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Common Fixed Point Theorems by Using Two Mappings in b-Rectangular Metric Space

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Abstract

In this paper, the aim to obtain the existence and uniqueness of some common fixed point theorem on b-rectangular metric space by using new conditions including rational, maximum, minimum and combination for two mappings.

Keywords: Common fixed-point, b-Rectangular metric space

1. Introduction

One of the most important topics in the development of nonlinear analysis is fixed point theory. Fixed point theory has also been successfully applied to a variety of other fields of research, including chemistry, biology, economics, computer science, engineering, and a variety of others. It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping $Q: E \to E$ where (E, d) is a metric space, is said to be a contraction if there exists $\alpha \in [0, 1)$ such that for all $t, u \in E$;

$$d(Qt, Qu) \leq \alpha d(t, u) \tag{1.1}$$

The mapping fulfilling (1.1) has a unique fixed point if the metric space (E, d) is complete. Inequality (1.1) implies continuity of Q.

In 1989, Backhtin [1] introduced the concept of b-metric space. In 1993, Czerwik [7] extended the results of b-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the b-metric space. Boriceanu [3], Czerwik [7], Bota [4] extended the fixed point theorem in b-metric space., many authors initiated and studied many existing fixed point theorems in such spaces. Also, the concept of b-rectangular metric space is introduced as a generalization of b-metric space and rectangular (generalized) metric space by George et al. [8].

Common fixed point theorems have been studied by many researchers like Chandok [6].

A point $t \in E$ is a common fixed point of self mappings $Q, R: E \to E$ of a metric space (E, d) if $Qt = Rt = t$.

2. Preliminaries

Definition 2.1. [2] Let X be a nonempty set and $\mathrm{Q}: \mathrm{E} \to \mathrm{E}$ a self map. We say that $t \in E$ is a fixed point of Q if $\mathrm{Q}(\mathrm{t}) = \mathrm{t}$ and denote by $\mathrm{F_Q}$ or Fix (Q) the set of all fixed points of Q.
Let E be any set and $\mathrm{Q}: \mathrm{E} \to \mathrm{E}$ a self map. For any given $t \in E$, we define $\mathrm{Q_n}(\mathrm{t})$ inductively by $\mathrm{Q}^0(\mathrm{t}) = \mathrm{t}$ and $\mathrm{Q}^{\mathrm{n}+1}(\mathrm{t}) = \mathrm{Q}(\mathrm{Q}^{\mathrm{n}}(\mathrm{t}))$; we recall $\mathrm{Q}^{\mathrm{n}}(\mathrm{t})$ the nth iterative of t under Q.
For any $\mathrm{t_0} \in \mathrm{E}$, the sequence $\{t_n\}_{n \geq 0} \subset \mathrm{E}$ given by

$$t_n = Qt_{n-1} = Q^n t_0, n = 1, 2, \ldots$$

is called the sequence of successive approximations with the initial value t_0. It is also known as the Picard iteration starting at t_0.

Definition 2.2. [1,7] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d: X \times X \to [0, \infty)$ is a b-metric on X if, for all x, y, $z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;

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(3) $d(x,z) \le s[d(x,y) + d(y,z)]$ (b-triangular inequality).

In this case, the pair (X,d) is called a b-metric space.

Definition 2.3. [5] Let X be a nonempty set, and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x, y, z \in X$ and all distinct points $u, v \in X$; each distinct from x and y:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \le d(x,u) + d(u,v) + d(v,z)$ (rectangular inequality).

Then (X,d) is called rectangular or generalized metric space.

Definition 2.4. [8] Let X be a nonempty set, $s \ge 1$ be a given real number and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x,y,z \in X$ and distinct points $u,v \in X$; each distinct from x and y:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \le s[d(x,u) + d(u,v) + d(v,z)]$ (b-rectangular inequality).

Then (X,d) is called a b-rectangular metric space or a b-generalized metric space (b-g.m.s.).

Definition 2.5. [8] Let (X,d) be a b-rectangular metric space and $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x; if for every $\epsilon > 0$; there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) < \epsilon$ forall $n > n_0$ and this fact is represented by

$\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$ is said to be b-rectangular-Cauchy in (X, d) if for every > 0 ; there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > n_0$; $p > 0$ or equivalently, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.
(iii) (X,d) is said to be complete if every b-rectangular-Cauchy sequence in (X,d) converges to an element of X.

Lemma 2.1. [9] Let (E,d) be a b- metric space with coefficient $s \ge 1$ and $Q : E \to E$ be a mapping. Suppose that $\{t_n\}$ is a sequence in E induced by $t_{n+1} = Qt_n$ such that

$d(t_n, t_{n+1}) \le \alpha d(t_{n-1}, t_n)$,

for all $n \in \mathbb{N}$, where $\alpha \in [0,1)$ is a constant. Then $\{t_n\}$ is a Cauchy sequence.

Definition 2.6. Let $Q, R : E \to E$ be self-mappings on complete b-rectangular metric space (E,d) with $s \ge 1$. The following condition is called (α, β, γ)-contraction:

$$d(Qt,Ru) \le \alpha d(t,u) + \beta \frac{d(t,Qt)d(t,QRu) + [d(u,Qt)]^2}{d(t,QRu) + d(u,Qt)} + \gamma d(t,Qt) \tag{2.1}$$

Where $d(t, QRu) + d(u, Qt) \ne 0$, $\alpha, \beta, \gamma > 0$, *with* $0 \le (\alpha + s\beta + \gamma) < 1$ for all t,u *in* E.

3. Main results

Theorem 3.1. Let (E,d) be a complete b-rectangular metric space with $s \ge 1$, and $Q, R : E \to E$ be two mappings on E satisfying the condition

$$d(Qt,Ru) \le \alpha M(t,u) + \beta N(t,u) \quad (3.1)$$

Where $M(t,u) = \max\left\{ d(t,u), \frac{d(t,Qt)}{1+d(RQt,Ru)}, \frac{d(u,Ru)}{1+d(u,Ru)}, \frac{d(u,QRu)d(u,Qt)}{2} \right\}$

And

$$N(t,u) = \min\{d(t,Qt), d(t,Ru), d(u,Qt)\}$$

For all t,u in E and $\alpha,\beta \in [0,1)$ with <1 . then Q and R have a unique common fixed point.

Proof: for any arbitrary t_0 in E. Define the sequence $\{t_n\}$ in E such that

$$t_{2n+1} = Qt_{2n}$$

$$t_{2n+2} = Rt_{2n+1} = RQt_{2n}$$

$$t_{2n+3} = Qt_{2n+2} = QRt_{2n+1}, \text{for all } n \in \mathbb{N} \tag{3.2}$$

Assume that there is some $n \in \mathbb{N}$ such that $t_n = t_{n+1}$.

If $n = 2k$, then $t_{2k} = t_{2k+1}$ and from (3.1),

$$d(t_{2k+1}, t_{2k+2}) = d(Qt_{2k}, Rt_{2k+1})$$

$$\le \alpha M(t_{2k}, t_{2k+1}) + \beta N(t_{2k}, t_{2k+1})$$

Where

$$M(t_{2k}, t_{2k+1})$$

$$= \max\left\{ d(t_{2k}, t_{2k+1}), \frac{d(t_{2k}, Qt_{2k})}{1 + d(RQt_{2k}, Rt_{2k+1})}, \frac{d(t_{2k+1}, Rt_{2k+1})}{1 + d(t_{2k+1}, Rt_{2k+1})}, \right.$$

$$\left.\frac{d(t_{2k+1},QRt_{2k+1})d(t_{2k+1},Qt_{2k})}{2}\right\}$$

$$=\max\left\{d(t_{2k},t_{2k+1}),\frac{d(t_{2k},t_{2k+1})}{1+d(t_{2k+2},t_{2k+2})},\frac{d(t_{2k+1},t_{2k+2})}{1+d(t_{2k+1},t_{2k+2})},\right.$$

$$\left.\frac{d(t_{2k+1},t_{2k+3})d(t_{2k+1},t_{2k+1})}{2}\right\}$$

$$=\max\left\{0,0,\frac{d(t_{2k+1},t_{2k+2})}{1+d(t_{2k+1},t_{2k+2})},0\right\}$$

$$=\frac{d(t_{2k+1},t_{2k+2})}{1+d(t_{2k+1},t_{2k+2})}$$

And

$$N(t_{2k},t_{2k+1})=\min\{d(t_{2k},Qt_{2k}),d(t_{2k},Rt_{2k+1}),\\ d(t_{2k+1},Qt_{2k})\}$$

$$=\min\{d(t_{2k},t_{2k+1}),d(t_{2k},t_{2k+2}),d(t_{2k+1},t_{2k+1})\}=0$$

Thus we have

$$d(t_{2k+1},t_{2k+2})\leq\alpha\frac{d(t_{2k+1},t_{2k+2})}{1+d(t_{2k+1},t_{2k+2})}\leq\alpha d(t_{2k+1},t_{2k+2})$$

$$d(t_{2k+1},t_{2k+2})\leq\alpha d(t_{2k+1},t_{2k+2})$$

Which is a contradiction

$$d(t_{2k+1},t_{2k+2})=0$$

Therefore $t_{2k+1}=t_{2k+2}$, Hence we have $t_{2k}=t_{2k+1}=t_{2k+2}$.
It mains that $t_{2k}=Qt_{2k}=Rt_{2k}$ t_{2k} is a common fixed point of, R.
If $n=2k+1$, then using same influences, can be presented that t_{2k+1} is a common fixed point of Q and R.
Now; assume $t_n\neq t_{n+1}$ for all $n\in\mathbb{N}$.

$$d(t_{2n+1},t_{2n+2})=d(Qt_{2n},Rt_{2n+1})$$

$$d(t_{2n+1},t_{2n+2})\leq\alpha M(t_{2n},t_{2n+1})+\beta N(t_{2n},t_{2n+1})\quad(3.3)$$

Where

$$M(t_{2n},t_{2n+1})=\max\left\{d(t_{2n},t_{2n+1}),\frac{d(t_{2n},Qt_{2n})}{1+d(RQt_{2n},Rt_{2n+1})},\right.\\ \frac{d(t_{2n+1},Rt_{2n+1})}{1+d(t_{2n+1},Rt_{2n+1})},$$

$$\left.\frac{d(t_{2n+1},QRt_{2n+1})d(t_{2n+1},Qt_{2n})}{2}\right\}$$

$$=\max\left\{d(t_{2n},t_{2n+1}),\frac{d(t_{2n},t_{2n+1})}{1+d(t_{2n+2},t_{2n+2})},\frac{d(t_{2n+1},t_{2n+2})}{1+d(t_{2n+1},t_{2n+2})},\right.$$

$$\left.\frac{d(t_{2n+1},t_{2n+3})d(t_{2n+1},t_{2n+1})}{2}\right\}$$

$$=\max\left\{d(t_{2n},t_{2n+1}),d(t_{2n},t_{2n+1}),\frac{d(t_{2n+1},t_{2n+2})}{1+d(t_{2n+1},t_{2n+2})},0\right\}$$

$$=\max\left\{d(t_{2n},t_{2n+1}),\frac{d(t_{2n+1},t_{2n+2})}{1+d(t_{2n+1},t_{2n+2})}\right\}$$

And

$$N(t_{2n},t_{2n+1})=\min\{d(t_{2n},Qt_{2n}),d(t_{2n},Rt_{2n+1}),\\ d(t_{2n+1},Qt_{2n})\}$$

$$=\min\{d(t_{2n},t_{2n+1}),d(t_{2n},t_{2n+2}),d(t_{2n+1},t_{2n+1})\}$$

$$=0$$

If $(t_{2n},t_{2n+1})=d(t_{2n},t_{2n+1})$, then by (3.3)

$$d(t_{2n+1},t_{2n+2})\leq\alpha\, d(t_{2n},t_{2n+1})$$

If $(t_{2n},t_{2n+1})=\frac{d(t_{2n+1},t_{2n+2})}{1+d(t_{2n+1},t_{2n+2})}$, then by (3.3)

$$d(t_{2n+1},t_{2n+2})\leq\alpha\frac{d(t_{2n+1},t_{2n+2})}{1+d(t_{2n+1},t_{2n+2})}\leq\alpha d(t_{2n+1},t_{2n+2})$$

Which is a contradiction
By induction we get

$$d(t_{n+1},t_n)\leq\alpha^2 d(t_1,t_0)\quad(3.4)$$

Thus from (3.4), we obtain that

$$\lim_{n\to\infty}d(t_n,t_{n+1})=0,$$

Hence $\{t_n\}$ is a b-rectangular-Cauchy sequence in (E,d). By completeness of (E,d), there exists $r\in E$ such that $t_n=Qt_{n-1}\to r$ *as* $n\to\infty$.

$$d(r,Qr)\leq s[d(r,t_n)+d(t_n,t_{n+1})+d(t_{n+1},Qr)]$$

$$\frac{1}{s}d(r,Qr)\leq d(r,t_n)+d(t_n,t_{n+1})+d(Rt_n,Qr)$$

$$\frac{1}{s}d(r,Qr)\leq d(r,t_n)+d(t_n,t_{n+1})+\alpha M(r,t_n)+\beta N(r,t_n)\quad(3.5)$$

Where

$$M(r,t_n)=\max\left\{d(r,t_n),\frac{d(r,Qr)}{1+d(RQr,Rt_n)},\frac{d(t_n,Rt_n)}{1+d(t_n,Rt_n)},\right.\\ \left.\frac{d(t_n,QRt_n)d(t_n,Qr)}{2}\right\}$$

$$= \max\left\{d(r,t_n), \frac{d(r,Qr)}{1+d(Qr,r)}, \frac{d(r,r)}{1+d(r,r)}, \frac{d(r,r)d(r,Qr)}{2}\right\}$$

$$= \max\left\{0, \frac{d(r,Qr)}{1+d(Qr,r)}, 0, 0\right\}$$

$$= \frac{d(r,Qr)}{1+d(Qr,r)}$$

And

$$N(r,t_n) = \min\{d(r,Qr), d(r,Rt_n), d(t_n,Qr)\}$$

$$= 0$$

Now, (3.5) became

$$\frac{1}{s}d(r,Qr) \le d(r,t_n) + d(t_n,t_{n+1}) + \alpha\,\frac{d(r,Qr)}{1+d(Qr,r)}$$

$$\le \alpha d(r,Qr)$$

$$d(r,Qr) \le s\alpha d(r,Qr)$$

$$d(r,Qr) = 0 \Longrightarrow Qr = r$$

Now; we prove that Q and R have a unique common fixed point, Suppose r and v are common fixed points of Q and R with $r \ne v$. By (1),

$$d(r,v) = d(Qr,Rv) \le \alpha M(r,v) + \beta N(r,v) \quad (3.6)$$

Where

$$M(r,v) = \max\left\{d(r,v), \frac{d(r,Qr)}{1+d(RQr,Rv)}, \frac{d(v,Rv)}{1+d(v,Rv)}, \frac{d(v,QRv)d(v,Qr)}{2}\right\}$$

$$= \max\{d(r,v), 0, 0, 0\}$$

$$= d(r,v)$$

And

$$N(r,v) = \min\{d(r,Q_r), d(r,Rv), d(v,Qr)\}$$

$$= \min\{0, d(r,v), d(v,r)\}$$

$$= 0$$

Now, (3.6) became

$$d(r,v) = d(Qr,Rv) \le \alpha d(r,v)$$

Which is a contradiction
So $d(r,v) = 0 r = v$,
Therefore Q and R have a unique common fixed point r.

Corollary 3.1. Let (E,d) be a complete b-rectangular metric space with $s \ge 1$, and $Q, R : E \to E$ be two mappings on E satisfying the condition

$$d(Qt,Ru) \le \alpha M(t,u) + \beta N(t,u) \quad (3.1)$$

Where

$$M(t,u) = \max\left\{d(t,u), \frac{d(t,Qt)}{1+d(RQt,Ru)}, \frac{d(u,QRu)d(u,Qt)}{2}\right\}$$

And

$$N(t,u) = \min\{d(t,Qt), d(t,Ru), d(u,Qt)\}$$

For all t, u in E and $\alpha, \beta \in [0,1)$ with <1 . then Q and R have a unique common fixed point.

Theorem 3.2. Let (E,d) *be a* complete b-rectangular metric space. Then (α, β, γ)-contraction satisfy a unique common fixed point.
Proof: For any arbitrary point, $t_0 \in E$. Define sequence $\{t_n\}$ in E such that

$$t_{2n+1} = Qt_{2n}$$

$$t_{2n+2} = Rt_{2n+1}$$

$$t_{2n+3} = Qt_{2n+2} = QRt_{2n+1}, \text{for all } n \in \mathrm{N} \quad (3.1)$$

Suppose that there is some $n \in \mathbb{N}$ such that $t_n = t_{n+1}$. If $n = 2k$, then $t_{2k} = t_{2k+1}$ and from the condition (2.1) with $t = t_{2k}$ and $u = t_{2k+1}$ we have

$$d(t_{2k+1}, t_{2k+2}) = d(Qt_{2k}, Rt_{2k+1})$$

$$\le \alpha d(t_{2k}, t_{2k+1}) + \beta\frac{d(t_{2k}, Qt_{2k})d(t_{2k}, QRt_{2k+1}) + [d(t_{2k+1}, Qt_{2k})]^2}{d(t_{2k}, QRt_{2k+1}) + d(t_{2k+1}, Qt_{2k})}$$

$$+\gamma d(t_{2k}, Qt_{2k}) \quad (3.2)$$

$$= \beta\,\frac{d(t_{2k}, t_{2k+1})d(t_{2k}, t_{2k+3}) + [d(t_{2k+1}, t_{2k+1})]^2}{d(t_{2k}, t_{2k+3}) + d(t_{2k+1}, t_{2k+1})} + \gamma d(t_{2k}, t_{2k+1})$$

$$= 0$$

therefore $d(t_{2k+1}, t_{2k+2}) = 0$. Hence $t_{2k+1} = t_{2k+2}$. Thus we have $t_{2k} = t_{2k+1} = t_{2k+2}$. By (3.1), it means $t_{2k} = Qt_{2k} = Rt_{2k}$, that is, t_{2k} is a common fixed point of Q and R.
If $n = 2k+1$, then using the same arguments as in the case $t_{2k=}t_{2k+1}$, it can be shown that t_{2k+1} is a common fixed point of Q and R.
From now on, we suppose that $t_n \ne t_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We will show that

$$d(t_n,t_{n+1})\le(\alpha+\beta+\gamma)d(t_{n-1},t_n),\text{for all } n\in\mathrm{N} \qquad (3.3)$$

There are two cases which we have to consider.

Case 1. $n=2k+1, k\in\mathbb{N}$.
From the condition (2.1) with $t=t_{2k}$ and $u=t_{2k+1}$ we have

$$d(t_{2k+1},t_{2k+2})=d(Qt_{2k},Rt_{2k+1})$$

$$\le\alpha\, d(t_{2k},t_{2k+1})+\beta\,\frac{d(t_{2k},Qt_{2k})d(t_{2k},QRt_{2k+1})+[d(t_{2k+1},Qt_{2k})]^2}{d(t_{2k},QRt_{2k+1})+d(t_{2k+1},Qt_{2k})}+\gamma d(t_{2k},Qt_{2k})$$

$$=\alpha\, d(t_{2k},t_{2k+1})+\beta\,\frac{d(t_{2k},t_{2k+1})d(t_{2k},t_{2k+3})+[d(t_{2k+1},t_{2k+1})]^2}{d(t_{2k},t_{2k+3})+d(t_{2k+1},t_{2k+1})}+\gamma d(t_{2k},t_{2k+1})$$

$$=\alpha\, d(t_{2k},t_{2k+1})$$

$$+\beta\,\frac{d(t_{2k},t_{2k+1})[sd(t_{2k},t_{2k+1})+sd(t_{2k+1},t_{2k+2})+sd(t_{2k+2},t_{2k+3})]}{sd(t_{2k},t_{2k+1})+sd(t_{2k+1},t_{2k+2})+sd(t_{2k+2},t_{2k+3})}$$

$$+\gamma d(t_{2k},t_{2k+1})$$

$$=\alpha\, d(t_{2k},t_{2k+1})+\beta d(t_{2k},t_{2k+1})+\gamma d(t_{2k},t_{2k+1})$$

$$d(t_{2k+1},t_{2k+2})\le(\alpha+\beta+\gamma)\,\mathrm{d}(t_{2k},t_{2k+1})$$

Thus we obtain that

$$d(t_n,t_{n+1})\le(\alpha+\beta+\gamma)d(t_{n-1},t_n),n=2k+1,k\in\mathrm{N} \qquad (3.4)$$

Case 2. $n=2k$, $k\in\mathbb{N}$. By using the same argument as in Case 1, it can be proved that (3.3) holds for $n=2k$, that is

$$d(t_n,t_{n+1})\le(\alpha+\beta+\gamma)d(t_{n-1},t_n),n=2k,k\in\mathrm{N} \qquad (3.5)$$

From (3.4) and (3.5) we can conclude that $d(t_n,t_{n+1})\le(\alpha+\beta+\gamma)d(t_{n-1},t_n)$, for all $n\in\mathbb{N}$.
Thus we obtain that (3.3) holds.
Since $0\le(\alpha+\beta+\gamma)<1$, by Lemma (2.1) we can say that $\{t_n\}$ is a Cauchy sequence in (E,d). since (E,d) is a complete b-rectangular metric space, $\{t_n\}$ converges to same $r\in E$ as $n\to+\infty$.

Step 2. we will prove that $Qr=Rr=r$.
Using b-rectangular inequality and (2.1), we have

$$d(r,Qr)\le s[d(r,t_{2n+1})+d(t_{2n+1},t_{2n+2})+d(t_{2n+2},Qr)]$$

$$=sd(r,t_{2n+1})+sd(t_{2n+1},t_{2n+2})+sd(t_{2n+2},Qr)$$

$$=sd(r,t_{2n+1})+sd(t_{2n+1},t_{2n+2})+sd(Rt_{2n+1},Qr)$$

$$\le sd(r,t_{2n+1})+sd(t_{2n+1},t_{2n+2})$$

$$+s\alpha\, d(r,t_{2n+1})+s\beta\,\frac{d(r,Qr)d(r,QRt_{2n+1})+[d(t_{2n+1},Qr)]^2}{d(r,QRt_{2n+1})+d(t_{2n+1},Qr)}$$

$$+\gamma d(r,Qr)$$

$$=sd(r,t_{2n+1})+sd(t_{2n+1},t_{2n+2})$$

$$+s\alpha\, d(r,t_{2n+1})+s\beta\,\frac{d(r,Qr)d(r,t_{2n+3})+[d(t_{2n+1},Qr)]^2}{d(r,t_{2n+3})+d(t_{2n+1},Qr)}$$

$$+\gamma d(r,Qr)$$

Take the limit as $n\to\infty$, we obtain that $d(r,Qr)\le(s\beta+\gamma)d(r,Qr)$, which is a contradiction since $0\le(s\beta+\gamma)<1$
Hence $d(r,Qr)=0 \quad Qr=r$.
Similarly, we obtain $Rr=r$, thus r is common fixed point of Q and R.

Step 3. we will prove that Q and R have a unique common fixed point.
Suppose now that r and v are different common fixed points of Q and R, then from (2.1), we have

$$d(r,v)=d(Qr,Rv)$$

$$\le\alpha\, d(r,v)+\beta\,\frac{d(r,Qr)d(r,QRv)+[d(v,Qr)]^2}{d(r,QRv)+d(v,Qr)}$$

$$+\gamma d(r,Qr)$$

$$= \alpha\, d(r,v) + \beta\, \frac{d(r,r)d(r,QRv) + [d(v,r)]^2}{d(r,v) + d(v,r)} + \gamma d(r,r)$$

$$= \alpha\, d(r,v) + \beta\, \frac{d(v,r)^2}{2d(v,r)}$$

$$d(r,v) \leq \left(\alpha + \frac{\beta}{2}\right) d(r,v)$$

Which is a contradiction
Since $0 \leq \left(\alpha + \frac{\beta}{2}\right) < 1$, we have $d(r,v) = 0$.
Thus proved that Q and R have a unique common fixed point r in E.
Now, if $\alpha, \gamma = 0$ in theorem 3.2, we get the following corollary:

Corollary 3.2. Let (E,d) *be a* complete b-rectangular metric space with ≥ 1 , and $Q,R : E \to E$ be two mappings on E satisfying the condition

$$d(Qt,Ru) \leq \beta\, \frac{d(t,Qt)d(t,QRu) + [d(u,Qt)]^2}{d(t,QRu) + d(u,Qt)}$$

for all, t,u in E and $d(t,QRu) + d(u,Qt) \neq 0$, with $\beta > 0$, $0 \leq s\beta < 1$. Then Q and R have a unique common fixed point.

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