Solving Some Types of Non-Linear Ordinary Differential Equations by Using a New Assumption

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Abstract

In this paper ,we solved some sets of non-linear ordinary differential equations by using a new procedure .This is done through finding the function z(x) and using the assumption y= $e^{\int z(x)\,dx}$ which gives the general solution for the non-linear differential equation required to be solved . We have applied this method on some sets of non-linear differential equations of first order and 2nd, 3nd and fourth degree - second order and 1nd and 2 degree and also the third order and first degree whether one of the variable is missing or not .

حل بعض أنواع من المعادلات التفاضلية الاعتيادية اللاخطية باستخدام فرضية جديدة

الملخص

في هذا البحث قمنا بحل المعادلات التفاضلية اللاخطية بأجراء جديد وذلك من خلال إيجاد دالة z(x) واستخدام الفرضية $y=e^{\int z(x)\,dx}$ التي تمثل الحل العام للمعادلة التفاضلية اللاخطية المطلوب حلها . حيث قمنا بتطبيق هذه الطريقة على مجموعة من المعادلات التفاضلية اللاخطية من الرتبة الأولى والدرجة الثانية ، الرتبة الأولى والدرجة الثالثة ، الرتبة الأولى والدرجة الرابعة ، الرتبة الثانية والدرجة الأولى ، الرتبة الثانية والدرجة الثانية وكذلك الرتبة الثالثة والدرجة الأولى سواء كان احد المتغيرين فيها مفقودا أو لا .

Introduction

Kathem A.N.,[1] **used a new function in a specific assumption for finding the solution of some types of the linear second order homogenous differential equations with variable coefficients which have the form**

$$y'' + P(x)y' + Q(x)y = 0 \ .$$

Kudaer R.A.,[2] also used the assumption in solving some kinds of linear second order non-homogenous differential equations with variable coefficients which have the following formula

$$y'' + P(x)y' + Q(x)y = f(x) \ .$$

In this paper ,we have used the same assumption in solving some types of non-linear ordinary differential equations .

The General Solution of the Non-linear Differential Equations

In this paper the method of solving non-linear differential equations depends on investigating a new function z(x),such that the assumption

$$y = e^{\int z(x)\,dx} \quad \dots \quad (1)$$

represents the general solution of the required equations .This assumption will transform the non-linear differential equations to the one of a simple formula for the differential equations which we can solve .Through finding y', y'', y''' from equation (1) ,we get

$$y' = z(x)\, e^{\int z(x)\,dx} \quad \dots \quad (2)$$

$$y'' = (z'(x) + z^2(x))\, e^{\int z(x)\,dx} \quad \dots \quad (3)$$

$$y''' = (z''(x) + 3z'(x)z(x) + z^2(x))\, e^{\int z(x)\,dx} \quad \dots \quad (4)$$

and by substituting (y, y', y'', y''') in the required equation , we find a simple formula of an ordinary differential equation which we can solve by using the previous methods in order to find the function $z(x)$,and by substituting the result in equation (1) ,we get the general solution of the original equation .

Problem (1) : Consider the differential equation

$$(x\,y')^2 = y^2 \;,$$

to solve this differential equation by the above method ,we use the formula (1) to get

$$(x\, z\, e^{\int z(x)\,dx})^2 = (e^{\int z(x)\,dx})^2$$

$$e^{2\int z(x)\,dx}(x^2 z^2 - 1) = 0 \;,$$

since $e^{2\int z(x)\,dx} \neq 0$

so $(x^2 z^2 - 1) = 0$

and $z = \pm \frac{1}{x}$,

by substituting z in equation (1) , we find

$$y_1 = e^{\int \frac{1}{x} dx} = Ax \;,$$

where A is an arbitrary constant .

Also

$$y_2 = e^{\int \frac{-1}{x} dx} = \frac{A}{x} \;.$$

Where A is an arbitrary constant .

So the set of the general solution of the original equation is

$$(y - Ax)\left(y - \frac{A}{x}\right) = 0$$

Problem (2) : Consider the differential equation

$$(y')^2 + y\,y' + y^2 = 0 \;,$$

for finding the solution of the above differential equation we use equation (1) to get

$$e^{2\int z(x)\,dx}(z^2 + z + 1) = 0$$

$$\because e^{2\int z(x)dx} \neq 0$$

$$(z^2 + z + 1) = 0$$

$$\therefore z = \frac{-1}{2} \pm \sqrt{3}i .$$

Now, we can find y from equation (1) ,since

$$y_1 = e^{\int (\frac{-1}{2} - \sqrt{3}i)dx} = Ae^{(\frac{-1}{2} - \sqrt{3}i)x} \quad ,$$

where A is an arbitrary constant .
Also

$$y_2 = e^{\int (\frac{-1}{2} + \sqrt{3}i)dx} = Ae^{(\frac{-1}{2} + \sqrt{3}i)x} \quad .$$

So the set of the general solution of the required equation is

$$(y - Ae^{(\frac{-1}{2} - \sqrt{3}i)x})(y - Ae^{(\frac{-1}{2} + \sqrt{3}i)x}) = 0$$

Problem (3) :

Again we consider the first order non-linear ordinary differential equation of degree three

$$x((y')^3 + xy(y')^2 - 3y^3) = 3y^2 y' \quad ,$$

by using equation (1) , we get

$$e^{3\int z(x)\, dx} (xz^3 + (xz)^2 - 3x - 3z) = 0$$

$$\because e^{3\int z(x)dx} \neq 0$$

$$(xz^3 + (xz)^2 - 3x - 3z) = 0$$

$$(x z^2 - 3) (z + x) = 0 \quad .$$

If $(x z^2 - 3) = 0$

$$z = \pm \sqrt{\frac{3}{x}} \quad ,$$

or $(z + x) = 0$

$$z = -x \quad .$$

So by substituting z in equation (1) , we get

$$y_1 = e^{\int \sqrt{\frac{3}{x}} dx} = Ae^{2\sqrt{3}x},$$

in a similar way ,we find

$$y_2 = e^{\int -\sqrt{\frac{3}{x}} dx} = Ae^{-2\sqrt{3}x},$$

also

$$y_3 = e^{\int -x dx} = Ae^{\frac{-x^2}{2}} \quad .$$

Where A is an arbitrary constant .
So the set of the general solution of the original equation is

$$(y - Ae^{2\sqrt{3}x})(y - Ae^{-2\sqrt{3}x})(y - Ae^{\frac{-x^2}{2}}) = 0$$

Problem (4) : Consider the differential equation

$$x^2(y')^4 - (1 - 3x)x^2 (y')^3 y - (3x^3 + 4)(y')^2y^2 + 4(1- 3x)y'y^3 + 12x\, y^4 = 0,$$

to find the solution of the above differential equation we can solve it by using equation (1) ,we get

$$e^{4\int z(x)\, dx} (x^2 z^4 - (1-3x)x^2z^3 - (3x^3+4)z^2 + 4 (1 - 3x)z + 12x) = 0$$

since $e^{4\int z(x)dx} \neq 0$

so $(x^2 z^4 - (1-3x)x^2z^3 - (3x^3+4)z^2 + 4 (1 - 3x)z + 12x) = 0$

$(x z - 2) (x z + 2) (z + 3x) (z - 1) = 0$,

if $(x z - 2) = 0$ **implies that** $z = 2/x$,

or $(x z + 2) = 0$ **implies that** $z = -2/x$,

or $(z - 1) = 0$ **implies that** $z = 1$.

By substituting the values of z in equation (1) , we find

$$y_1 = e^{\int \frac{2}{x} dx} = Ax^2 \quad ,$$

$$y_2 = e^{\int \frac{-2}{x} dx} = \frac{A}{x^2} \quad ,$$

$$y_3 = e^{\int -3x dx} = Ae^{\frac{-3}{2}x^2} \quad ,$$

$$y_4 = Ae^{x} \quad ,$$

where A is an arbitrary constant .

So the set of the general solutions of the required differential equation is

$$(y - Ax^2)(y - \frac{A}{x^2})(y - Ae^{\frac{-3}{2}x^2})(y - Ae^{x}) = 0 \quad .$$

Problem (5) : To solve the differential equation

$$(x^2 + 3x) y y'' - (2x + 3) y y' = (x^2 + 3x) (y')^2 \; ,$$

by using the assumption (1) , we get

$$e^{2\int z(x)\, dx} ((x^2 + 3x) (z' + z) - (2x + 3) z - (x^2 + 3x) z^2) = 0 \; ,$$

since $e^{2\int z(x)dx} \neq 0$

so $(x^2 + 3x) z' - (2x + 3) z = 0$,

$$\int \frac{dz}{z} = \int \frac{(2x+3)}{(x^2 + 3x)} dx$$

$$z = A (x^2 + 3x) \quad .$$

Where A is an arbitrary constant .

Now, we can find the general solution of the original equation by substituting z in equation (1) , we get

$$y = e^{\int A(x^2 + 3x)dx} = Be^{Ax^2(\frac{1}{3}x + \frac{3}{2})} \quad .$$

Where B is an arbitrary constant .

Problem (6) : To find the solution of the differential equation

$$y\,y'' - 2(y')^2 + y^2 = 0 \quad ,$$

where one of the variables is absent .
By using equation (1) ,we get

$$e^{2\int z(x)\,dx}\,(z' - z^2 + 1) = 0 \quad ,$$

since $\quad e^{2\int z(x)dx} \neq 0 \quad ,$

so $\quad (z' - z^2 + 1) = 0$

$$\int \frac{dz}{(1-z^2)} = -\int dx$$

$$z = \tanh(-x + c) \quad ,$$

where c is an arbitrary constant .
Also, by using equation (1) we find the general solution of the required equation

$$y = e^{\int \tanh(c-x)dx}$$

$$y = \frac{A}{\cosh(c-x)} \quad .$$

Where A is an arbitrary constant .

Problem (7) :
Again as an application of the above method ,we are going to solve the non-linear ordinary differential equation of the second order and degree

$$x^2 y^2 (y'')^2 - y'' y^3 - (y')^2 y^2 - x (y')^4 = 0 \quad ,$$

by using the assumption (1) , we find

$$e^{4\int z(x)\,dx}\,(x(z')^2 + 2x\,z'\,z^2 - z' - 2z^2) = 0 \quad ,$$

since $\quad e^{4\int z(x)\,dx} \neq 0 \quad ,$

so $\quad (x\,z' - 1)(z' + 2z^2) = 0 \quad ,$

if $\quad (x\,z' - 1) = 0$

$$z = (\ln x + c) \quad ,$$

or $\quad (z' + 2z^2) = 0$

$$z = \frac{1}{2x - c} \quad ,$$

where c is an arbitrary constant .
Now ,by substituting the values of z in equation (1) , we find

$$y_1 = e^{\int (\ln x + c)\,dx} = A\,x^x\,e^{x(c-1)} \quad ,$$

where A is an arbitrary constant .
And

$$y_2 = e^{\int \frac{dx}{2x-c}} = A\sqrt{2x-c} \quad ,$$

So the set of the general solutions of the above equation is

$$(y - A\,x^x\,e^{x(c-1)})\,(y - A\sqrt{2x-c}\,) = 0$$

Problem (8) :

For solving the differential equation

$$y''' y' - (y'')^2 = 0 \quad ,$$

by using equation (1) , we find

$$e^{2\int z(x)\,dx}\,(z'' z + z' z^2 - (z')^2) = 0$$

$$\because e^{2\int z(x)dx} \neq 0$$

$$\therefore (z'' z + z' z^2 - (z')^2) = 0 \quad ,$$

where the resulting equation is a non-linear equation of second order, we can reduce to first order by using the assumption

let $z' = p \quad \rightarrow \quad z'' = p\dfrac{dp}{dz}$

$$p\left(\frac{dp}{dz} z + z^2 - p\right) = 0$$

If $p = 0 \quad \rightarrow z' = 0 \quad \rightarrow z = A$

by substituting the values of z in equation (1), we find the singular solution of the original equation

$$y = B\,e^{Ax} \; .$$

Where A , B are arbitrary constants .

Or $\dfrac{dp}{dz} z + z^2 - p = 0$

$$\frac{dp}{dz} - \frac{p}{z} = -z \quad ,$$

this is linear equation [Mohammed, A.H.(2002)] and it has the general solution

$$p = -z^2 + cz \quad ,$$

where c is an a arbitrary constant.

$$z' = -z^2 + cz \; ,$$

where is Bernoulli equation and it has the general solution

$$z = \frac{ce^x}{ce^x + A} \quad ,$$

Also, by using equation (1) we get

$$y = e^{\int \frac{ce^X}{(ce^X + A)} de} = B(e^{cx} + A) \; ,$$

References

[1] Kathem, Athera Nema, (2005) .<u>Solving special Kinds of Second Order Differential Equations by using the substitution $y = e^{\int z(x)\,dx}$</u> . Msc, thesis, University of Kufa, College of Education, Department of Mathematics.

[2] Kudaer, Rehab Ali, (2006) .<u>Solving Some Kinds of Linear Second Order Non-Homogeneous Differential Equations with Variables Coefficients</u> . Msc, thesis, University of Kufa, College Education , Department of Mathematics .