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On Differential Subordination and Superordination Results of Multivalent Functions Defined by a Linear Operator

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Abstract :

The main idea of this paper is to derive some subordination and superordination results defined by a linear operator for multivalent functions in the open unit disk . Several sandwich-type results are also obtained.

Keywords : Analytic function, multivalent function, subordination, superordination , linear operator .

2019 Mathematics Subject Classification: 30C45, 30C50.

1- Introduction :

Let A(p) denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k , \ p \in \mathbb{N} = \{1,2,3,\dots\}, \ z \in U \qquad (1.1)$$

which are analytic multivalent in the open unit disk $U = \{ z \in \mathbb{C}: |z| < 1 \}$.

For two functions f and g are analytic in U , we say that the functions f is subordinate to g in U , written $f \prec g$, if there exists Schwars function w ,analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z))$, $z \in U$. If g is univalent and $g(0) = f(0)$, then $f(U) \subset g(U)$.

If $f \in A(p)$ is given by (1.1) , then the linear operator

$I_p(n, \lambda) : A(p) \to A(p)$ ([2]) is defined by

$$I_p(n,\lambda)f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right) a_n z^n , \lambda > -p, \ p \in \mathbb{N} = \{1,2,3,\dots\}. \qquad (1.2)$$

It is easily verified from (1.2) that

$$z[I_p(n, \lambda)f(z)]' = (p+\lambda)\, I_p(n+1,\lambda)f(z) - \lambda\, I_p(n, \lambda)f(z). \qquad (1.3)$$

The main object of this idea is to find sufficient conditions for certain normalized analytic functions f to satisfy :

$$q_1(z) \prec \left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)f(z)}{(t_1+t_2)\quad z^p}\right)^{\frac{1}{\delta}} \prec q_2(z)$$

$$q_1(z) \prec \left(\frac{Ip\,(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Several authors studied differential subordination and superordination for different conditions (see [3,4,5,6,7,8,9]) .

2- Preliminaries : In order to prove our subordinations and superordinations results, we need the following definitions and lemmas.

Definition (2.1)[11] : Let Q the set of all functions q that are analytic and injective on $\overline{U}$ / E(q) where $\overline{U} = U \cup \{z \in \partial U\}$, and $E(q) = \{ \varepsilon \in \partial U : \lim_{z\to\varepsilon} q(z) = \infty\}$ (2.1)

such that $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U$ / E(q). Further , let the subclass

of Q for which $q(0) = a$ be denoted by Q(a), and $Q(0) = Q_0$, $Q(1) = Q_1$.

Lemma (2.1) [1]: Let q(z) be convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} / \{0\}$ and suppose that

$$\mathrm{Re}\left\{ 1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If p(z) is analytic in U , and

$$\alpha\, p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z),$$

then $p(z) \prec q(z)$ and q is the best dominant .

Lemma (2.2)[10]: Let q be univalent function in U and let ϕ and θ be analytic in the domain D containing q(U) with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\,\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$.Suppose that,

(i) Q is starlike univalent in U .

(ii) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U with p(0) = q(0) , $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\,\Phi(p(z)) \prec \theta(q(z)) + zp'(z)\,\Phi(q(z)),$$

then $p \prec q$ and q is the best dominant .

Lemma (2.3) [12]: Let q(z) be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing q(U). Suppose that

(i) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,

(ii) $zq'(z)\,\phi(q(z))$ is starlike univalent in $z \in U$.

If $p \in H[q(0),1] \cap Q$ with $p(U) \subseteq D$,and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U , and $\theta(q(z)) + zq'(z)\,\phi(q(z)) \prec \theta(p(z)) + zp'(z)\,\phi(p(z))$, (2.2)

then $q \prec p$ and q is the best subordinant .

Lemma (2.4) [12]: Let q(z) be convex univalent in U and q(0) = 1 .Let $\beta \in \mathbb{C}$,that $\mathrm{Re}(\beta) > 0$. If $\rho(z) \in H[q(0), 1] \cap Q$ and $P(z) + \beta zp'(z)$ is univalent in U, then $q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z)$, which implies that $q(z) \prec p(z)$ and q(z) is the best subordinant.

3- Subordination Results :

Theorem (3.1) : Let q(z) be convex univalent function in U wth $q(0) = 1, \eta, \delta \in \mathbb{C} \setminus \{0\}$. Suppose that

$$\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{1}{\delta\,\eta}\right)\right\}. \tag{3.1}$$

If $f \in A(p)$ is satisfies the subordination $G(z) \prec q(z) + \delta\eta zq'(z)$,

(3.2)

where

$$G(z) = \left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)f(z)}{(t_1+t_2)\quad z^p}\right)^{\frac{1}{\delta}} \times \Bigg(1 + \eta\left(\frac{t_1(p+\lambda)\, I_p\,(n+1,\lambda)f(z) - \lambda\, I_p\,(n,\lambda)f(z) + t_2(p+\lambda)\, I_p\,(n+1,\lambda)f(z) - \lambda\, I_p\,(n,\lambda)f(z)}{t_1\, Ip\,(n,\lambda)\, f(z) + t_2\, Ip\,(n,\lambda)\, f(z)}\right)\Bigg), \tag{3.3}$$

then

$$\left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)f(z)}{(t_1+t_2)\quad z^p}\right)^{\frac{1}{\delta}} \prec q(z)\,, \tag{3.4}$$

and q(z) is the best dominant .

Proof : Define a function k(z) by

$$k(z) = \left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)\, f(z)}{(t_1+t_2)\quad z^p}\right)^{\frac{1}{\delta}}, \tag{3.5}$$

then the function k(z) is analytic in U and q(0) = 1, therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.3) in the resulting equation,

$$G(z) = \left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)f(z)}{(t_1+t_2)\quad z^p}\right)^{\frac{1}{\delta}} \times \Bigg(1 + \eta\left(\frac{t_1(p+\lambda)\, I_p\,(n+1,\lambda)f(z) - \lambda\, I_p\,(n,\lambda)f(z) + t_2(p+\lambda)\, I_p\,(n+1,\lambda)f(z) - \lambda\, I_p\,(n,\lambda)f(z)}{t_1\, Ip\,(n,\lambda)\, f(z) + t_2\, Ip\,(n,\lambda)f(z)}\right)\Bigg).$$

Thus The subordination (3.2) is equivalent to

$$k(z) + \delta\eta\, zk'(z) \prec q(z) + \delta\eta\, zq'(z).$$

An application of Lemma (2.1) with $\beta = \delta\eta$ and $\alpha = 1$, we obtain (3.4).

Taking $q(z) = (\frac{1+Az}{1+Bz})$ $(-1\leq B < A \leq 1)$ in Theorem (3.1) , we obtain the following corollary.

Corollary (3.1): Let $\eta ,\delta \in \mathbb{C}\backslash\{0\}$ and $(-1\leq B<A \leq 1)$. Suppose that

$$\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max \{0, -\operatorname{Re}\left(\frac{1}{\delta\eta}\right)\}.$$

If $f \in A(p)$ is satisfy the following subordination condition :

$$G(z) \prec \frac{1+Az}{1+Bz} + \delta\eta \frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ is given by (3.3) , then

$$\left(\frac{t_1 I_p(n,\lambda)f(z)+ t_2 I_p(n,\lambda) f(z)}{(t_1+ t_2)\ z^p}\right)^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking A = 1and B = -1 in Corollary(3.1), we get following result.

Corollary (3.2) : Let $\eta ,\delta \in \mathbb{C} \setminus \{0\}$ and suppose that

$$\operatorname{Re}\left\{\left(\frac{1+z}{1-z}\right) > \max\{0, -\operatorname{Re}\left(\frac{1}{\delta\eta}\right)\}\right\}.$$

If $f \in A(p)$ satisfy, the following subordination ,

$$G(z) \prec \frac{1+z}{1-z} + \delta\eta \frac{2z}{\delta\,(1-z)^2},$$

where $G(z)$ is given by (3.3) then

$$\left(\frac{t_1 I_p(n,\lambda)f(z)+ t_2 I_p(n,\lambda)f(z)}{(t_1+ t_2)\ z^p}\right)^{\frac{1}{\delta}} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant .

Theorem (3.2): Let $q(z)$ be convex univalent in unit disk U with $q(0) =1,\delta > 0$, let $\eta ,\delta, s \in \mathbb{C} \setminus\{0\}$, $\gamma, t, \psi, \tau \in \mathbb{C}$, $f \in A(p)$, and suppose that f and q satisfy the following conditions :

$$\operatorname{Re}\left\{ \frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)} \right\} > 0, \tag{3.6}$$

and

$$\frac{I_p(n,\lambda)f(z)}{z^p} \neq 0. \tag{3.7}$$

$$\text{If } r(z) \prec t + \psi\, q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)}, \tag{3.8}$$

where

$$r(z) = \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \left(\psi + t\gamma \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right) + t \right.$$

$$\left. + s\frac{1}{\delta}(p+\lambda)\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} -1\right)\right), \tag{3.9}$$

then

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \prec q(z), \text{ and } q(z) \text{ is the best dominant.}$$

Proof : Define analytic function $k(z)$ by

$$k(z) = \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}}, \tag{3.10}$$

then the function $k(z)$ is analytic in U and $g(0) = 1$,

differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zk'(z)}{k(z)} = \frac{1}{\delta}(p+\lambda)\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right). \tag{3.11}$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Also , if we let .

$$\Phi(z) = zq'(z)\, \phi(q(z)) = s\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)},$$

we find $Q(z)$ is starlike univalent in U, we have

$$h'(z)= \psi q'(z) +2\tau\gamma q(z)q'(z) + s\frac{q'(z)}{q(z)} +sz\frac{q''(z)}{q(z)} - sz\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} -z\frac{q'(z)}{q(z)},$$

hence that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(Z)}\right)= \operatorname{Re}\left(\frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} -z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\Psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} = \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} (\psi +$$

$$\Upsilon\tau\gamma\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}}) + t +\left(s\frac{1}{\delta}(p+\lambda)\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right)\right).$$

By using (3.8), we have

$$\Psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} \prec \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)},$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $k(z) \prec q(z)$ and the function $q(z)$ is the best dominant .

and

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Theorem (3.2), the condition (3.6) becomes.

$$\operatorname{Re}\left(\frac{\psi}{s}\frac{1+AZ}{1+Bz} + \frac{2\tau\gamma}{s}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0, \tag{3.12}$$

hence, we have the following Corollary.

Corollary (3.3) : Let $(-1 \leq B < A \leq 1)$, $s , \delta \in \mathbb{C} \setminus \{0\}$, $\gamma, t ,\tau ,\psi \in \mathbb{C}$ .Assume that (3.12) holds. If $f \in A(p)$ and

$$r(z) \prec t + \psi\frac{1+Az}{1+Bz} + \tau\gamma\left(\frac{1+Az}{1+Bz}\right)^2 + s\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9) , then

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}, \text{ and } \frac{1+Az}{1+Bz} \text{ is the best dominant.}$$

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$,$(0 <\rho\leq1)$,In Theorem (3.2) , the condition (3.6) becomes

$$\text{Re}\,\{\frac{\psi}{s}(\frac{1+z}{1-z})^{\rho} + \frac{2\tau\gamma}{s}(\frac{1+z}{1-z})^{2\rho} + \frac{2z^2}{1-z^2}\} > 0, (s \in \mathbb{C}\backslash\{0\}), \quad (3.13)$$

hence , we have the following Corollary.

Corollary (3.4): Let $0 < \rho \leq 1$, s ,$\delta \in \mathbb{C}\backslash\{0\}$,γ ,t ,τ , $\psi \in \mathbb{C}$. Assume that (3.13) holds. If $f \in A(p)$ and

$$r(z) \prec t + \psi\,(\frac{1+z}{1-z})^{\rho} + \tau\gamma\,(\frac{1+z}{1-z})^{2\rho} + s\,\frac{2pz}{1-z^2},$$

where r(z) is defined in (3.9), then

$$(\frac{I_p\,(n,\lambda)f(z)}{z^p})^{\frac{1}{\delta}} \prec (\frac{1+z}{1-z})^{\rho}, \text{ and } (\frac{1+z}{1-z})^{\rho} \text{ is the best dominant.}$$

4- Superordination Results :

Theorem (4.1): Let q(z) be convex univalent in U with q(0) = 1 , $\delta \in \mathbb{C} \setminus \{0\}$, Re $\{\eta\} > 0$, if $f \in A(p)$, such that

$$\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)f(z)}{(t_1 + t_2)\quad z^p} \neq 0,$$

and

$$(\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)\,f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}}\ H[q(0),1] \cap Q. \quad (4.1)$$

If the function G(z) defined by (3.3) is univalent and the following superordination condition:

$$q(z) + \delta\eta z q'(z) \prec G(z), \quad (4.2)$$

holds , then

$$q(z) \prec (\,(\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)\,f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}} \quad (4.3)$$

and q(z) is the best subordinant .

Proof : Define a function k(z) by

$$k(z) = (\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)\,f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}}. \quad (4.4)$$

Differentiating (4.4) with respect to z logarithmically ,we get.

$$\frac{zk(z)}{k(z)} = \frac{1}{\delta}(\frac{t_1(z(\,Ip\,(n,\lambda)f(z))') + t_2(z(\,Ip\,(n,\lambda)f(z))')}{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)f(z)} - \frac{pt_1\,Ip\,(n,\lambda)f(z) + pt_2\,Ip\,(n,\lambda)f(z)}{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)f(z)}). \quad (4.5)$$

A simple computation and using (1.3) from (4.5), we get

$$G(z) = (\frac{t_1\,Ip\,(n,\lambda)f\,(z) + t_2\,Ip\,(n,\lambda)\,f\,(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}} \times$$

$$(1 + \boldsymbol{\eta}\,(\frac{t_1\,(p+\lambda)\,I_p\,(n+1,\lambda)f(z) - \lambda\,I_p\,(n,\lambda)f(z) + t_2(p+\lambda)\,I_p\,(n+1,\lambda)f(z) - \lambda\,I_p\,(n,\lambda)f(z)}{t_1\,Ip\,(n,\lambda)\,f\,(z) + t_2\,Ip\,(n,\lambda)\,f(z)}))$$

$= k(z) + \delta\eta\, zk'(z)$.

Now , by using Lemma (2.4), we get the desired result .

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$), in Theorem (4.1), we get the following Corollary.

Corollary (4.2): Let Re$\{\eta\}$>0 ,$\delta \in \mathbb{C}\backslash\{0\}$ and $-1 \leq B < A \leq 1$, such that

$$(\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}} \in H[q(0),1] \cap Q.$$

If the function G(z) given by (3.3) is univalent in U and $f \in A(p)$ satisfies the following superordination condition :

$$\frac{1+Az}{1+Bz} + \delta\eta\,\frac{(A-B)Z}{(1+BZ)^2} \prec G(z),$$

then

$$\frac{1+Az}{1+Bz} \prec (\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)\,f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem (4.2): Let q(z) be convex univalent in unit disk U, let δ, s$\in \mathbb{C}\backslash\{0\}$,$\gamma$, t, ψ ,$\tau \in \mathbb{C}$, $q(z) \neq 0$ and $f \in A(p)$ .Suppose that

$$\text{Re}\,\{\frac{q(z)}{s}(2\tau\gamma q(z) + \psi)\}\,q'(z) > 0,$$

and satisfies the next conditions

$$(\frac{Ip\,(n,\lambda)f\,(z)}{z^p})^{\frac{1}{\delta}} \in H[q(0), 1] \cap Q, \quad (4.6)$$

and

$$\frac{Ip\,(n,\lambda)f(z)}{z^p} \neq 0.$$

If the function r(z) is given by (3.9) is univalent in U ,

$$t + \psi q(z) + \tau\gamma q^2(z) + s\,\frac{zq'(z)}{q(z)} \prec r(z) \quad (4.7)$$

implies

$$q(z) \prec (\frac{Ip\,(n,\lambda)f(z)}{z^p})^{\frac{1}{\delta}}, \text{ and } q(z) \text{ is the best subordinant.}$$

Proof : Let the function k(z) defined on U by (4.1).

then a computation show that

$$\frac{zk'(z)}{k(z)} = \frac{1}{\delta}(p + \lambda)\,(\frac{Ip\,(n+1,\lambda)f(z)}{Ip\,(n,\lambda)f(z)} - 1). \quad (4.8)$$

By setting $\theta(w) = t + \psi\omega + \tau\gamma\omega^2$ and $\phi(w) = \frac{s}{\omega}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$ $w \in \mathbb{C}\backslash\{0\}$).

Also ,we get $Q(z) = zq'(z)\,\phi\,(q(z)) = s\,\frac{zq'(z)}{q(z)}$, it observed that Q(z) is starlike univalent in U, since q(z) is convex , it follows that

$$\text{Re}\,(\frac{z\theta'(q(z))}{\phi(q(z))}) = \text{Re}\,\{\frac{q(z)}{s}(2\tau\gamma q(z)) + \psi\}\,q'(z) > 0.$$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$\theta(q(z) + zq'(z)\,\phi\,(q(z))) = \theta\,(k(z) + zk'(z)\,\phi\,(k(z)))$, thus , by applying Lemma (2.3) , the proof is complete.

5- Sandwich Results :

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem.

Theorem (5.1): Let q_1 and q_1 be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1).Suppose that Re$\{\eta\} > 0$, η , $\delta \in \mathbb{C} \setminus \{0\}$. If $f \in A(p)$, such that

$$(\frac{t_1\,Ip\,(n,\lambda)f(z) + t_2\,Ip\,(n,\lambda)f(z)}{(t_1 + t_2)\quad z^p})^{\frac{1}{\delta}} \in H[\,q(0)\,, 1] \cap Q,$$

and the function G(z) defined by (3.3) is univalent and satisfies

$$q_1(z) + \delta\eta z q'_1(z) \prec G(z) \prec q_2(z) + \frac{\eta}{\delta} z q'_2(z), \qquad (5.1)$$

then

$$q_1(z) \prec \left(\frac{t_1\, Ip\,(n,\lambda)f(z) + t_2\, Ip\,(n,\lambda)f(z)}{(t_1+t_2)\;\; z^p}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

where q_1 and q_2 are respectively , the best subordinant and the best dominant of (5.1).

Combining Theorem (3.2) with Theorem (4.2), we obtain the following sandwich theorem.

Theorem (5.2): Let q_i be two convex univalent function in U , such that $q_i(0) = 1$ and $q_i(0) \neq 0$ (i=1,2).Suppose that q_1 and q_2 satisfies (3.8) and (4.8), respectively. If $f \in A(p)$ and suppose that f satisfies the next conditions :

$$\left(\frac{Ip\,(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \in H[Q(0), 1] \cap Q,$$

and

$$\frac{Ip\,(n,\lambda)f(z)}{z^p} \neq 0,$$

and r(z) is univalent in U , then

$$t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq'_1(z)}{q_1(z)} \prec t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq'_1(z)}{q_1(z)},$$

implies

$$q_1(z) \prec \left(\frac{Ip\,(n,\lambda)f(z)}{z^p}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

and q_1 and q_2 are the best subordinant and the best dominant respectively and r(z) is given by (3.9).

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