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ABSTRACT

Some relations using in new subclass of meromorphically ς -valent functions $TK(\delta,\gamma,\mu,\lambda)$ defined by integral operator involving $\flat$ -function. We get some thing properties, like, coefficient inequality $\flat_\varsigma^{\alpha,\beta}$, growth and distortion bounds, Partial sums, convex set, radii of starlikeness and radii convexity.

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Introduction.

$$Q(z)=\frac{1}{z^{\varsigma}}+\sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma}\ ;\ \ (t>\varsigma,\varsigma\in\mathbb{N}=\{1,2,\dots\}) \tag{1}$$

be class of functions denote by *TK*, including analytic and ς - valent in the punctured unit disk $\Delta^*=\{z: z\in\mathbb{C} \text{ and } o<|z|<1\}$

Also, denoted by $TK^+\subseteq TK$ of the form

$$Q(z)=\frac{1}{z^{\varsigma}}-\sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma}\quad (a_{t-\varsigma}\geq o,\varsigma\in\mathbb{N}) \tag{2}$$

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Definition (1): Saxena [12] introduced the $\flat$ - function as follows:

$$\flat(z)=\flat^{m,t}_{\varsigma i,qi:r}[z]=\flat^{m,t}_{\varsigma i,qi:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,t}(a_j,\alpha_j)_{t+1,\varsigma i}\\(b_j,\beta_j)_{1,m}(b_j,\beta_j)_{m+1,qi}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_c \sigma(s)z^s\,ds, \quad (3)$$

where

$$\sigma(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{t}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=n+1}^{qi}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=t+1}^{pj}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}$$

$\varsigma_i(i=1,2,3,\dots,r)$, $q_i(i=1,2,3,\dots,r), m,t$ are integers satisfying $o\le t\le \varsigma i, o\le m\le qi(i=1,2,3,\dots,r), r$ is finite $\alpha_i,\beta_i,\alpha_{ij},\beta_{ij}$ are real and positive and a_i,b_i,a_{ij},b_{ij} are complex numbers such that

$$\alpha_j(b_h+v)\neq\beta_j(a_h-1-w).$$

We define the integral operator $\flat^{\alpha,\beta}_{\varsigma}$ of $Q\in TK^+$ $for\ \alpha,\beta>0$

$$\flat^{\alpha,\beta}_{\varsigma}Q(z)=\frac{z^{\beta-\varsigma}}{\Gamma(\alpha-\beta)I^{m,t+1}_{\varsigma_i^{+1},q_i^{+1}:r}[z]}\int_0^z\sigma^{\varsigma-\alpha}(z-\sigma)^{\alpha-\beta-1}\flat^{m,t}_{\varsigma_i^{+1},q_i^{+1}:r}[\sigma]h(\sigma)d\sigma, \quad (4)$$

where

$$\flat^{m,t+1}_{\varsigma_i^{+1},q_i^{+1}:r}[z]=\flat^{m,t+1}_{\varsigma_i^{+1},q_i^{+1}:r}\left[z\left|\begin{matrix}(\alpha,1)(a_j,\alpha_j)\ \ (a_{ji},\alpha_{ji})\\(b_j,\beta_j)\ \ (b_{ji},\beta_{ji})(\beta,1)\end{matrix}\right.\right],$$ when

Real$(b_j)<Real(a_j)<1+\min_{1\le j\le m}Re(\frac{b_j}{\beta_j})$),

then $\flat^{\alpha,\beta}_{\varsigma}Q(z)$ can be written by

$$Q(z)=\frac{1}{z^{\varsigma}}+\sum_{t=1}^{\infty}a_{t-\varsigma}z^{t-\varsigma}\,.$$

So

$$\mathbb{I}_{\varsigma}^{\alpha,\beta} Q(z) = \frac{1}{z^{\varsigma}} + \sum_{t=1}^{\infty} \left[\frac{\mathbb{I}_{\varsigma_i^{+1},q_i^{+1}:r}^{m,t+1}\left[z \left| \begin{matrix} (\alpha-\varsigma-t,1) \ \ (a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta-\varsigma-k,1) \end{matrix} \right.\right]}{\mathbb{I}_{\varsigma_i^{+1},q_i^{+1};r}^{m,t+1}\left[z \left| \begin{matrix} (\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1) \end{matrix} \right.\right]} \right] a_{t-\varsigma} z^{t-\varsigma}$$

$$= \frac{1}{z^{\varsigma}} + \sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)\, a_{t-\varsigma} z^{t-\varsigma}, \qquad (5)$$

where

$$\psi(m,t,\alpha,\beta) = \frac{\mathbb{I}_{\varsigma_i^{+1},q_i^{+1}:r}^{m,t+1}\left[z \left| \begin{matrix} (\alpha-\varsigma-t,1) \ \ (a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta-\varsigma-t,1) \end{matrix} \right.\right]}{\mathbb{I}_{\varsigma_i^{+1},q_i^{+1}:r}^{m,t+1}\left[z \left| \begin{matrix} (\alpha,1)(a_j,\alpha_j)(a_{ji},\alpha_{ji}) \\ (b_j,\beta_j)(b_{ji},\beta_{ji})(\beta,1) \end{matrix} \right.\right]}. \qquad (6)$$

The I-function extension of Fox's H- function

Now,we study a subclass of a function (2) define below.

Definition (2): Let $Q \in TK^{+}$ given by (2). Then Q be in the class $TK\,(\delta,\gamma,\mu,\lambda)$ if it satisfy the condition:

$$\left| \frac{\frac{(-\varsigma-1)z\lambda\left(\mathbb{I}_{\varsigma}^{\alpha,\beta} Q(z)\right)'}{z^{-\varsigma}} - (\lambda\varsigma^2 + \lambda\varsigma)}{(-\varsigma^2-\varsigma) + \frac{z^2\left(\mathbb{I}_{\varsigma}^{\alpha,\beta} Q(z)\right)''}{z^{-\varsigma}} - \mu\gamma(\varsigma+\lambda)z^{\varsigma}} \right| < \delta, \qquad (7)$$

where z ∈ Δ^{*}; o≤ $\delta < \varsigma$, o $< \lambda, \gamma <$ 1, o $< \mu < \varsigma$ *and* $\varsigma \in \mathbb{N}$.

Several authors studied geometric properties of this function subclass for other classes, like, M. K. Aouf [2],[3], W. G. Atshan [4],[5],[6]and[7] and another authors [1],[8],[9],[10] and [11].

Theorem (1): Let $Q \in TK^{+}$. Then Q is in the class $TK\,(\delta,\gamma,\mu,\lambda)$ iff

$$\sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)\ \ a_{t-\varsigma} z^{t-\varsigma} \le \delta\mu\gamma(\varsigma+\lambda), \qquad (8)$$

where z ∈ Δ^{*}; o≤ $\delta < \varsigma$, o $< \lambda, \gamma <$ 1, o $< \mu < \varsigma$ *and* $\varsigma \in \mathbb{N}$.

For the function the result is sharp

$$Q(z)=\frac{1}{z^{\varsigma}}+\frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}\,z^{t-\varsigma}. \qquad (9)$$

Proof: By $|z|=1$, and let the inequality (8) true .Then, we get

$$\left|(-\varsigma-1)z\lambda\left(\mathfrak{b}_{\varsigma}^{\alpha,\beta}Q(z)\right)'-(\lambda\varsigma^2+\lambda\varsigma)z^{\varsigma}\right|-\delta\left|(-\varsigma^2-\varsigma)z^{\varsigma}-z^2\left(\mathfrak{b}_{\varsigma}^{\alpha,\beta}Q(z)\right)''-\mu\gamma(\varsigma+\lambda)\right|$$

$$=\left|\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)\lambda(-\varsigma-1)(t-\varsigma)\;a_{t-\varsigma}\,z^{t-\varsigma}\right|$$

$$-\delta\left|-\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)\,(t-\varsigma)(t-\varsigma-1)\,a_{t-\varsigma}z^{t-\varsigma}-\mu\gamma(\varsigma+\lambda)\right|$$

$$=\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma-1)(t-\varsigma)a_{t-}\varsigma\,z^{t-\varsigma}\;-\mu\gamma(\varsigma+\lambda)\le o,$$

by hypothesis. Thus by maximum modulus principle, the function

Q(z) ∈ TK (δ, γ, μ, λ).

Conversely. Let the function $Q(z)\in$ TK $(\delta,\gamma,\mu,\lambda)$. Then by condition (7), we have

$$\left|\frac{\frac{(-\varsigma-1)z\lambda\left(\mathfrak{b}_{\varsigma}^{\alpha,\beta}Q(z)\right)'}{z^{-\varsigma}}-(\lambda\varsigma^2+\lambda\varsigma)}{\frac{z^2\left(\mathfrak{b}_{\varsigma}^{\alpha,\beta}Q(z)\right)''}{z^{-\varsigma}}-\mu\gamma(\varsigma+\lambda)z^{\varsigma}+(-\varsigma^2-\varsigma)}\right|$$

$$=\left|\frac{\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)\lambda(-\varsigma-1)(t-\varsigma)\;a_{t-\varsigma}z^{t-\varsigma}}{\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)(t-\varsigma)(t-\varsigma-1)\,a_{t-\varsigma}z^{t-\varsigma}-\mu\gamma(\varsigma+\lambda)}\right|<\delta.$$

Since Real (z) ≤ $|z|$ for all z (z ∈ Δ*), we get

$$Re\left\{\frac{\sum_{k=1}^{\infty}\psi(m,t,\alpha,\beta)\lambda(-\varsigma-1)(t-\varsigma)\,a_{t-\varsigma}z^{t-\varsigma}}{\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)(t-\varsigma)(t-\varsigma-1)\,a_{t-\varsigma}z^{t-\varsigma}-\mu\gamma(\varsigma+\lambda)}\right\}<\delta. \quad (10)$$

by choose z on the real axis so that $z\left(\mathfrak{b}_{\varsigma}^{\alpha,\beta}Q(z)\right)'$ is real.

by clearing the denominator of inequality (10) and let z → 1^{-}, through real values

now we can write (10) as,

$$\sum_{t=1}^{\infty}\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)\;\;a_{t-\varsigma}z^{t-\varsigma}\le\delta\mu\gamma(\varsigma+\lambda).$$

Sharpness of the result follows by setting

$$Q(z) = z^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,y,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}\ z^{t-\varsigma}, (y \geq \varsigma). \qquad (11)$$

Corollary (1): Let $Q(z) \in TK\ (\delta,\gamma,\mu,\lambda)$.Then

$$a_{t-\varsigma} \leq \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}. \qquad (12)$$

Theorem (2): If $b_n \geq b_1 (n \geq 1)$ and $Q \in TK(\delta,\gamma,\mu,\lambda)$ then

$$\frac{1}{r^{\varsigma}} - \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(1-\varsigma)(-\varsigma)} r^{-\varsigma} \leq \left| b_{\varsigma}^{\alpha,\beta} h(z) \right|$$

$$\leq \frac{1}{r^{\varsigma}} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(1-\varsigma)(-\varsigma)} r^{-\varsigma}, (|z| = r < 1). \qquad (13)$$

The result is sharp:

$$Q(z) = z^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,y,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}\ z^{t-\varsigma}. \qquad (14)$$

Proof: Let $Q \in TK(\delta,\gamma,\mu,\lambda)$, then by Theorem (1), we get

$$\sum_{t=1}^{\infty} a_{t-\varsigma} \leq \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,1,\alpha,\beta)(\lambda+\delta)(1-\varsigma)(-\varsigma)}. \qquad (15)$$

then

$$\left| b_{\varsigma}^{\alpha,\beta} Q(z) \right| \leq |z|^{-\varsigma} + \sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)\, a_{t-\varsigma} |z|^{t-\varsigma}$$

$$\leq |z|^{-\varsigma} + \psi(m,1,\alpha,\beta)|z|^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$= r^{-\varsigma} + \psi(m,t,\alpha,\beta) r^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$\leq r^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)(-\varsigma)} r^{-\varsigma}. \qquad (16)$$

By the same way

$$\left|\mathfrak{b}_{\varsigma}^{\alpha,\beta} h(z)\right| \geq \frac{1}{|z|^{\varsigma}} - \sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)\, a_{t-\varsigma} |z|^{t-\varsigma}$$

$$\geq \frac{1}{|z|^{\varsigma}} - \psi(m,1,\alpha,\beta)|z|^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$= r^{-\varsigma} - \psi(m,1,\alpha,\beta) r^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$\leq r^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)(-\varsigma)} r^{-\varsigma} \tag{17}$$

we get (13) by (16) and (17).■

Theorem (3): Let $b_n \geq b_1 (n \geq 1)$ and $Q \in TK(\delta,\gamma,\mu,\lambda)$,

$$\frac{1}{\varsigma r^{\varsigma+1}} - \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)} r^{-\varsigma-1} \leq \left|\mathfrak{b}_{\varsigma}^{\alpha,\beta} Q(z)'\right|$$

$$\leq \frac{1}{\varsigma r^{\varsigma+1}} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)} r^{-\varsigma-1} \ ,(|z| = r < 1). \tag{18}$$

For the function the result is sharp

$$Q(z) = z^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)} z^{t-\varsigma}. \tag{19}$$

Proof: By Theorem (1) and let $Q \in TK(\delta,\gamma,\mu,\lambda)$, get

$$\sum_{t=1}^{\infty} a_{t-\varsigma} \leq \frac{\delta\mu\gamma(\varsigma+\lambda)}{\psi(m,1,\alpha,\beta)(\lambda+\delta)(1-\varsigma)(-\varsigma)}. \tag{20}$$

Hence

$$\left|\mathfrak{b}_{\varsigma}^{\alpha,\beta} Q(z)'\right| \leq \frac{1}{|-\varsigma z|^{\varsigma+1}} + \leq \sum_{t=1}^{\infty} (t-\varsigma) a_{t-\varsigma} |z|^{t-\varsigma-1}$$

$$\leq \frac{1}{|\varsigma z|^{\varsigma+1}} + \psi(m,1,\alpha,\beta)(-\varsigma)|z|^{-\varsigma-1} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$= \frac{1}{\varsigma r^{\varsigma+1}} + \psi(m,1,\alpha,\beta)(-\varsigma) r^{-\varsigma-1} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$\leq \frac{1}{\varsigma r^{\varsigma+1}} + \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)} r^{-\varsigma-1} \ . \tag{21}$$

By the same way

$$\left| \flat_{\varsigma}^{\alpha,\beta} Q(z)' \right| \geq \frac{1}{|-z\varsigma|^{\varsigma+1}} - \sum_{y=1}^{\infty} (t-\varsigma) a_{t-\varsigma} |z|^{t-\varsigma-1}$$

$$\geq \frac{1}{|z\varsigma|^{\varsigma+1}} - \psi(m,1,\alpha,\beta)(-\varsigma)|z|^{-\varsigma-1} \sum_{t=1}^{\infty} a_{t-\varsigma}$$

$$= \frac{1}{\varsigma r^{\varsigma+1}} - \psi(m,1,\alpha,\beta)(-\varsigma) r^{-\varsigma-1} \sum_{y=1}^{\infty} a_{t-\varsigma}$$

$$\geq \frac{1}{\varsigma r^{\varsigma+1}} - \frac{\delta\mu\gamma(\varsigma+\lambda)}{(\lambda+\delta)(t-\varsigma)} r^{-\varsigma-1} \tag{22}$$

we get (18), from (22) and (21) ■

Theorem (4): The partial sums is

S_1(z) and S_w(z) as follows S_1(z) $=z^{-\varsigma}$ and let $Q \in TK^+$ be given by (2) and

$$S_w(z) = z^{-\varsigma} + \sum_{t=1}^{w-1} a_{t-\varsigma} z^{t-\varsigma} \qquad (w \in \mathbb{N}\backslash\{1\}). \tag{23}$$

and suppose that

$$\sum_{t=1}^{\infty} c_{t-\varsigma}\ a_{t-\varsigma} \leq 1, \quad \left(c_{k-\varsigma} = \frac{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{\delta\mu\gamma(\varsigma+\lambda)} \right). \tag{24}$$

we have

$$Re\left(\frac{Q(z)}{s_w(z)}\right) > 1 - \frac{1}{c_w}, (w \in \mathbb{N}) \tag{25}$$

and

$$Re\left(\frac{s_w(z)}{Q(z)}\right) > \frac{c_w}{1 + c_w} \ \ (k \in \mathbb{N}). \tag{26}$$

The bounds in (25) and (26),the good possible for $t \in \mathbb{N}$

Proof: By (24), that

$$c_{t-\varsigma+1} > c_{t-\varsigma} > 1, (t \in \mathbb{N})$$

then, we have

$$\sum_{t=1}^{w-1} a_{t-\varsigma} + c_w \sum_{t=w}^{\infty} a_{t-\varsigma} \leq \sum_{y=1}^{\infty} \ c_{t-\varsigma} \ a_{t-\varsigma} \leq 1. \tag{27}$$

By

$$g_1(z) = c_w\left(\frac{h(z)}{s_w(z)} - \left(1 - \frac{1}{c_w}\right)\right) = 1 + \frac{c_w \sum_{n=t}^{\infty} \ a_{t-\varsigma} z^{t-\varsigma+1}}{1 + \sum_{t=1}^{w-1} \ a_{t-\varsigma} z^{t-\varsigma+1}}, \tag{28}$$

then show that

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq 1, \ \ (z \in \Delta^*) \tag{29}$$

by applying (28)

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq \frac{c_w \sum_{n=t}^{\infty} \ a_{t-\varsigma}}{2 - 2\sum_{y=1}^{w-1} a_{t-\varsigma} - c_w \sum_{t=w}^{\infty} \ a_{t-\varsigma}}, \tag{30}$$

which yields (25), if we take

$$Q(z) = z^{-\varsigma} - \frac{z^{w-\varsigma}}{c_w}, \tag{31}$$

So, (25) is the good possible for each $k \in \mathbb{N}$.

By the same way

$$g_2(z) = (1+c_k)\left(\frac{s_w(z)}{h(z)} - \frac{c_w}{1+c_w}\right) = 1 - \frac{(1+c_w)\sum_{w=t}^{\infty} a_{t-\varsigma} z^{t-\varsigma+1}}{1+\sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma+1}}, \qquad (32)$$

and use of (27), we obtain

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+c_k)\sum_{k=w}^{\infty} a_{t-\varsigma}}{2-2\sum_{t=1}^{w-1} a_{t-\varsigma} - (1-c_w)\sum_{n=t}^{\infty} a_{t-\varsigma}} \le 1\,, \qquad (33)$$

which leads us to (26). The bound in (26) is sharp for each $w \in \mathbb{N}$, when function is given by (31). ■

Theorem (5): Convex set is the class $TK\ (\delta, \gamma, \mu, \lambda)$.

Proof: For all Q_1 and $Q_2 \in TK\ (\delta, \gamma, \mu, \lambda)$, and for every ƺ $(o \le ƺ \le 1)$.

To prove $(1-ƺ)Q_1 + ƺQ_2 \in TK\ (\delta, \gamma, \mu, \lambda)$.

Thus,

$$(1-ƺ)Q_1 + ƺQ_2 = z^{-\varsigma} + \sum_{t=1}^{\infty}\left[(1-ƺ)a_{t-\varsigma} + bƺ_{t-\varsigma}\right] z^{t-\varsigma}.$$

then,

$$\sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)\left[(1-ƺ)a_{t-\varsigma} + ƺb_{t-\varsigma}\right]$$

$$= (1-ƺ)\sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)a_{t-\varsigma}$$

$$+ƺ\sum_{t=1}^{\infty} \psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)b_{t-\varsigma}$$

$$\le (1-ƺ)\delta\mu\gamma(\varsigma+\lambda) + ƺ\delta\mu\gamma(\varsigma+\lambda) = \delta\mu\gamma(\varsigma+\lambda). \ ■$$

Theorem (6): If the function $Q(z) \in TK(\delta, \gamma, \mu, \lambda)$, then function $Q(z)$ is ς -valent meromorphic starlike of order $\varphi(0 \le \varphi < \varsigma)$ in the disk $|z| < r$,if

$$r = inf_n \left\{ \frac{(\varsigma - \varphi)}{(n + \varsigma - \varphi)} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} a_{t-\varsigma} \right\}^{\frac{1}{n}}, n \geq \varsigma.$$

For the function $Q(z)$ the result is sharp.

Proof: We need to show that

$$\left| \frac{zQ'(z) + \varsigma Q(z)}{Q(z)} \right| \leq \varsigma - \varphi \quad \text{for } |z| < r. \tag{34}$$

Therefore

$$\left| \frac{\sum_{t=1}^{\infty}(t) a_{t-\varsigma} z^n}{1 + \sum_{t=1}^{\infty} a_{t-\varsigma} z^n} \right| \leq \frac{\sum_{t=1}^{\infty}(t) a_{t-\varsigma} |z|^n}{1 - \sum_{t=1}^{\infty} a_{t-\varsigma} |z|^n}$$

So, (34) satisfied if

$$\frac{\sum_{t=1}^{\infty}(t) a_{t-\varsigma} |z|^n}{1 - \sum_{t=1}^{\infty} a_{t-\varsigma} |z|^n} \leq \varsigma - \varphi,$$

or satisfied if

$$\sum_{t=1}^{\infty} \frac{(t + \varsigma - \varphi) a_{t-\varsigma}}{\varsigma - \varphi} |z|^n \leq 1. \tag{35}$$

Since $Q(z) is\ in\ the\ class\ TK(\delta, \gamma, \mu, \lambda)$, then

$$\sum_{t=1}^{\infty} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} \ a_{t-\varsigma} \leq 1.$$

So, (35) true if

$$\frac{(n + \varsigma - \varphi)}{\varsigma - \varphi} |z|^n \leq \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)}$$

or equivalent

$$|z| \leq \left\{\frac{(\varsigma-\varphi)}{(n+\varsigma-\varphi)} \frac{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{\delta\mu\gamma(\varsigma+\lambda)} \; a_{t-\varsigma}\right\}^{\frac{1}{n}}, n \geq \varsigma,$$

Setting $|z| = r$.■

Theorem (7): If $Q(z) \in TK(\delta,\gamma,\mu,\lambda)$, then $Q(z)$ is ς -valent meromorphic convex of order $\varphi(0 \leq \varphi < \varsigma)$ in the disk $|z| < \mathfrak{r}$,where

$$\mathfrak{r} = inf_n \left\{\frac{\varsigma(\varsigma-\varphi)\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{(t+\varsigma-\varphi)\delta\mu\gamma(\varsigma+\lambda)}\right\}^{\frac{1}{n+\varsigma}}, n \geq \varsigma.$$

For the function $Q(z)$ the result is sharp.

Proof: We need to prove

$$\left|(\varsigma+1) + \frac{zQ''(z)}{Q'(z)}\right| \leq \varsigma - \varphi \quad \text{for } |z| < \mathfrak{r}. \tag{36}$$

So,

$$\left|\frac{zQ''(z) + (1+\varsigma)Q'(z)}{Q'(z)}\right| \leq \frac{\sum_{y=1}^{\infty} t(t-\varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}{\varsigma - \sum_{t=1}^{\infty}(t-\varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}.$$

(36) satisfied if

$$\frac{\sum_{t=1}^{\infty} t(t-\varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}{\varsigma - \sum_{t=1}^{\infty}(t-\varsigma)\, a_{t-\varsigma}|z|^{t-\varsigma}} \leq \varsigma - \varphi,$$

or if

$$\sum_{t=1}^{\infty} \frac{(t+\varsigma-\varphi)a_{t-\varsigma}}{\varsigma(\varsigma-\varphi)} |z|^{t-\varsigma} \leq 1. \tag{37}$$

Since $(z) \in TK(\delta,\gamma,\mu,\lambda)$, , we have

$$\sum_{t=1}^{\infty} \frac{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{\delta\mu\gamma(\varsigma+\lambda)} \; a_{t-\varsigma} \leq 1.$$

Hence, (37) will be true if

$$\frac{(t+\varsigma-\varphi)}{\varsigma(\varsigma-\varphi)}|z|^{t-\varsigma} \leq \frac{\psi(m,t,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{\delta\mu\gamma(\varsigma+\lambda)},$$

or equivalently $a_{t-\varsigma}$

$$|z| \leq \left\{\frac{\varsigma(\varsigma-\varphi)\psi(m,y,\alpha,\beta)(\lambda+\delta)(t-\varsigma)(t-\varsigma-1)}{(t+\varsigma-\varphi)\delta\mu\gamma(\varsigma+\lambda)}\right\}^{\frac{1}{n}}, n \geq \varsigma,$$

put $|z| = r$.■

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