Journal of Thi-Qar University number4 Vol.6 september/2011

# A MODULES WITH FUZZY ZARISKI TOPOLOGY

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## الموديولات مع تبولوجي زارسكي الضبابي

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**المستخلص**

**في هذا البحث قدمنا مفهوم الموديول مع تبولوجي زارسكي الضبابي كتعميم لتبولوجي زارسكي الاعتيادي ، وقد تم دراسة واعطاء العديد من الخواص الاساسية المتعلقة بهذا المفهوم .**
**وايضا دراسة العديد من الخواص الخاصة بالموديولات الضبابية العظمى والتي كان هذا البحث بحاجة اليها .**

**ABSTRACT**

**In this paper, the concept module with Zariski topology have been investigated .Also, we give some results and some properties of it.**
**Moreover, some properties of prime fuzzy submodule have been given which are needed in this research.**

## INTRODUCTION

**The present paper introduces and studies module with fuzzy Zariski topology.**

**In section one, some basic definitions and results are recalled which will be needed later. Several results about prime fuzzy submodules of R-module M and maximal fuzzy submodules of R-module M are given which are necessary in proving some resultes in the following sections.**

**In section two, we introduce the definition about fuzzy spectrum of R-module M and we give and prove some properties about fuzzy spectrum of R-module M .**

**Section three is devoted for studying module with fuzzy Zariski topology, where R-module M is called module with fuzzy Zariski topology, if T is closed under finite intersection.**

**Throughout this paper R is commutative ring with unity and M is an unitary R-module . Finally, A (0) =X (0), for any fuzzy submodule A of fuzzy module X of R-module M .**

## S. 1 PRELIMINARY:

**In this section some basic definitions and resultes which we will be used in the next section.**

**Let (R, +,· ) be a commutative ring with identity. A fuzzy subset of R is a function from R into [0, 1], ([1], [2]).**

**Let A and B be fuzzy subset of R. We write $A \subseteq B$ if $A(x) \leq B(x)$, for all $x \in R$. If $A \subseteq B$ and there exists $x \in R$ such that $A(x) < B(x)$, then we write $A \subset B$ and we say that A is a proper fuzzy subset of B, [2]. Note that A = B if and only if A(x) = B(x), for all $x \in R$, [1].**

**Let I be a subset of a set R. The characteristic function of I denoted by $X_I$ which define by $X_I(x) = 1$ if $x \in I$ and $X_I(x) = 0$ otherwise , ( [ 1 ],[ 2 ] ).**

**Let $\lambda_R$ denote the characteristic function of R defined by $\lambda_R(x) = 1$ if $x \in R$ and $\lambda_R(x) = 0$ if $x \notin R$ ,( [ 3] , [ 4] ).**

**Let $f : R \to R'$ ,A and B be two fuzzy subsets of R and R' respectively ,the fuzzy subset f(A) of R' defined by : $f(A)(y) = \sup A(y)$ if $f(y) \neq 0$ , $y \in R'$ and $f(A)(y) = 0$ , otherwise .It is called the image of A under f and denoted by f(A) . The fuzzy subset $f^{-1}$ (B) of R**

**defined by : $f^{-1}(B)(y) = B(f(x))$ ,for all $x \in R$ .Is called the inverse image of B and denoted by $f^{-1}(B)$ , [2] .**

**Let R, R' be any sets and f: R→ R' be any function .A fuzzy subset A of R is called f-invariant if f(x) = f(y) implies A(x) =A(y), where x, y ∈R, [3].**

**For each $t \in [0,1]$, the set $A_t = \{ x \in R \mid A(x) \geq t \}$ is called a level subset of R and the set $A_* = \{ x \in R \mid A(x) = A(0) \}$,**
**and A=B if and only if $A_t = B_t$ for all $t \in [0,1]$ ( [ 2], [ 4 ] ).**

**Let $x \in R$ and $t \in [0, 1]$, let $x_t$ denote the fuzzy subset of R defined by $x_t(y) = 0$ if $x \neq y$ and $x_t(y) = t$ if $x = y$ for all $y \in R$. $x_t$ is called a fuzzy singleton, [4].**

**If $x_t$ and $y_s$ are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x . y)_\lambda$ , where $\lambda = \min \{ t ,s \}$ , ( [ 1],[ 4 ] ).**

**Let $I^R = \{A_i \mid i \in \Lambda\}$ be a collection of fuzzy subset of R. Define the fuzzy subset of R (intersection) by $(\cap_{i \in \Lambda} A_i)(x) = \inf \{ A_i(x) \mid i \in \Lambda \}$, for all $x \in R$,([ 2] ,[ 4 ] ). Define the fuzzy subset of R (union) by $(\cup_{i \in \Lambda} A_i)(x) = \sup\{ A_i(x) \mid i \in \Lambda \}$, for all $x \in R$,([ 3] ,[ 4 ] ).**
**Let $\phi$ denote $\phi(x) = 0$ for all $x \in R$, the empty fuzzy subset of R, ([1], [5]).**

**Note that throughout our work any fuzzy subset is a nonempty fuzzy subset.**

**Let A and B be a fuzzy subsets of R , the product $A \circ B$ define by : $A \circ B(x) = \sup\{ \min\{A(y), B(z) \} \mid x = y \cdot z\}$ $y, z \in R$ , for all $x \in R$ , [6] .**

**The addition A + B define by $(A + B)(x) = \sup \{\min \{A(y), B(z) \mid x = y + z\}$ $y, z \in R$ , for all $x \in R$ , [ 6] .**

**Let A be a fuzzy subset of R, A is called a fuzzy subgroup of R if for all $x, y \in R$, $A(x + y) \geq \min \{A(x), A(y)\}$ and $A(x) = A(-x)$, [6].**
**Let A be a fuzzy subset of R , A is called a fuzzy ring of R if for all $x, y \in R$ ,**
**$A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \min \{A(x), A(y)\}$, ([5],[6]).**

**A fuzzy subset A of R is called a fuzzy ideal of R if and only if for all $x, y \in R$, $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \max \{A(x), A(y)\}$ , ([5],[6]) .**
**Let X be a fuzzy ring of R and A be a fuzzy ideal of R such that $A \subseteq X$. Then A is a fuzzy ideal of the fuzzy ring X [6]. But let X be a fuzzy ring of R and A be a fuzzy subset. A is**

**called a fuzzy ideal of the fuzzy ring X if $A \subseteq X$ (that is $A(a) \leq X(a)$, for all $a \in R$) , $A(b-c) \geq \min \{A(b) , A(c)\}$ and $A(bc) \geq \min \{A(b),A(c)\}$,for all b ,c$\in$ R, [5].And A is a fuzzy ideal of the fuzzy ring X of R if $A(b-c) \geq \min \{A(b),A(c)\}$ and $A(bc) \geq \min \{\max\{A(b),A(c)\},X(bc )\}$ ,( [3],[5] ).**

**Let X be a fuzzy ring of R . A be a fuzzy subset of X is a fuzzy ideal of X if and only if $A_t$ is an ideal of $X_t$ ,for all $t \in [0,A(0)]$ , [5].**

**Let A be a fuzzy ideal of R. If for all $t \in [0, A (0)]$, then $A_t$ is an ideal of R is an ideal of R, ([3], [4]).**

**Let A and B be two fuzzy set of R ,then : $(A \cup B )(x) = \max \{A(x) ,B(x) \}$ and $(A \cap B )(x) = \min \{A(x) ,B(x) \}$ ,for all $x \in R$ .**

**In general, if $\{A_i \mid i \in \Lambda\}$ is a family of fuzzy ideals of R. Then $\bigcap_{i \in \Lambda} A_i$ is a fuzzy ideal of R, ([5], [6],[7] ).**

**Let A and B are fuzzy ideals of R ,then $A \circ B$ is a fuzzy ideal of R, [5] .**

**Let A and B are fuzzy ideals of R, then $A \cap B$, $A + B$ are fuzzy ideals of R , ([7],[8] ).**

**A non empty fuzzy subset A of M is called a fuzzy module of M if and only if for all x, y $\in$ M, then $A(x - y) \geq \min \{A(x), A(y)\}$ and $A( rx ) \geq A(x)$ and $A(0) = 1$,(0 is the zero element of M) , [10] .**

**A and B are fuzzy modules of an R-module M , B is called a fuzzy submodule in A if and only if $B \subseteq A$ , [10] .**

**Let A be a fuzzy subset of an R-module M . A is a fuzzy submodule of M if and only if $A_t$ is a submodule of M , for all $t \in [0, 1]$ , [11] .**

**PROPOSITION 1. 1 ([8] ,[9] ) :**

**Let A and B be two fuzzy subsets of R-module M .Then:**

1. **$A \circ B \subseteq A \cap B$.**
2. **$(A \circ B)_t = A_t \cdot B_t$, $t \in [0, 1]$.**
3. **$(A \cap B)_t = A_t \cap B_t$, $t \in [0, 1]$.**
4. **$(A \cup B)_t = A_t \cup B_t$, $t \in [0, 1]$ ,by [6] .**

**PROPOSITION 1. 2 [5] :**

Let $X : R \rightarrow [0,1]$, $Y : R' \rightarrow [0,1]$ are fuzzy rings $f : R \rightarrow R'$ be homomorphism between them and $A : R \rightarrow [0,1]$ a fuzzy ideal of X , $B : R' \rightarrow [0,1]$ a fuzzy ideal of Y, then :

1. f (A) is a fuzzy ideal of Y.
2. $f^{-1}(B)$ is a fuzzy ideal of X.

**PROPOSITION 1. 3 [9] :**

Let A and B be two fuzzy subsets of R-module M and f is inverse image function of B .Then:

1- $f(A) \cap f(B) = f(A \cap B)$ . 2- $f(A) \circ f(B) = f(A \circ B)$ .

3- $f(A_t) = (f(A))_t$ . 4- $f^{-1}(A_t) = (f^{-1}(A))_t$ .

We give concepts of a maximal fuzzy ideal and a prime fuzzy ideal . We give some basic properties of these concepts .

**DEFINITION 1. 4 [ 7 ] :**

Let A be a fuzzy ideal of R .Then A is called a maximal fuzzy ideal of R if either $A = \lambda_R$ or

1- A is not constant , and

2- For any fuzzy ideals B and C of R ,if $B \circ C \subseteq A$ , then either $B \subseteq A$ or $C \subseteq A$ .

**DEFINITION 1 .5 [ 9 ] ,[ 11 ] :**

Let A be a fuzzy ideal of R .Then A is called a prime fuzzy ideal of R if $A(xy) = A(0)$ ,then $A(x)=A(0)$ or $A(y)=A(0)$ .

In [12] , the authors explain the suitability of Definition (1.4) ,(1 .5) over the one which requires $A(xy) = \max\{A(x), A(y)\}$.

**DEFINITION 1 . 6 [ 13 ] :**

Let P be a non constant fuzzy ideal of R . P is said to be a prime fuzzy ideal of R if and only if for all fuzzy ideals A ,B of R , $A \circ B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, [11] .

**PROPOSITION 1.7 [ 14 ] :**

**Definitions ( 1.5)and (1.6 ) are equivalent .**

**PROPOSITION 1 .8 [15] :**

**A fuzzy ideal A is a prime ideal if and only if $0_1$ is a maximal fuzzy ideal of A .**

**PROPOSITION 1 .9 [15] :**

**A fuzzy ideal of prime ring is prime fuzzy ideal.**

**REMARKS and EXAMPIES 1 .10 [15] :**

**1- Every fuzzy ideal of the Z , $Z_n$ is a maximal fuzzy ideal if and only if n is a prime number.**

**2- Every fuzzy ideal of Q as a Z is a prime fuzzy Z-module.**

**3- Every fuzzy ideal of the Z , M=Z $\oplus$ Z is a maximal fuzzy ideal .**

**PROPOSITION 1 .11 [15] :**

**(1) A is a maximal fuzzy ideal of R if and only if $A_t$ is a maximal ideal for all t $\in$ [0,A(0)] .**

**(2) A is a prime fuzzy ideal of R if and only if $A_t$ is a prime ideal for all t $\in$ [0,A(0)] .**

**PROPOSITION 1 .12 [15] :**

**Let I be a ideal of R-module M and let $A_I$ be the fuzzy ideal of R .Then I is a prime ideal of R if and only if $A_I$ is a maximal fuzzy ideal of R .**

**DEFINITION 1 .13 [ 14 ] :**

**Let P be a non constant fuzzy ideal of R . P is said to be an L- prime fuzzy ideal of R if and only if for all x ,y $\in$ R , either P(x y)=max {P(x) ,P(y)},[8] .**

**PROPOSITION 1.14 ([11],[12]) :**

**1- P is a prime fuzzy ideal of R if and only if Im(P)={t,1} with 0 $\leq$ t < 1 and $P_1$ is a prime ideal of R .**

**2- P is an L- prime fuzzy ideal of R if and only if $P_t$ is a prime ideal of R ,for all t $\in$[0,P(0)]**

**PROPOSITION 1 .15 [15] [14] :**

**Let X be a fuzzy ring of a ring $R_1$ and Y be a fuzzy ring of $R_2$ . Let f: $R_1 \to R_2$ be an epimorphism such that the fuzzy ideal $0_1$ of X is f-invariant . Then Y is a prime fuzzy ideal , if X is a prime fuzzy ideal .**

**REMARKS 1 .16 [15] :**

**The converse of proposition (1.15) is not true in general, the condition ($0_1$ is f-invariant) is necessary for example:**

**Let f: $Z \to Z/<8> = Z_8$ defined as: f(a) = a ,f is an epimorphism .**

**Let X: $Z \to [0, 1]$ defined by:**

$$X(a)=\begin{cases} 1 & if a \in 2Z \\ 0 & otherwise \end{cases}$$

**Thus X is a prime fuzzy ideal . Let Y : $Z_8 \to [0,1]$ defined by :**

$$Y(a)=\begin{cases} 1 & if\ a \in \{0,2,4,6\} \\ 0 & otherwise \end{cases}$$

**Then Y is not prime fuzzy ideal since $2_{1/2} \circ 4_{1/2} \subseteq 0_1$ . But $4_{1/2} \not\subset 0_1$ and $2_{1/2} \subseteq$ F-Ann Y since $2_{1/2} \circ 2_{1/2} = 4_{1/2} \not\subset 0_1$ .**

**Moreover, note that $0_1$ is not f-invariant since f (8) =f(0) ,but $0_1$ (8) =$0_1 \neq 0_1(0)$ = 1.**

**PROPOSITION 1 .17 [14] ,[15] :**

**Let X be a fuzzy module of R-module $M_1$ and A and B be two fuzzy submodules in X and Y be a fuzzy module of R-module $M_2$ and C and D be two fuzzy submodules in Y. Let f : $M_1 \to M_2$ be a homomorphism . Then:**

1. **f (A∩B) = f (A)∩ f (B) , where f is a monomorphism .**
2. **$f^{-1}$(C∩D) = $f^{-1}$(C) ∩ $f^{-1}$(D) .**

**PROPOSITION 1. 18 [14] ,[15] :**

**Let X : M → [0,1], Y : M′→ [0,1] are fuzzy modules . Let f : M → M′ be homomorphism between them and A : M → [0,1] a fuzzy submodule in X , B : M′→ [0,1] a fuzzy submodule in Y, then :**

1. **f (A) is a maximal fuzzy submodule in Y.**
2. **$f^{-1}$(B) is a maximal fuzzy submodule in X.**

**Now we give the concept of the maximal fuzzy module . We give some basic properties of it .**

**DEFINITION 1.19 [16] :**

**A fuzzy module A of a R-module M is called a maximal fuzzy submodule (module) of an R-module M if and only if $A_t$ is maximal module of M , for all $t \in (0, 1]$ .**

**PROPOSITION 1.20 [16] :**

**Let N be a submodule of R-module M and let $A_N$ be the fuzzy module of M determined by N . Then N is a maximal submodule of M if and only if $A_N$ is a maximal fuzzy submodule of M .**

**PROPOSITION 1.21 [16] :**

**Let A be a maximal fuzzy module of R-module $M_1$ and B be a maximal fuzzy module of R-module $M_2$ . Let $f : M_1 \rightarrow M_2$ be a R-epimorphism . Then :**

1) **f (A) is a maximal fuzzy submodule of $M_2$ if f is f-invariant .**
2) **$f^{-1}$ (B) is a maximal fuzzy submodule of $M_1$ .**

**PROPOSITION 1.22 [16] :**

**Let X be a fuzzy module of R-module $M_1$ and Y be a fuzzy module of R-module $M_2$ . Let $f : M_1 \rightarrow M_2$ be an epimorphism. If $0_t$ is a maximal fuzzy submodule in X , then Y is a maximal fuzzy module , if F-Ann X is a maximal fuzzy module .**

**REMARKS 1.23 [16] :**

**The proposition (2.7) is not true in general , the condition ($0_t$ is maximal fuzzy module ) is necessary for example:**

**Let $f : Z \rightarrow Z/ <8> = Z_8$ defined as f(x) = x , f is an epimorphism , let $X : Z \rightarrow [0,1]$ defined by :**

$$X(a) = \begin{cases} 1 & if a \in 2Z \\ 0 & otherwise \end{cases}$$ **. Thus X is a maximal fuzzy module of Z.**

**And let $Y : Z_8 \rightarrow [0,1]$ defined by :** $$Y(a) = \begin{cases} 1 & if a \in \{0,2,4,6\} \\ 0 & otherwise \end{cases}$$ **.**

**Thus Y is not maximal fuzzy module of $Z_8$ . Moreover , $0_1$ is not maximal since f(8) = f(0) , but $0_1(8) = 0_1 \neq 0_1(0) = 1$ .**

**PROPOSITION 1.24 [16] :**

Let A and B are fuzzy ideals of R , then (A:B) is a fuzzy ideal of R .

**PROPOSITION 1.25 [16] :**

Let A be maximal fuzzy submodule of a fuzzy module X of R-module M and let I be a fuzzy ideal of R such that I (0) = 1 , then ($A:_X$ I) is a maximal fuzzy module of R-module M .

## S.2 FUZZY SPECTRUM OF MODULES :

We give the basic concept about fuzzy spectrum of modules and we give and prove new results.

Also, we introduce the definition about fuzzy spectrum of R-module M and we give and prove some properties about fuzzy spectrum of R .

**DEFINITION 2.1 :**

Let R be a ring. The collection of the set of all maximal fuzzy submodules of an R-module M is called the fuzzy spectrum of an R-module M and denoted by F-spec(R) .That is : F-spec( R) = { A ∣ A is a maximal fuzzy submodule of an R-module M } .

**REMARKS 2 .2 :**

Let R be a ring and M be a module of R .

1. X = {A ∣ A is a maximal fuzzy submodule of an R-module M } = F-spec(R), (for simplicity).
2. The variety of the fuzzy ideal B denoted by V(B ) and it is defined by : V(B ) = { A $\in$ X ∣ B $\subseteq$ A }.
3. X (B) =X- V (B), the complement of V (B ) in X .

We shall give in the following some properties of the variety of fuzzy ideals.

**PROPOSITION 2 .3 :**

Let A and B be two fuzzy submodules of an R-module M .Then:

1. If A $\subseteq$ B ,then V(B ) $\subseteq$ V( A ) .
2. V(A ) U V( B ) $\subseteq$ V( A $\cap$B ) .
3. If {$A_i$ ∣ i$\in$ $\Lambda$ } is a family of fuzzy submodules of an R-module M , then : V (U {$A_i$ ∣ i$\in$ $\Lambda$ }) = $\cap$ V{$A_i$ ∣ i$\in$ $\Lambda$ }.
4. V ($\phi$ )= X and V( $X_R$ )= $\phi$ .

**PROOF:**

1. It is easy.
2. Let $C \in V(A) \cup V(B)$ .Then $C \in V(A)$ or $C \in V(B)$ . If $C \in V(A)$ ,then C is a maximal fuzzy submodule of an R-module M and $A \subseteq C$ . But $A \cap B \subseteq A$ , which implies that $A \cap B \subseteq C$ .Thus $C \in V(A \cap B)$ .

   Similarly, If $C \in V(B)$, which implies that $C \in V(A \cap B)$ . Therefore , $V(A) \cup V(B) \subseteq V(A \cap B)$ .

3. Let $C \in V(\cup \{A_i \mid i \in \Lambda\})$ .Then $\cup \{A_i \mid i \in \Lambda\} \subseteq C$ , where C is a maximal fuzzy submodule of an R-module M .Thus $A_i \subseteq C$ ,for all $i \in \Lambda$ . Implies that $C \in V(A_i)$, for all $i \in \Lambda$ . Therefore $C \in \cap V\{A_i \mid i \in \Lambda\}$.

   Similarly ,we prove that $\cap V\{A_i \mid i \in \Lambda\} \subseteq V(\cup \{A_i \mid i \in \Lambda\})$ . Therefore , $V(\cup \{A_i \mid i \in \Lambda\}) = \cap V\{A_i \mid i \in \Lambda\}$.

4. We must prove that $V(\phi) = X$ , where $\phi$ is an empty fuzzy set of an R-module M . But $V(\phi) = \{A \in X \mid \phi \subseteq A\}$, and $\phi(x) = 0 \leq A(x)$ , for all $x \in R$ and for all $A \in X$ . Then $\phi \subseteq A$ ,for all $A \in X$ .Therefore $V(\phi) = X$ .

   Also ,we must prove that $V(X_R) = \phi$ ,where $X_R$ is a characteristic function of R defined by $X_R(x) = 1$ ,for all $x \in R$ . But $V(X_R) = \{A \in X \mid X_R \subseteq A\}$,since $A \subset X_R$ , for all $A \in X$ .Therefore, $V(X_R) = \phi$ .

**DEFINITION 2 . 4 [4] :**

Let B be a fuzzy set of an R-module M and <B> the intersection of all fuzzy submodules A of an R-module M such that $B \subseteq A$ .Then <B> is called the fuzzy submodule of R generated by B. That is : $\langle B \rangle = \{A \mid B \subseteq A$ , A is a fuzzy submodule of an R-module M $\}$ .

It's clear that $B \subseteq \langle B \rangle$ and if $B \subseteq C$ , then $\langle B \rangle \subseteq C$ .

**PROPOSITION 2 .5 :**

Let B be a fuzzy submodule of an R-module M .Then : $V(B) = V(\langle B \rangle)$ .

**PROOF:**

We must prove that $V(B) \subseteq V(<B>)$ and $V(<B>) \subseteq V(B)$ .If $C \in V(B)$ ,then C is a maximal fuzzy submodule of an R-module M and $B \subseteq C$ .Thus $C \in V(<B>)$ .Since $B \subseteq <B>$ . Then $V(<B>) \subseteq V(B)$ by proposition (2.3(1)) .Therefore $V(B) = V(<B>)$ .

**COROLLARY 2 .6 [17] :**

Let A and B be two fuzzy submodules of an R-module M .Then $V(A \cap B) = V(A) \cup V(B)$ .

**PROOF:**

It's clear by proposition (2.3(4)) .

**COROLLARY 2 .7 :**

Let $\{A_i \mid i \in \Lambda\}$ be a family of fuzzy submodule of an R-module M. Then:

$\bigcup_{i \in \Lambda} X(A_i) = X(< \bigcup_{i \in \Lambda} A_i >)$ .

**PROOF:**

$\bigcup_{i \in \Lambda} X(A_i) = \bigcup_{i \in \Lambda} (X-V(A_i)) = X- \bigcap_{i \in \Lambda} V(A_i)) = X - V(\bigcup_{i \in \Lambda} A_i)$ , by proposition (2.3) $= X-V(<\bigcup_{i \in \Lambda} A_i >)$ , by proposition (2.5) $\bigcup_{i \in \Lambda} X(A_i) = X(< \bigcup_{i \in \Lambda} A_i >)$ .

**PROPOSITION 2 .8 :**

Let A be a fuzzy submodule of an R-module M and B be a maximal fuzzy submodule of an R-module M. Then $B \in V(A)$ if and only if $B_t \in V(A_t)$ , for each $t \in (0,A(0)]$ .

**PROOF:**

Since A is a fuzzy submodule of an R-module M and B is a maximal fuzzy submodule of an R-module M , then $A_t$ is a prime submodule of an R-module M ,for each $t \in (0,A(0)]$ by (proposition (1.11) and proposition (2.3)) .

$B \in V(A) \leftrightarrow A \subseteq B$

$\leftrightarrow A_t \subseteq B_t$ , for each $t \in (0,A(0)]$ .

$\leftrightarrow B_t \in V(A_t)$ , for each $t \in (0,A(0)]$ .

**PROPOSITION 2 .9 :**

**Let I be an ideal of R-module M and J be a prime submodule of an R-module M.Then $J \in V(I)$ if and only if $Q_J \in V(A_I)$ .where $Q_J$ and $A_I$ are the fuzzy submodules of an R-module M determine by I and J respectively .That is :**

**$A_I(x) = t$ $x \in I$ and $A_I(x) = s$ otherwise and $Q_J(x) = t$ $x \in J$ and $Q_J(x) = s$ otherwise ,where $t, s \in [0,A(0)]$ and $t > s$ .**

**PROOF:**

**If $J \in V(I)$ , then $I \subseteq J$ .And $Q_J$ is a maximal fuzzy submodule of an R-module M by proposition (1.11) and proposition (1.12) .We have to show that $A_I \subseteq Q_J$ .**

**Let $x \in R$, then either $x \in I$ or $x \in I$. If $x \notin I$, then $A_I(x) = t$ and $Q_J(x) = t$ (since $I \subseteq J$ ) . If $x \notin I$, then either $x \in J$ or $x \notin J$ implies that $A_I(x) = s$ and $Q_J(x) = t$ and $A_I(x) = s$ and $Q_J(x) = s$. Hence $A_I(x) \leq Q_J(x)$, for all $x \in R$ .Therefore $A_I \subseteq Q_J$ .Hence $Q_J \in V(A_I)$ .**

**Conversely, if $Q_J \in V(A_I)$ ,then $A_I \subseteq Q_J$ .Thus $A_I(x) \leq Q_J(x)$ , for all $x \in R$ . If $x \in I$ implies that $A_I(x) = t = Q_J(x)$ .Hence $I \subseteq J$ implies that $J \in V(I)$ .**

## S.3 A MODULE WITH FUZZY ZARISKI TOPOLOGY :

**In ([17] ,[18]),the collection of all V(K) , K is a submodule of an R-module M is denote by T . M is called a ring with Zariski topology if:**

1. **The empty set and M are in T .**
2. **T closed under arbitrary intersection.**
3. **T is closed under finite union.**

**Our concern in this section is to introduce the concept of a ring with fuzzy Zariski topology.**

**We put T = {X (B) | B is a fuzzy submodule of an R-module M } ,then each of the empty fuzzy set and X are belong to T . Also T closed under arbitrary intersection and T closed under finite union .**

**However T need not be closed under finite intersection in general . This lead us to introduce the following definition.**

**DEFINITION 3 . 1 :**

**R-module M is called a module with fuzzy Zariski topology, if T is closed under finite intersection.**

**That mean for any fuzzy submodules B and C of an R-module M , there exists a fuzzy submodule D of an R-module M such that $X(B) \cap X(C) = X(D)$ .**

**DEFINITION 3 . 2 [18] :**

1. **An submodule I of R-module M is called semiprime if I is an intersection of prime submodule.**
2. **A prime submodule J of R-module M is called extraordinary if whenever I and K are semiprime submodules of an R-module M with $I \cap K \subseteq J$ , then either $I \subseteq J$ or $K \subseteq J$ .**

**In order to get necessary and sufficient conditions for a module to be a module with fuzzy Zariski topology we introduce the following concepts .**

**DEFINITION 3 . 3 :**

**A fuzzy submodule A of an R-module M is called semiprime fuzzy submodule of an R-module M if A is an intersection of prime fuzzy submodule of an R-module M .**

**PROPOSITION 3 .4 :**

**Let A be a fuzzy submodule of an R-module M .Then A is a semiprime fuzzy submodule if and only if $A_t$ is a semiprime submodule of an R-module M, for all $t \in (0, A(0)]$.**

**PROOF:**

**If A is a semiprime fuzzy submodule of an R-module M.$A = \cap_{i\in \Lambda} A_i$, where Ai is a maximal fuzzy submodule of an R-module M , for all $i\in \Lambda$.**

**Since $A_t = (\cap_{i\in \Lambda} A_i)_t$ , for all $t \in (0,A(0)]$. But $(\cap_{i\in \Lambda} Ai)_t = \cap_{i\in \Lambda} (A_i)_t$ by proposition (1.1(3)).**

**Thus $A_t = \cap_{i\in \Lambda} (A_i)_t$ and $(A_i)_t$ is a prime submodule of an R-module M,for all $t \in (0,A(0)]$, then $A_t$ is a semiprime submodule of an R-module M by definition (3.2) .**

**Conversely , let $t \in (0,A(0)]$ , $A_t$ be a semiprime submodule of an R-module M . Then $A_t = \cap_{i\in \Lambda} (I_i)$ , where $I_i$ is a prime submodule of an R-module M , for all $i\in \Lambda$**

Now , for all $i \in \Lambda$ ,define $A_{Ii} : R \to [0,1]$ by : $A_{Ii}(x) = t$ if $x \in I_i$ and $A_{Ii}(x) = s$ otherwise , where $t,s \in [0,1]$ and $t > s$ .

Then $A_{Ii}$ is a maximal fuzzy submodule of an R-module M , for all $i \in \Lambda$ by proposition (1.11).

Clearly $A_{Ii} = I_i$ ,for all $i \in \Lambda$ .Therefore $A_t = \cap_{i \in \Lambda} (A_{Ii})_t = (\cap_{i \in \Lambda} A_{Ii})_t$ implies that $A = (\cap_{i \in \Lambda} A_{Ii})$ . Hence A is a semiprime fuzzy submodule of an R-module M .

**DEFINITION 3 . 5 :**

A maximal fuzzy submodule A of R-module M is called extraordinary if whenever B and C are semiprime fuzzy submodules of an R-module M with $B \cap C \subseteq A$ , then either $B \subseteq A$ or $C \subseteq A$ .

**PROPOSITION 3 .6 :**

Let A be a fuzzy submodule of an R-module M.Then A is an extraordinary fuzzy submodule of an R-module M if and only if $A_t$ is an extraordinary submodule of an R-module M ,for all $t \in (0,A(0)]$ .

**PROOF:**

If A is a extraordinary fuzzy ideal of R . Let for all $t \in (0,A(0)]$ , suppose that $I \cap J \subseteq A_t$ ,where I and J are semiprime ideal of R .Let $s, k \in [0,A(0)]$ with $s \neq k$, $s < k$ and $k < t$ .

Define $A_I : R \to [0,1]$ and $A_J : R \to [0,1]$ by : $A_I(x) = t$ if $x \in I$ and $A_I(x) = s$ otherwise and $A_J(x) = t$ if $x \in J$ and $A_J(x) = k$ otherwise .

Then $A_I$ and $A_J$ are fuzzy submodules of an R-module M .Clearly $(A_I)_t = I$ and $(A_J)_t = J$ . Therefore $(A_I)_t$ and $(A_J)_t$ are semiprime submodules of an R-module M and by proposition (3.4) implies that $A_I$ and $A_J$ are semiprime fuzzy submodules of an R-module M.

Now, $(A_I)_t \cap (A_J)_t \subseteq A_t$ . Hence $(A_I \cap A_J)_t \subseteq A_t$ by proposition (1.1(3)) . Therefore $A_I \cap A_J \subseteq A$ .Since A is a extraordinary fuzzy submodule of an R-module M , then $A_I \subseteq A$ or $A_J \subseteq A$ . Hence $(A_I)_t \subseteq A_t$ or $(A_J)_t \subseteq A_t$ which completes the proof.

Conversely, let A be a maximal fuzzy submodule of an R-module M such that $A_t$ is an extraordinary submodule of an R-module M,for all $t \in (0,A(0)]$ .

**Suppose that $B \cap C \subseteq A$, where B and C are semiprime fuzzy submodule of an R-module M . $(B \cap C)_t \subseteq A_t$ implies that $B_t \cap C_t \subseteq A_t$ by proposition (1.1(3)) . And according to proposition (3.4) , $B_t$ and $C_t$ are semiprime submodules of an R-module M and since $A_t$ is an extraordinary submodule of an R-module M ,for all $t \in (0,A(0)]$ by hypothesis .**

**Then either $B_t \subseteq A_t$ or $C_t \subseteq A_t$ . Hence $B \subseteq A$ or $C \subseteq A$ by ([8] ,[9]) .**

**Thus A is a extraordinary fuzzy submodule of an R-module M .**

**DEFINITION 3 . 7 :**

**Let A be a fuzzy submodule of an R-module M. The prime radical fuzzy of A denoted by F-rad (A) is the intersection of all maximal fuzzy submodules of an R-module M which contains A.**

**PROPOSITION 3 .8 :**

**If A and B are fuzzy submodules of an R-module M, then :**

1. **$A \subseteq$ F-rad (A).**
2. **F-rad (A) is a semiprime fuzzy submodule of an R-module M .**
3. **V (A) = V (F-rad (A)).**
4. **V (A) $\subseteq$ V (B) if and only if F-rad (B) $\subseteq$ F-rad (A).**

**PROOF:**

1. **It's obvious.**
2. **Since F-rad (A) is the intersection of all maximal fuzzy submodules of an R-module M which contains A .Then F-rad (A) is a semiprime fuzzy submodule of an R-module M , by definition (3.3) .**
3. **Let $B \in$ V (A) , then B is a maximal fuzzy submodule of an R-module M and $A \subseteq B$ implies that F-rad (A) $\subseteq$ B by definition (3.3) . Thus $B \in$ V (F-rad (A) ) ----(1) .**

**Now, let $B \in$ V(F-rad (A) ) , then B is a maximal fuzzy submodule of an R-module M and F-rad (A) $\subseteq$ B . Since $A \subseteq$ F-rad (A) by part (1) , then $A \subseteq B$ . Thus $B \in$ V (A) . Therefore V (F-rad (A) ) $\subseteq$ V (A) ---(2).**

**From (1) and (2) , we have V(A) = V(F-rad (A) ).**

4. **Suppose V (A) $\subseteq$ V(B), then $\cap\{C \mid C \in V(B)\} \subseteq \cap\{C \mid C \in V(A)\}$. But F-rad (A) $=\cap\{C \mid C \in V(A)\}$ and F-rad (B) = $\cap\{C \mid C \in V(B)\}$. Thus (F-rad (B)) $\subseteq$ (F-rad (A)) .**

**Conversely, Since F-rad (B) $\subseteq$ F-rad (A), then V(F-rad (A)) $\subseteq$ V(F-rad (B)) by proposition (2.3(1)) . Therefore V (A) $\subseteq$ V (B) by part (3) .**

**THEOREM 3.9 :**

**Let R be a ring and M be a module of R . Then the following statements are equivalent:**

1. **M is a module with fuzzy Zariski topology.**
2. **Every prime fuzzy submodule of an R-module M is a fuzzy extraordinary.**
3. **V (A) U V (B) =V (A $\cap$B) for any semiprime fuzzy submodules A and B of an R-module M .**

**PROOF:**

**(1)→ (2), let C be a maximal fuzzy submodule of an R-module M and A and B be two semiprime fuzzy submodules of an R-module M such that A $\cap$ B $\subseteq$ C . Then by (1) ,there exists a fuzzy submodule D of an R-module M such that V(A) U V(B) = V(D) . Since A is a semiprime fuzzy submodule of an R-module M, then A = $\cap$ i$\in \Lambda$ $A_i$ , for some $\{A_i \mid i\in \Lambda\}$ of prime fuzzy submodule of an R-module M by definition (3.3) .**

**Now, for all i$\in\Lambda$ , $A_i \in$ V(A) $\subseteq$ V(D) . So that D $\subseteq A_i$ for all i$\in\Lambda$. Thus D $\subseteq \cap$ i$\in\Lambda$ Ai = A.**

**Similarly, D $\subseteq$ B. Thus D $\subseteq$ A $\cap$ B. Therefore V(A $\cap$ B) $\subseteq$ V(D) by proposition (2.3 (1)) .Since V(A) U V(B) $\subseteq$ V(A $\cap$B) , hence V(A) U V(B) $\subseteq$ V(A $\cap$B) ) $\subseteq$ V(D) = V(A) U V(B) . Therefore V(A) U V(B) = V(A $\cap$B) . But C is a maximal fuzzy submodule of an R-module M containing A $\cap$ B therefore C $\in$ V (A $\cap$B) = V (A) U V(B) . Hence either C $\in$ V (A) or C $\in$ V (B). That is, either A $\subseteq$ C or B $\subseteq$ C . Hence the result follows .**

**(2) → (3), let A and B be two semiprime fuzzy submodules of an R-module M . Clearly V(A) U V(B) $\subseteq$ V(A $\cap$B) by proposition (2.3 (2)) .To prove V(A $\cap$B) $\subseteq$ V(A) U**

**V(B) .Let C $\in$ V(A $\cap$B) , then C is a maximal fuzzy submodule of an R-module M and (A $\cap$B) $\subseteq$ C . By (2) ,C is extraordinary fuzzy . Hence either A $\subseteq$ C or B $\subseteq$ C. That is either C $\in$ V(A) or C $\in$ V(B) . Therefore, C $\in$ V (A) U V(B) which implies (3) .**

**(3) $\rightarrow$ (1) , let A and B be two fuzzy submodules of an R-module M. Then F-rad (A) and F-rad (B) are semiprime fuzzy submodules of an R-module M by proposition (3.8 (2)) and V(A) U V(B) = V(F-rad (A)) U V(F-rad (B)) by proposition (3.8 (3)) = V[(F-rad (A)) $\cap$ (F-rad (B))] by (3) which proves (1) .**

**PROPOSITION 3 .10 :**

**Let $M_1$ and $M_2$ be two modules of R and f be a homomorphism from $M_1$ to $M_2$ . If A is a semiprime fuzzy submodule of an R-module $M_2$ , then $f^{-1}$ (A) is a semiprime fuzzy submodule of an R-module $M_1$ .**

**PROOF:**

**Since A is a fuzzy submodule of an R-module $M_2$ , then $f^{-1}$ (A) is a fuzzy submodule of an R-module $M_1$ by proposition (1.2 ) .And since A is a semiprime fuzzy submodule of an R-module $M_2$ , then A = $\cap_{i\in \Lambda} A_i$ , where $A_i$ is a maximal fuzzy submodule of an R-module $M_2$ , for all $i\in \Lambda$ .**
**$f^{-1}(A) = f^{-1}(\cap_{i\in \Lambda} A_i) = \cap_{i\in \Lambda} (f^{-1} A_i)$ by proposition ( 1.17 ) . But $f^{-1}(A_i)$ is a maximal fuzzy submodule of an R-module $M_2$ , for all $i\in \Lambda$ ,by proposition ( 1.18 ) . Therefore, $f^{-1}$ (A) is a semiprime fuzzy submodule of an R-module $M_1$ .**

**COROLLARY 3 .11 :**

**Any homomorphism image of a module with fuzzy Zariski topology is a module with fuzzy Zariski topology.**

**PROOF:**

**Let f : $M_1 \rightarrow M_2$ be a homomorphism image such that $M_1$ is a module with fuzzy Zariski topology . We have to prove that $f(M_1)$ is a module with fuzzy Zariski topology.**

**Let A be a maximal fuzzy submodule of $f(M_1)$ and B, C be two semiprime fuzzy submodules of $f(M_1)$ such that $B \cap C \subseteq A$ .**

**Now, $f^{-1}(B) \cap f^{-1}(C) = f^{-1}(B \cap C) \subseteq f^{-1}(A)$ by proposition (1.3,(2)) . By proposition (1.18), $f^{-1}(A)$ is a maximal fuzzy submodule of an R-module $M_1$. And by proposition (3.10) , $f^{-1}(B)$ and $f^{-1}(C)$ are two semiprime fuzzy submodule of an R-module $M_1$ . Therefore, $f^{-1}(B) \subseteq f^{-1}(A)$ or $f^{-1}(C) \subseteq f^{-1}(A)$ since $R_1$ is a module with fuzzy Zariski topology.**

**Then $B \subseteq A$ or $C \subseteq A$ by [19] ,which proves that $f(M_1)$ is a module with fuzzy Zariski topology by proposition (3.9) .**

**PROPOSITION 3 .12 :**

**Let I be a submodule of an R-module M. Then I is a semiprime i submodule of an R-module M if and only if $A_I$ is a semiprime fuzzy submodule of an R-module M.**

**PROOF:**

**Since I is a submodule of an R-module M, then $I = \cap_{i\in \Lambda} I_i$ , where $I_i$ is a prime submodule of an R-module M , for all $i\in \Lambda$ by definition (3.2) . Since $A_{I_i}$ is a fuzzy submodule of an R-module M , for all $i\in \Lambda$ by proposition (1.12) . Note that:**

**$\cap_{i\in \Lambda} A_{I_i}(x) = \inf \{ A_{I_i}(x) \mid i\in \Lambda \}$ , for each $x \in R$ .**

$$= \begin{cases} t & \text{if } x\in Ii \\ s & \text{otherwise} \end{cases} \quad \text{, for some } i\in \Lambda$$

$$= \begin{cases} t & \text{if } x\in \bigcap_{i\in\Lambda} Ii \\ s & \text{otherwise} \end{cases}$$

**$= A_I(x)$.**

**Therefore, $A_I$ is a semiprime fuzzy submodule of an R-module M.**

**Conversely, if $A_I$ is a semiprime fuzzy submodule of an R-module M, then $A_I = \cap_{i\in \Lambda} A_i$ , where $A_i$ is a maximal fuzzy submodule of an R-module M for each $i\in \Lambda$ . Therefore $(A_i)_t$ is a prime submodule of an R-module M for each $i\in \Lambda$ with $t\in(0,A(0)]$ by proposition (1.11) . We claim that :**

**$I = \cap_{i\in \Lambda} (A_i)_t = (\cap_{i\in \Lambda} A_i)_t = (A_I)_t$ . Thus I is a semiprime submodule of an R-module M.**

**THEOREM 3 .13 :**

**Let R be a ring and M be module of R . Then the following statements are equivalent:**

**(1) M is a module with Zariski topology.**

**(2) M is a module with fuzzy Zariski topology.**

**PROOF:**

**(1)→ (2), let A and B be two semiprime fuzzy submodules of an R-module M. Then $A_t$ and $B_t$ are semiprime submodules of an R-module M , for each $t\in(0,A(0)]$ by proposition (3.4). By (1) , M is a module with Zariski topology , therefore $V(A_t) \cup V(B_t) = V(A_t \cap B_t)$ by [18] .But $A_t \cap B_t = (A_t \cap B)_t$ , for each $t\in(0,A(0)]$ by proposition (1.1(4)) .Hence $V(A_t) \cup V(B_t) = V(A_t \cap B_t)$---(*)**

**We have to prove that V (A) U V (B) =V (A ∩B). Let C ∈ V (A) U V (B), then C ∈V (A) or C ∈V (B).**

**Therefore, $C_t \in V(A_t)$ or $C_t \in V(B_t)$. , for each $t \in (0, A(0)]$ by proposition (2.8) . Thus, $C_t \in V(A_t) \cup V(B_t)$. , for each $t \in (0, A(0)]$ implies that, $C_t \in V(A \cap B)_t$ , for each $t \in (0, A(0)]$ by (*) . , Hence C ∈V (A ∩B) , (1.1(3)) . Therefore V (A) U V (B) ⊆ V (A ∩B).**

**Similarly, we prove that V (A ∩B) ⊆ V (A) U V (B) .**

**Hence V (A) U V (B) =V (A ∩B) which implies that M is a module with fuzzy Zariski topology.**

**(2)→ (1) , let I and J be two semiprime submodules of an R-module M, we have to prove that V (I) U V (J) =V (I ∩J) .**

**Let $A_t$ and $B_t$ be the fuzzy submodules of an R-module M determined by I and J respectively , such that :**

$$A_I(x)=\begin{cases} t & if\ x\in I \\ s & otherwise \end{cases} \quad \text{and} \quad B_J(x)=\begin{cases} t & if\ x\in J \\ s & otherwise \end{cases}$$

, for each t, s $\in[0,A(0)]$ and t > s .

By proposition (3.12),$A_I$ and $B_J$ are semiprime fuzzy submodules of an R-module M .

Now , if C $\in$V (I) U V (J) , then C is a prime submodule of an R-module M and C $\in$V (I) or C $\in$V(J) .

Let $Q_C$ be a fuzzy submodule of an R-module M determined by C, then $Q_C$ is a maximal fuzzy submodule of an R-module M by proposition (1.12) and according to proposition (2.9), $Q_C \in V(A_I)$ or $Q_C \in V(B_J)$ .

That is $Q_C \in V(A_t) \cup V(B_t) = V(A_t \cap B_t)$ by proposition (3.9) .Note that:

$$(A_I \cap B_J)(x) = \begin{cases} t & \text{if } x \in I \cap J \\ s & \text{otherwise} \end{cases}$$

Which means $(A_I \cap B_J) = (A \cap B)_{I\cap J}$ .Therefore $Q_C \in (A \cap B)_{I\cap J}$ and by proposition (3. 8), we get that C$\in$V (I$\cap$J ) . Hence V (I) U V (J) $\subseteq$ V (I $\cap$ J).

Similarly, we prove that V (I $\cap$J) $\subseteq$ V (I) U V (J) .

Hence V (I) U V (J) =V (I $\cap$J) which implies that M is a module with Zariski topology by [18] .

## COROLLARY 3 .14 :

Let M be a multiplication module .Then M is a module with fuzzy Zariski topology .

**PROOF:**

Since M a multiplication ring . Then M is a module with Zariski topology by ([18],[20]) .

Therefore, M is a module with fuzzy Zariski topology by proposition(3.13) .

## COROLLARY 3 .15 :

Every cyclic ring is a module with fuzzy Zariski topology.

**PROOF:**

It's clearly.

**COROLLARY 3 .16 :**

If M is a finitely generated module .Then M is a module with fuzzy Zariski topology if and only if for all prime submodule I,J and K of an R-module M with $I \cap J \subseteq K$ either $I \subseteq K$ or $J \subseteq K$ .

**PROOF:**

Let M be a module with fuzzy Zariski topology. Then M be a module with Zariski topology by theorem (3.13) .

Let I ,J and K are prime submodules of an R-module M such that $I \cap J \subseteq K$ either $I \subseteq K$ or $J \subseteq K$ by [18] .

Conversely, let I, J and K are prime submodules of an R-module M with $I \cap J \subseteq K$ either $I \subseteq K$ or $J \subseteq K$ . Then M is a module with Zariski topology by [19]. Therefore M be a module with fuzzy Zariski topology by theorem (3.13).

**COROLLARY 3 .17 :**

If M is any finitely generated module such that every prime submodule of an R-module M is irreducible, then M is a module with fuzzy Zariski topology.

**PROOF:**

It's clearly.

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