique.

$$\left[\int_{t_0}^{t_f} (x(t) - x_m(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

since $x(t)$ is unknown, it can be replaced the presumably better approximation $x_{n+m}(t)$, where $m \geq 0$

$$\left[\int_{t_0}^{t_f} (x_{n+m}(t) - x_m(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

Using the Scaling Boubaker polynomials for approximating the state variables, then

$$\left[\int_{t_0}^{t_f} (\sum_{i=0}^{n+m} a_i SB_i(t) - \sum_{i=0}^{m} a_i\, SB_i(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \left[\int_{t_0}^{t_f} (\sum_{i=m+1}^{n+m} a_i SB_i(t))^2 dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \left[\int_{t_0}^{t_f} (\sum_{i=m+1}^{n+m} a_i SB_i(t))(\sum_{i=m+1}^{n+m} a_i SB_i(t))\, dt\right]^{\frac{1}{2}} < \varepsilon$$

$$= \sum_{i=m+1}^{n+m} \sum_{j=m+1}^{l+m} a_i a_j \int_{t_0}^{t_f} SB_i\, SB_j(t)\ dt < \varepsilon$$

when the sum of the squares of the remaining coefficients becomes negligible, a satisfactory approximation to the solution is achieved.

6.2. The Control Technique:

In the proposed method, the control vector is approximated as,

$u_m(t) = \sum_{i=0}^{\infty} b_i\, SB_i(t) \qquad t_0 \leq t \leq t_f$

The truncated series would be

$$u_m(t) = \sum_{i=0}^{m} b_i\, SB_i(t)$$

There using the same convergence method as in the previous section (state convergence), we get

$$= \sum_{i=m+1}^{n+m} \sum_{j=m+1}^{l+m} b_i b_j \int_{t_0}^{t_f} SB_i\, SB_j(t)\ dt < \varepsilon$$

when the squares of the remaining coefficients become negligible, a satisfactory approximation to the solution will be reached.

7. Numerical Examples

Example 1:

Min $j = 0.5 \int_0^1 (2x^2 + u^2)dt$

with $\dot{x} = 0.5x + u$, $x(0) = 1$.

where exact solution for this example is $x(t) = \frac{2e^{3t} + e^3}{e^{\frac{3t}{2}}(2+e^3)}$, and $u(t) = \frac{2(e^{3t} - e^3)}{e^{\frac{3t}{2}}(2+e^3)}$

and the optimal value of J = 0.8641644977.

a. Solving by using state parameterization

By using the state technique illustrated in section 4, we found the following coefficients

a_0=0.460412396461131, a_1=-0.127823658910297, a_2=0.037726555649684,

a_3=-0.003032245966087,a_4=0.000373433066107.

Then the state and control would be:

$x(t) = 1 - (919/749)\, t + (937/844)\, t^2 - (807/1970)t^3 + (353/2611)t^4$.

and $u(t) = -(1031/597) + (2678/945)t - (983/551)t^2 + (2336/3133)t^3 - (353/5222)\, t^4$.

and Jappro. = 0.86416456896. With Abs (Error (J)) =7.119486900020178e-08.

b. Solving by using control parameterization

Applying the same example1 by using the control technique illustrated in section 5, we found the following coefficients b_i s as follows:

b_0= -0.680251932678915, b_1= 0.358325053630541, b_2= -0.055675910199709,

b_3= 0.008291627207229, b_4= -0.000396395227143.

we can write approximate solution of state and control as follows:

$x(t) = (2834/61) + (6601/307)\, t - (2773/61)\ e^{\frac{t}{2}} + (4435/652)\, t^2 + (2262/4451)\, t^3 + (816/2843)\, t^4$.

$u(t) = -(2089/1209) + (779/273)\, t - (4937/2631)\, t^2 + (312/349)\, t^3 - (408/2843)\, t^4$.

and Jappro. = 0.864160768327565. With Abs (Error (J)) =3.729433434962459e-06

Table1 state results using state Para. And control Para. for example1

t	$x_{appr}(t)$ using St. Par.	Abs. Error using St. Par.	$x_{appr}(t)$ using Co. Par.	Abs. Error using Co. Par.
0	0.99999999999	0.00000000001	1.00000000000	0
0.1	0.88800888825	0.00003182008	0.88798763065	0.00001056248
0.2	0.79595296674	0.00001814327	0.79596996999	0.00000114002
0.3	0.72186107669	0.00004703036	0.72190263669	0.00000547036
0.4	0.66408653265	0.00003198221	0.66411596847	0.00000254638
0.5	0.62130712247	0.00000749470	0.62129891629	0.00000071147
0.6	0.59252510732	0.00003889426	0.59248211274	0.00000410030
0.7	0.57706722166	0.00003846926	0.57702007236	0.00000868002
0.8	0.57458467326	0.00000587293	0.57457247932	0.00000632100
0.9	0.58505314321	0.00002798628	0.58508451573	0.00000338623
1.0	0.60877278590	0.00000030019	0.60876618135	0.00000630435

Table2 control results using state Par. And control Par. for example1

t	$u_{appr}(t)$ using St. Par.	Abs. Error using St. Par.	$u_{appr}(t)$ using Co. Par.	Abs. Error using Co. Par.
0	-1.72696878100	0.00136021453	-1.72787421730	0.00045477823
0.1	-1.46068395138	0.00036649078	-1.46041120036	0.00009373976
0.2	-1.22570067102	0.00047594913	-1.22531465614	0.00008993425
0.3	-1.01778862747	0.00003736009	-1.01773732514	0.00001394223
0.4	-0.83287974492	0.00034044005	-0.83317637280	0.00004381216
0.5	-0.66706818424	0.00040355034	-0.66747338954	0.00000165494
0.6	-0.51661034293	0.00015923538	-0.51681439071	0.00004481239
0.7	-0.37792485516	0.00020830002	-0.37772981667	0.00001326153
0.8	-0.24759259173	0.00041449041	-0.24709453271	0.00008356860
0.9	-0.12235666013	0.00014506970	-0.12212782910	0.00008376132
1.0	0.00087759552	0.00087759552	-0.00039342107	0.00039342107

Remark: Since the results for the state and control parameterizations are approximately the same, Fig.1 would be sufficient for illustrating the exactness the method.

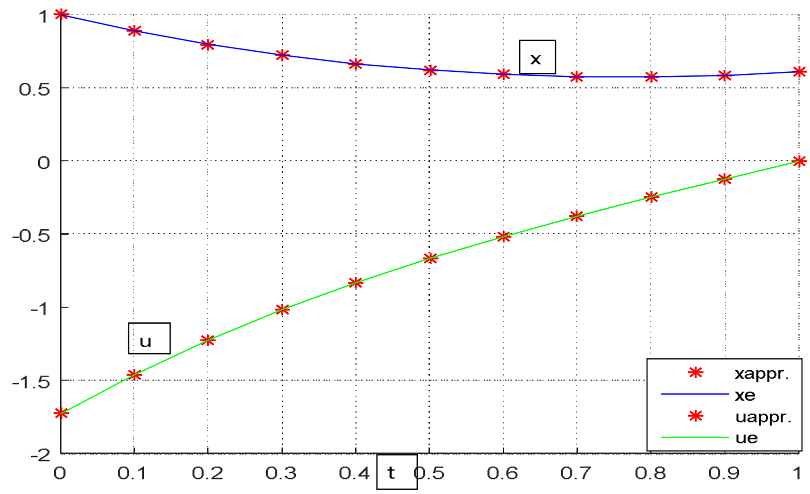

Fig.1 comparison for *x* and *u* with exact solution for example1

Example 2:

Min $j = \int_0^1 (\frac{5}{8}x^2 + \frac{1}{2}xu + \frac{1}{2}u^2)dt$

with $\dot{x} = 0.5x + u$, $x(0) = 1$.

where exact solution for this example is $x(t) = \frac{\cosh(1-t)}{\cosh(1)}$, and $u(t) = \frac{-(\tanh(1-t)+0.5)(\cosh(1-t))}{\cosh(1)}$ and the optimal value of $J_{exact} = 0.38797077977882$.

a. Solving by using state parameterization

The same procedure of state technique in the above example are used to solve this problem we get

a_0=0.556333395838366, a_1=-0.093241205628115, a_2=0.018556405972549;

a_3=-0.000974119658567, a_4=0.000085277514714.

Then the state and control would be as follows:

$x(t)= 1- (5912/7765)t+(1134/2279)t^2-(427/3587)t^3+(259/8389)t^4$.

$u(t) = - (9739/7721)+ (2211/1607)t- (635/1048)t^2+ (528/2885)t^3-(309/20017)t^4$.

Jappro=0.380797080316190. Abs (Error (J)) = 2.338307991678335e-09.

b. Solving by using control parameterization

Applying the same example2 by using control parameterization, the control with coefficients b_i s would be as follows:

b_0=-0.631520147177317, b_1=0.193446478784298, b_2=-0.021052830981420,

b_3=0.001999520525119, b_4=-0.000080517706408.

$x(t)=(827/90)+(2613/784)t-(737/90)\ e^{\frac{t}{2}}+ (629/413)t^2+(458/9923)t^3+(256/4391)t^4$;

$u(t)=(796/577)\, t - (1585/2544)\, t^2 + (83/395)\, t^3 - (128/4391)\, t^4.- (11041/8752)$.

$j$appro = 0.380793599989886. Abs(Error(j)) =3.477987995992304e-06

Table3 state results using state Para. And control Para. for example2

t	$x_{appr}(t)$ using St. Par.	Abs. Error using St. Par.	$x_{appr}(t)$ using Co. Par.	Abs. Error using Co. Par.
0	1.00000000000	0.00000000000	0.99999344085	0.00000655914
0.1	0.92872340268	0.00000564606	0.92871261917	0.00000513744
0.2	0.86672751521	0.00000291748	0.86672404295	0.00000638974
0.3	0.81340923708	0.00000840118	0.81341084851	0.00000678975
0.4	0.76823956480	0.00000623615	0.76823950787	0.00000629308
0.5	0.73076359192	0.00000076608	0.73075692744	0.00000589840
0.6	0.70060050899	0.00000693828	0.70058739805	0.00000617265
0.7	0.67744360360	0.00000751209	0.67742938850	0.00000670300
0.8	0.66106026033	0.00000163993	0.66105217482	0.00000644557
0.9	0.65129196080	0.00000528534	0.65129229659	0.00000494956
1.0	0.64805428366	0.00000001000	0.64804983167	0.00000444198

Table4 control results using state Par. And control Par. for example2

t	$u_{appr}(t)$ using St. Par.	Abs. Error using St. Par.	$u_{appr}(t)$ using Co. Par.	Abs. Error using Co. Par.
0	-1.26136509985	0.00022905610	-1.26154021709	0.00005393885
0.1	-1.12965721023	0.00005978084	-1.12960834946	0.00001092007
0.2	-1.00899116564	0.00008507207	-1.00891722551	0.00001113194
0.3	-0.89832444563	0.00001221659	-0.89831102836	0.00000120068
0.4	-0.79665157827	0.00005739696	-0.79670390235	0.00000507287
0.5	-0.70300414012	0.00007531251	-0.70307995313	0.00000050050
0.6	-0.61645075626	0.00003582923	-0.61649324757	0.00000666206
0.7	-0.53609710030	0.00003337697	-0.53606781381	0.00000409047
0.8	-0.46108589434	0.00007992725	-0.46099764123	0.00000832585
0.9	-0.39059690898	0.00003479547	-0.39054668049	0.00001543301
1.0	-0.32384696335	0.00018017347	-0.32404884349	0.00002170666

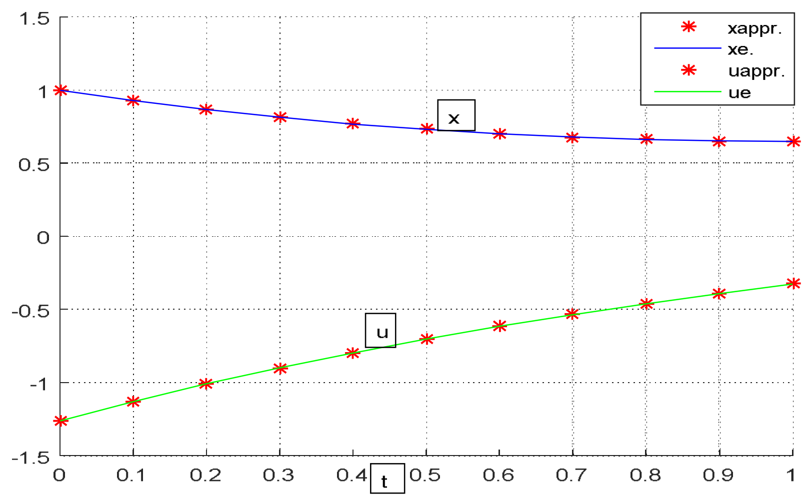

Fig.2 comparison for *x* and *u* with exact solution for example2

8. Conclusion

Utilizing Scaling Boubaker functions as an aid in parameterization techniques in two forms (state, control) were proved to be a powerful tool for solving optimal control problems. Only few number of Scaling Boubaker polynomials were needed to reach a good accuracy. All the approximate results were compared with the exact solution showing good efficiency and capability of this method for solving these problems.

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