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Legendre Collocation for Six Order Boundary Value Problems

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Abstract

The study aims to employ Legendre polynomials as basis functions for numerically solving six-order Boundary Value Problems (BVPs). Through the utilization of collocation points, the method transforms the differential problem into a system of algebraic equations, subsequently solved through matrix inversion. The research delves into the convergence and error analysis of this technique and validates its effectiveness through discussion of three test problems. The proposed approach provides highly accurate results, closely approximating the exact solution. The study illustrates its precision and efficacy through tables and figures, showcasing its superiority over other methods discussed in the literature.

Keywords: Legendre polynomials, Boundary value problems, Collocation approach approximate solution

1. Introduction

In the realm of differential equations, a Boundary Value Problem (BVP) in mathematics involves both a differential equation and a set of additional constraints known as boundary conditions. A differential equation that meets these boundary conditions and has a solution is categorized as a boundary value problem. Given that nearly all physical differential equations give rise to boundary value problems, these problems emerge in various branches of physics (see Tables 1–3, Figs. 1–3).

Boundary Value Problems (BVPs) provide a prevalent framework for addressing wave equations, such as the determination of normal modes. The realm of Sturm-Liouville problems encompasses a substantial class of significant BVPs. The exploration of these problems often involves employing the eigenfunctions of a differential operator.

Numerous numerical techniques have been developed to tackle BVPs [16]. Some notable researchers have offered numerical solutions for such problems. For instance, the Daftardar Jafari approach was utilized by Ullah et al. [26] to analyze numerical solutions for fifth and sixth-order nonlinear boundary value problems. Solutions for sixth-order BVPs were attained by Noor et al. [18,19] using the variational iteration method and the homotopy perturbation approach.

In the realm of two-point BVPs, Caglar et al. [5] introduced B-spline interpolation as an alternative to finite difference, finite element, and finite volume methods. To address sixth-order boundary value problems, a non-polynomial spline approach was employed by (Akram & Siddiqi [1]; Tirmizi & Khan [24]; Pervaiz et al. [21]). The optimal homotopy asymptotic approach was applied to a unique sixth-order BVP by Idrees et al. [8].

Researchers like (Lang & Xu [14]; Kalyani et al. [10]; Khalid & Naeem [13]; Khalid et al., [13]) tackled linear fifth-order and linear/nonlinear sixth-order BVPs using the cubic B-spline approach. Subdivision methods based on collocation algorithms were explored by (Arshed & Hussain, [3]; Mustafa & Ejaz,

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Table 1. displays numerical results for example 5.1 with $n = 14$.

υ	Exact	Approximate solution	Absolute error of the proposed method	Absolute error of [12]
0.0	1.00000000000000	0.99999996230000	3.770E-08	–
0.1	0.99465382620000	0.99465383440000	8.300E-09	1.18E-05
0.2	0.97712220640000	0.97712222320000	1.670E-08	4.29E-05
0.3	0.94490116560000	0.94490117280000	7.100E-09	8.53E-05
0.4	0.89509481880000	0.89509481460000	4.300E-09	1.28E-04
0.5	0.82436063550000	0.82436062550000	1.050E-08	1.59E-04
0.6	0.72884752000000	0.72884751160000	7.900E-09	1.67E-04
0.7	0.60412581210000	0.60412581070000	1.800E-09	1.45E-04
0.8	0.44510818560000	0.44510819170000	5.900E-09	9.47E-05
0.9	0.24596031110000	0.24596031180000	1.100E-09	3.33E-05
1	0.00000000000000	−0.00000004270000	4.270E-08	–

Table 2. displays numerical results for Example 5.2 with $n = 14$.

υ	Exact	Approximate solution	Absolute error of the proposed method	Absolute error of [12]
0.0	0.00000000000000	−0.00000015184594	1.518E-07	–
0.1	0.09946538262000	0.09946530719000	7.543E-08	3.81E-05
0.2	0.19542444130000	0.19542434100000	1.005E-07	1.59E-04
0.3	0.28347034970000	0.28347026090000	8.892E-08	3.41E-04
0.4	0.35803792750000	0.35803790490000	2.261E-08	5.33E-04
0.5	0.41218031780000	0.41218037250000	5.454E-08	6.74E-04
0.6	0.43730851200000	0.43730860750000	9.541E-08	7.08E-04
0.7	0.42288806850000	0.42288814990000	8.135E-08	6.08E-04
0.8	0.35608654850000	0.35608658720000	3.865E-08	3.91E-04
0.9	0.22136428000000	0.22136433140000	5.14Ee-08	1.35E-04
1	0.00000000000000	0.00000032679919	3.268E-07	–

Table 3. displays numerical results for Example 5.3 with $n = 14$.

υ	Exact	Approximate solution	Absolute error of the proposed method	Absolute error of [12]
0.0	0.00000000000000	−0.00000001669157	1.669E-08	–
0.1	0.00072900000000	0.00072942425740	4.243E-07	2.25E-04
0.2	0.00409600000000	0.00409464342700	1.357E-06	7.36E-04
0.3	0.00926100000000	0.00925694659700	4.053E-06	1.28E-03
0.4	0.01382400000000	0.01381737987000	6.620E-06	1.68E-03
0.5	0.01562500000000	0.01561687978000	8.120E-06	1.83E-03
0.6	0.01382400000000	0.01381612725000	7.873E-06	1.68E-03
0.7	0.00926100000000	0.00925521757400	5.782E-06	1.28E-03
0.8	0.00409600000000	0.00409338577300	2.614E-06	7.36E-04
0.9	0.00072900000000	0.00072923755920	2.376E-07	2.25E-04
1	0.00000000000000	0.00072923755920	2.265E-06	–

[17]; Mustafa et al., [17]; Manan et al. [15]) to solve third, fourth, and sixth-order BVPs.

Godspower & Ogeh [6] and Tsetimi et al. [25] applied canonical and Chebyshev polynomials within the modified variational iteration method to address seventh and twelfth-order problems. Kanwal et al. [11] elucidated the subdivision technique used for solving two-point BVPs. Oyedepo et al. [20] applied Chebyshev Computational Approach for Volterra-Fredholm integro-differential equations [23].

Further exploration of Legendre polynomials approximation was undertaken by (Sohaib et al. [22]. Al-Shaher & Mechee [2]; Islam et al. [9], Heydari et al. [7]; Ray & Gupta [22]; Xu & Xu [27] and Ayinde et al. [4] to analyze initial value problems, parabolic partial differential equations, fractional Poisson equation with Dirichlet boundary conditions, time-fractional generalized seventh order KdV equation, fractional order differential equations under multi-point boundary conditions, and Collocation approximation methods for solving higher-order ordinary differential equations.

This study is focused on investigating numerical solutions for sixth-order ordinary differential

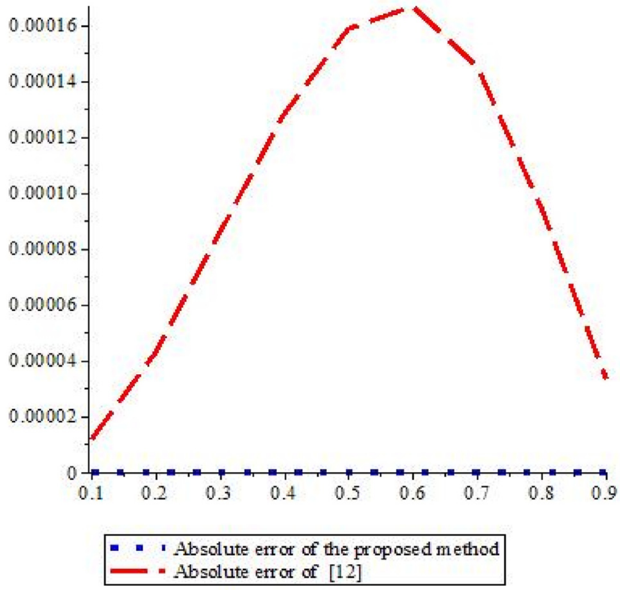

Fig. 1. depicts comparison of the absolute errors of example 1.

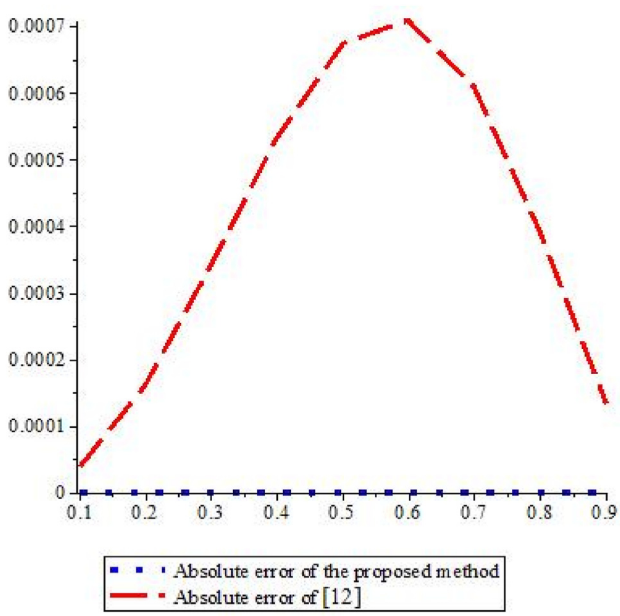

Fig. 2. depicts comparison of the absolute errors of example 5.2.

equations, which find applications in various scientific and engineering disciplines. The method employed utilizes Legendre polynomials as basis functions. The accuracy and efficacy of this technique are demonstrated through several numerical implementations. The literature contains numerous additional methods for addressing sixth-order BVPs.

This work take into consideration the numerical solution of 6th order BVP:

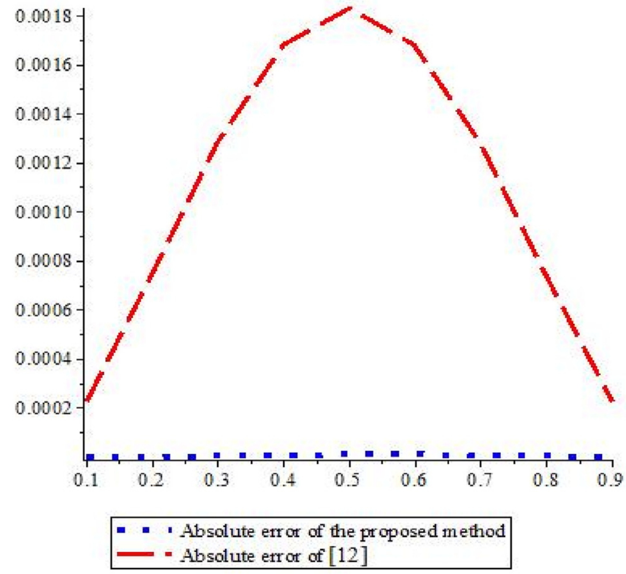

Fig. 3. Depicts comparison of the absolute errors of example 5.3.

$$\begin{aligned}&\zeta^{vi}(v)+\mu_1(v)\zeta^{v}(v)+\mu_2(v)\zeta^{iv}(v)+\mu_3(v)\zeta^{iii}(v)\\&\quad+\mu_4(v)\zeta^{ii}(v)+\mu_5(v)\zeta^{i}(v)+\mu_6(v)\zeta(v)\\&\quad=g(v),v\in[a,b]\end{aligned}\tag{1.1}$$

with boundary conditions

$$\zeta^i(a)=\alpha_i,\zeta^i(b)=\beta_i,i=0,1,2,\tag{1.2}$$

Here, α_0 , α_1 , α_2 and β_0 , β_1, β_2 represent provided real constants. $\mu_i(v)$, where i ranges from 0 to n, as well as $g(v)$, are known functions defined on the interval $[a, b]$. The unknown function to be determined is denoted as $\zeta(v)$.

2. Materials and method

Definition 2.1. Legendre's polynomial of degree n is denoted and defined by

$$\omega_n(v)=\sum_{j=0}^{\left[\frac{n}{2}\right]}(-1)^j\frac{(2n-2j)!}{2^n.j!(n-j)!(n-2j)!}v^{n-2j}$$

Where

$$\left[\frac{n}{2}\right]=\begin{cases}\dfrac{n}{2} & if\ \ n\,is\,even\\ \dfrac{n-1}{2}, & if\ n\,is\,odd\end{cases}$$

and the recurrence relation

$$n\omega_n(v) = (2n-1)v\omega_{n-1}(v) - (n-1)\omega_{n-2}(v); n \geq 2, \text{starting with}$$

$$\omega_0(v) = 1, \omega_1(v) = v$$

Hence, the first few of Legendre Polynomials on the interval [−1,1] is given below:

$$\left.\begin{aligned} \omega_0(\nu) &= 1 \\ \omega_1(\nu) &= \nu \\ \omega_2(\nu) &= \frac{1}{2}\left(3\nu^2 - 1\right) \\ \omega_3(\nu) &= \frac{1}{2}\left(5\nu^3 - 3\nu\right) \end{aligned}\right\} \tag{2.1}$$

"The shifted equivalent of it, denoted as $\omega_n{}^*(\nu)$, that valid in $\in [0,1]$ are given as: $\omega_n{}^*(\nu) = \omega_n(2\nu - 1)$, $n = 2, 3, \cdots$ with initial $\omega_0{}^*(\nu) = 1$, $\omega_1{}^*(\nu) = 2\nu - 1$." Hence, the first few of shifted Legendre Polynomials on the interval [0,1] is given below:

$$\left.\begin{aligned} \omega_0^*(\nu) &= 1 \\ \omega_1^*(\nu) &= 2\nu - 1 \\ \omega_2^*(v) &= 6v^2 - 6v + 1 \\ \omega_3^*(v) &= 20v^3 - 30v^2 + 20v - 1 \end{aligned}\right\} \tag{2.2}$$

Definition 3. Absolute Error: Absolute Error $= |\zeta(\nu) - \zeta_n(\nu)|$; $0 \leq \nu \leq 1$; $0 \leq \nu \leq 1$, where $\zeta(\nu)$ represents the exact solution and $\zeta_n(\nu)$ represents the approximate solution.

3. Proposed method

The study employed an approximate solution using shifted Legendre Polynomials in the following form:

$$\zeta_n(\nu) = \sum_{i=0}^{n} \omega_i(\nu) a_i \tag{3.1}$$

The unknown constants to be determined are denoted as a_i, where i ranges from 0 to n. Therefore, by differentiating equation (5) n times with respect to ν and substituting the resulting solution into equation (1.1), we obtain:

$$\sum_{i=0}^{n} \omega_i{}^{vi}(v) a_i + \mu_1(\nu) \sum_{i=0}^{n} \omega_i{}^{v}(\nu) a_i + \mu_2(\nu) \sum_{i=0}^{n} \omega_i{}^{iv}(\nu) a_i + \mu_3(\nu) \sum_{i=0}^{n} \omega_i{}^{iii}(\nu) a_i + \mu_4(v) \sum_{i=0}^{n} \omega_i{}^{ii}(v) a_i + \mu_5(v) \sum_{i=0}^{n} \omega_i{}^{i}(\nu) a_i + \mu_6(\nu) \sum_{i=0}^{n} \omega(\nu) a_i = g(\nu). \tag{3.2}$$

Let $\eta(\nu) = \sum_{i=0}^{n} \omega_i{}^{vi}(\nu) a_i, \eta^*(\nu) = \sum_{i=0}^{n} \omega_i{}^{v}(\nu) a_i, \xi(\nu) = \sum_{i=0}^{n} \omega_i{}^{iv}(\nu) a_i, \xi^*(\nu) = \sum_{i=0}^{n} \omega_i{}^{iii}(\nu) a_i, \chi(\nu) = \sum_{i=0}^{n} \omega_i{}^{ii}(\nu) a_i, \chi^*(\nu) = \sum_{i=0}^{n} \omega_i{}^{i}(\nu) a_i, \tau(\nu) = \sum_{i=0}^{n} \omega_i(\nu) a_i$

Thus, (3.2) becomes

$$\eta(\nu) + \mu_1(\nu)\eta^*(\nu) + \mu_2(\nu)\xi(\nu) + \mu_3(\nu)\xi^*(\nu) + \mu_4(v)\chi(\nu) + \mu_5(v)\chi^*(\nu) + \mu_6\tau(\nu) = g(\nu) \tag{3.3}$$

The set of equations in linear algebra with (n+1) unknown constants, denoted as constants a_i's, is derived by placing Eq. (3.3) at regularly spaced positions defined by point $\nu_i = a + \frac{(b-a)i}{n}$, where $(i = 0(1)n)$. Further equations are acquired from Eq. (1.2), and these are formulated using matrix representation:

$$\begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \cdots & \psi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{m1} & \psi_{m2} & \psi_{m3} & \psi_{m4} & \cdots & \psi_{mn} \\ {\psi_{11}}^0 & {\psi_{12}}^0 & {\psi_{13}}^0 & {\psi_{14}}^0 & \cdots & {\psi_{1n}}^0 \\ {\psi_{21}}^1 & {\psi_{22}}^1 & {\psi_{23}}^1 & {\psi_{24}}^1 & \cdots & {\psi_{2n}}^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {\psi_{m1}}^n & {\psi_{m2}}^n & {\psi_{m3}}^n & {\psi_{m4}}^n & \cdots & {\psi_{mn}}^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \varrho_{11} \\ \varrho_{21} \\ \vdots \\ \vdots \\ \varrho_{mn} \\ {\varrho_{11}}^0 \\ {\varrho_{22}}^1 \\ \vdots \\ \vdots \\ {\varrho_{mn}}^n \end{pmatrix} \tag{3.4}$$

where $\psi_i's$ and ${\psi_i}^{0'}s$ are the coefficients of $a_i's$ .given as

and $\varrho_i's$ are values of $g(\nu_i)$. Subsequently, the matrix inversion method is employed to solve the system of equations and determine the unknown constants.

The necessary approximate solution is derived by solving Equation (3.6) and subsequently substituting the values of the unknown constants into the assumed approximate solution.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \cdots & \psi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{m1} & \psi_{m2} & \psi_{m3} & \psi_{m4} & \cdots & \psi_{mn} \\ {\psi_{11}}^0 & {\psi_{12}}^0 & {\psi_{13}}^0 & {\psi_{14}}^0 & \cdots & {\psi_{1n}}^0 \\ {\psi_{21}}^1 & {\psi_{22}}^1 & {\psi_{23}}^1 & {\psi_{24}}^1 & \cdots & {\psi_{2n}}^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {\psi_{m1}}^n & {\psi_{m2}}^n & {\psi_{m3}}^n & {\psi_{m4}}^n & \cdots & {\psi_{mn}}^n \end{pmatrix}^{-1} \begin{pmatrix} \varrho_{11} \\ \varrho_{21} \\ \vdots \\ \vdots \\ \varrho_{mn} \\ {\varrho_{11}}^0 \\ {\varrho_{22}}^1 \\ \vdots \\ \vdots \\ {\varrho_{mn}}^n \end{pmatrix} \tag{3.5}$$

4. Error analysis

Theorem 4.1. Consider a compact operator $\mathfrak{H} : \mathscr{B}^*(\Omega) \to \mathscr{B}^*(\Omega)$ defined over the interval $\Omega \in [a, b]$, and let the equation

$$(I - \mathfrak{H})\zeta = g \tag{4.1}$$

Have a solution. Assume that the projection $\mathscr{L}_n : \mathscr{B}^*(\Omega) \to V_n$ satisfies $\|\mathscr{L}_n\mathfrak{H} - \mathfrak{H}\| \to 0, n \to \infty$.
Then, for sufficiently large n, the approximate

$$\zeta_n - \mathscr{L}_n\mathfrak{H}\zeta_n = \mathscr{L}_n g \tag{4.2}$$

has a unique solution for all $g \in \mathscr{B}^*(\Omega)$ and there holds an error estimate

$$a_0 = 0.718281820416451, a_1 = -0.464536480605602, a_2 = -0.214771471917629, a_3 = -0.0352141633629799,$$
$$a_4 = -0.00349662452936172, a_5 = -0.000248782336711884, a_6 = -0.0000137928873300552,$$
$$a_7 = -6.28642737865448 \times 10^{-7}, a_8 = -2.42143869400024 \times 10^{-8}, a_9 = -9.31322574615479 \times 10^{-10},$$
$$a_{10} = -5.82076609134674 \times 10^{-11}, a_{11} = -7.27595761418343 \times 10^{-12}, a_{12} = -1.81898940354586 \times 10^{-12},$$
$$a_{13} = -5.68434188608080 \times 10^{-14}, a_{14} = -2.66453525910038 \times 10^{-15}$$

$$\|\zeta - \zeta_n\| \le n\|\zeta - \mathscr{L}_n\zeta\|, \tag{4.3}$$

where M is a positive constant depending on $\mathfrak{H}$.

Proof. For sufficient large n, it is known that the operator $(I - \mathscr{L}_n\mathfrak{H})^{-1}$ exists and is uniformly bounded. To establish the error bound, we apply the projection operator $\mathscr{L}_n$ to equation (4.2) and obtain:

$$\mathscr{L}_n\zeta - \mathscr{L}_n\mathfrak{H}\zeta = \mathscr{L}_n g \tag{4.4}$$

This can be rewritten as:

$$\zeta - \mathscr{L}_n\mathfrak{H}\zeta = \mathscr{L}_n g + \zeta - \mathscr{L}_n\zeta \tag{4.5}$$

Subtracting equation (4.4) from equation (4.2), yields:

$$\zeta_n - \mathscr{L}_n\mathfrak{H}\zeta_n - (\zeta - \mathscr{L}_n\mathfrak{H}\zeta) = \mathscr{L}_n g - (\mathscr{L}_n g + \zeta - \mathscr{L}_n\zeta) \tag{4.6}$$

Simplifying, we get:

$$(I - \mathscr{L}_n\mathfrak{H})\zeta - (I - \mathscr{L}_n\mathfrak{H})\zeta_n = \zeta - \mathscr{L}_n\zeta \tag{4.7}$$

$$(I - \mathscr{L}_n\mathfrak{H})(\zeta - \zeta_n) = (I - \mathscr{L}_n)\zeta \tag{4.8}$$

Hence, the estimation in equation (4.3) follows.

5. Numerical examples and results

Example 5.1. [12]: Consider the sixth Order BVP

$$\zeta^6(\upsilon) - \zeta(\upsilon) = -6e^{-\upsilon}\zeta(\upsilon), 0 \le \upsilon \le 1,$$

Subject to the boundary conditions

$$\zeta(0) = 1, \zeta'(0) = 0, \zeta''(0) = -1$$

$$\zeta(1) = 0, \zeta'(1) = e, \zeta''(1) = -2e$$

With the exact solution $\zeta(\upsilon) = (1 - \upsilon)e^{\upsilon}$
The unknown constants were determined using the method described above, and the results are as follows:

Thus, (as in Table 1 and see Figure 1) the approximate solution is given as;

$$\begin{aligned}\zeta(\upsilon) = {} & 0.9999999623 + 6.925868010 \times 10^{-7}\upsilon \\ & - 0.5000024239\upsilon^2 - 0.3333339260\upsilon^3 \\ & - 0.1249832505\upsilon^4 - 0.03336688397\upsilon^5 \\ & - 0.006923016995\upsilon^6 - 0.001168977203\upsilon^7 \\ & - 0.0002328489082\upsilon^8 + 0.00004304310518\upsilon^9 \\ & - 0.00004678811941\upsilon^{10} + 0.00001763611474\upsilon^{11} \\ & - 0.000003417722985\upsilon^{12} + 1.570390040 \times 10^{-7}\upsilon^{13} \\ & - 3.106848112 \times 10^{-10}\upsilon^{14}\end{aligned}$$

Example 5.2. [12]: Consider the sixth Order BVP (as in Table 2 and see Figure 2)

$$\zeta^6(t) = -\upsilon\zeta(\upsilon) - (24 + 11\upsilon + \upsilon^3)e^{\upsilon}, 0 \le \upsilon \le 1,$$

Subject to the boundary conditions

$$\zeta(0) = 0, \zeta'(0) = 1, \zeta''(0) = 0$$

$$\zeta(1) = 0, \upsilon'(1) = e, \zeta''(1) = -4e,$$

with the exact solution $\zeta(\upsilon) = \upsilon(1 - \upsilon)e^{\upsilon}$

The unknown constants were determined using the method described above, and the results are as follows:

$$a_0=0.281718105077744, a_1=0.0839183330535889, a_2=-0.269777357578278, a_3=-0.0828157663345337,$$
$$a_4=-0.0118660926818848, a_5=-0.00109884142875671, a_6=-0.0000749528408050537,$$
$$a_7=-0.00000404566526412964, a_8=-1.82539224624634\times10^{-7}, a_9=-8.84756445884705\times10^{-9},$$
$$a_{10}=-4.65661287307739\times10^{-10}, a_{11}=-1.45519152283669\times10^{-11}, a_{12}=-7..27595761418343\times10^{-12},$$
$$a_{13}=-6.82121026329696\times10^{-13}, a_{14}=-2.84217094304040\times10^{-14}$$

Thus, the approximate solution is given as;

$$\begin{aligned}\zeta(v)= & -1.518459378\times10^{-7}+1.000002086v\\ & -0.00001830735487v^2-0.4999456807v^3\\ & -0.3333551221v^4-0.1251502944v^5\\ & -0.03308507361v^6-0.006986498775v^7\\ & -0..001451326255v^8+0.0001399675733v^9\\ & -0.0001727080656v^{10}+0.00002109216155v^{11}\\ & +0.00000146027798v^{12}+8.86808494\times10^{-7}v^{13}\\ & -3.313971320\times10^{-9}v^{14}\end{aligned}$$

Example 5.3. [12]: Consider the sixth Order BVP (as in Table 3 and see Figure 3)

$$\zeta^6(v)=-e^{-v}\zeta(v)-720+\left(v-v^2\right)^3e^{-v}, 0\le v\le1,$$

Subject to the boundary conditions

$$v(0)=0, v'(0)=0, v''(0)=0$$

$$v(1)=0, v'(1)=0, v''(1)=0,$$

with the exact solution $\zeta(t)=v^3(1-v)^3$
The unknown constants were determined using the method described above, and the results are as follows:

$$\begin{aligned}&a_0=0.00714242458343506,\\&a_1=0.0000762939453125000,\\&a_2=-0.0118875503540039,\\&a_3=0.0000109672546386719,\\&a_4=0.00584840774536133,\\&a_5=0.00000405311584472656,\\&a_6=-0.00107979774475098,\\&a_7=0.00000113248825073242,\\&a_8=3.57627868652344\times10^{-7},\\&a_9=9.31322574615479\times10^{-8},\\&a_{10}=1.95577740669250\times10^{-8},\\&a_{11}=3.14321368932724\times10^{-9},\\&a_{12}=4.3.14321368932724\times10^{-10},\\&a_{13}=3.27418092638254\times10^{-8},\\&a_{14}=1.13686837721616\times10^{-12}\end{aligned}$$

Thus, the approximate solution is given as;

$$\begin{aligned}\zeta(v)= & -1.669157495\times10^{-8}+0.00002136275495v\\ & -0.0002067259486v^2+1.000422636v^3\\ & -3.000556988v^4+3.000440453v^5-0.9991566623v^6\\ & -0.002902557956v^7+0.004184446926v^8\\ & -0.003888552873v^9+0.002189574906v^{10}\\ & -0.000571653721v^{11}+0.0000055336849v^{12}\\ & +0.0000212834039v^{13}+1.325588528\times10^{-7}v^{14}\end{aligned}$$

References

[1] Akram G, Siddiqi SS. Solution of sixth order boundary value problems using non-polynomial spline technique. Appl Math Comput 2006;181(1):708−20. https://doi.org/10.1016/j.amc.2006.01.053.

[2] Al-Shaher OI, Mechee MS. A study of Legendre polynomials approximation for solving initial value problems. J Phys Conf 2021;1897:1−9. https://doi.org/10.1088/1742-6596/1897/1/012058.

[3] Arshed AF, Hussain I. Solution of sixth-order boundary value problems by collocation method Haar wavelets. Int J Phys Sci 2012;7(43):5729−35. https://doi.org/10.5897/IJPS11.696.

[4] Ayinde MA, Ishaq AA, Oyedepo T, Alkali MA. Collocation approximation methods for solving higher-order ordinary differential equations. Int J Res Civil Eng Technol 2022;3(1): 26−35.

[5] Caglar H, Caglar N, Elfaituri K. B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems. Appl Math Comput 2006;175(1):72−9. https://doi.org/10.1016/j.amc.2005.07.019.

[6] Godspower CA, Ogeh KO. Modified variational iteration method for solution of seventh order boundary value problem Using Canonical Polynomials. Int J Modern Math Sci 2019;17(1):57−67.

[7] Heydari HM, Hooshmandasl MR, M Ghaini FM, Fereidouni F. Two-dimensional Legendre wavelets for solving fractional Poisson equation with Dirichlet boundary conditions. Eng Anal Bound Elem 2013;37:1331−8. https://doi.org/10.1016/j.enganabound.2013.07.002.

[8] Idrees M, Haq S, Islam S. Application of Optimal homotopy asymptotic method to special sixth order boundary value problems. World Appl Sci J 2010;9(2):138−43.

[9] Islam SU, Aziz I, Fhaid ASA, Shah A. A numerical assessment of parabolic partial differential equations using Haar and Legendre wavelets. Appl Math Model 2013;37(23): 9455−81. https://doi.org/10.1016/j.apm.2013.04.014. dio:.

[10] Kalyani P, Rama PS, Chandra R. Solution of boundary value problems by approaching spline techniques. Hindawi Publishing Corporation International Journal of Engineering Mathematics; 2013. p. 1−9. https://doi.org/10.1155/2013/482050I. Volume 2013, Article ID 482050.

[11] Kanwal G, Ghaffar A, Hafeezullah MM, Manan SA, Rizwan M, Rahman G. Numerical solution of 2-point boundary value problem by subdivision scheme. Commun Math Appl 2019;10(1):19−29. https://doi.org/10.26713/cma.v10i1.980 (2010).

[12] Khalid A, Naeem MN, Agarwal P, Ghaffar A, Ullah Z. Numerical approximation for the solution of linear sixth order boundary value problems by cubic B-spline. Adv Differ Equ 2019;492:1−16. https://doi.org/10.1186/s13662-019-2385-9.

[13] Khalid A, Naeem MN. Cubic B-spline solution of nonlinear sixth order boundary value problems. Punjab Univ J Math 2018;50(4):91−103.

[14] Lang FG, Xu XP. A new cubic B-spline method for linear fifth order boundary value problems. J Appl Math Comput 2011;36(1):101−16. https://doi.org/10.1007/s12190-010-0390-y.

[15] Manan SA, Ghaffar A, Rizwan M, Rahman G, Kanwal A. A subdivision approach to the approximate solution of 3rd order boundary value problem. Commun Math Appl 2018; 9(4):499−512. https://doi.org/10.26713/cma.v9i4.835.

[16] Mustafa G, Ejaz ST. Numerical solution of two-point boundary value problems by interpolating subdivision schemes. Abstr Appl Anal 2014;2014(2014):13. https://doi.org/10.1155/2014/721314. Article ID 721314.

[17] Mustafa G, Abbas M, Ejaz ST, Ismail AIM, Khan F. A numerical approach based on subdivision schemes for solving non-linear fourth order boundary value problems. J Comput Anal Appl 2017;23(4):607−23.

[18] Noor MA, Din STM. Homotopy perturbation method for solving sixth-order boundary value problems. Comput Math Appl 2008;55(12):2953−72. https://doi.org/10.1016/j.camwa.2007.11.026.

[19] Noor MA, Din STM. Variational iteration technique for solving higher order boundary value problems. Appl Math Comput 2007;189:1929−4192. https://doi.org/10.1016/j.amc.2006.12.071.

[20] Oyedepo T, Ayoade AA, Oluwayemi MO, Pandurangan R. Solution of Volterra-Fredholm integro-differential equations using the Chebyshev computational approach. In: International conference on science, engineering and business for sustainable development goals (SEB-SDG),Omu-Aran, Nigeria; 2023. p. 1−6. https://doi.org/10.1109/SEB-SDG57117.2023.10124647. 1(2023).

[21] Pervaiz A, Ahmad A, Zafar Z, Ahmad M. Numerical solution of sixth order BVPs by applying non-polynomial spline method. Pakistan J Sci 2014;66(2):110−6.

[22] Ray SS, Gupta AK. Two-dimensional Legendre wavelet method for travelling wave solutions of time-fractional generalized seventh order KdV equation. Comput Math Appl 2016;73(6):1118−33. https://doi.org/10.1016/j.camwa.2016.06.046 (2016).

[23] Sohaib M, Haq S, Mukhtar S, Khan I. Numerical solution of sixth-order boundary-value problems using Legendre wavelet collocation method. Results Phys 2018;8:1204−8. https://doi.org/10.1016/j.rinp.2018.01.065.

[24] Tirmizi IA, Khan MA. Non-polynomial splines approach to the solution of sixth-order boundary-value problems. Appl Math Comput 2008;195(1):270−84.

[25] Tsetimi J, Ogeh KO, Disu AB. Modified variational iteration method with Chebyshev Polynomials for solving 12th order Boundary value problems. J Nat Sci Math Res 2022;8(1):44−51. https://doi.org/10.21580/jnsmr.2022.8.1.2693.

[26] Ullah I, Khan H, Rahim MT. Numerical solutions of fifth and sixth order nonlinear boundary value problems by Daftardar Jafari method. J Comput Eng 2014;2014:1−8. https://doi.org/10.1155/2014/286039. Article ID 286039(2014.

[27] Xu X, Xu D. Legendre wavelets method for approximate solution of fractional order differential equations under multi-point boundary conditions. Int J Comput Math 2017;95(5):1−5. https://doi.org/10.1080/00207160.2017.1303139.