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# SOME RESULTS ON SOLVABLE FUZZY SUBGROUP OF A GROUP

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## Abstract

In this paper we introduce an alternative definition of a solvable fuzzy group and study some of its properties . Many of new results are also proved, which is useful and important in fuzzy mathematics .

## 1. Introduction

The concept of fuzzy sets is introduced by [1] .Rosenfeld introduced the notion of a fuzzy group as early as 1971.

The technique of generating a fuzzy group (the smallest fuzzy group) containing an arbitrarily chosen fuzzy set was developed only in 1992 by [2] .

Then many research study properties of fuzzy group , fuzzy subgroup of group ,fuzzy coset , and fuzzy normal sub group of group . In this paper we introduce the definition of solvable fuzzy group and study some of its properties .

Now we introduce the following definitions which is necessary and needed in the next section :

**Definition 1.1 [2], [3]**

A mapping from a nonempty set X to the interval [0, 1] is called a fuzzy subset of X .

Next, we shall give some definitions and concepts related to fuzzy subsets of G.

**Definition 1.2**

Let $\mu, v$ be fuzzy subsets of G, if $\mu(x) \leq v(x)$ for every $x \in G$, then we say that $\mu$ is contained in $v$ (or $v$ contains $\mu$) and we write $\mu \subseteq v$ (or $v \supseteq \mu$).

If $\mu \subseteq \nu$ and $\mu \neq \nu$, then $\mu$ is said to be properly contained in $\nu$ (or $\nu$ properly contains $\mu$) and we write $\mu \subset \nu$ ( or $\nu \supset \mu$ ).[4]

Note that: $\mu = \nu$ if and only if $\mu(x) = \nu(x)$ for all $x \in G$.[5]

**Definition 1.3 [4]**

Let $\mu, \nu$ be two fuzzy subsets of G. Then $\mu \cup \nu$ *and* $\mu \cap \nu$ are fuzzy subsets as follows:

(i) $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$

(ii) $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$, for all $x \in G$

Then $\mu \cup \nu$ *and* $\mu \cap \nu$ are called the union and intersection of $\mu$ and $\nu$, respectively.

**Definition 1.4 [5],[6]**

For $\mu, \nu$ are two fuzzy subsets of G, we define the operation $\mu \circ \nu$ as follows:

$(\mu \circ \nu)(x) = sup\{min\{\mu(a), \nu(b)\} \mid a, b \in G \ \ and \ \ x = a * b\}$ For all $x \in G$.

We call $\mu \circ \nu$ the product of $\mu$ and $\nu$.

Now, we are ready to give the definition of a fuzzy subgroup of a group :

**Definition 1.5[2], [7]**

A fuzzy subset $\mu$ of a group G is a fuzzy subgroup of G if:

(i) $min\{\mu(a), \mu(b)\} \leq \mu(a * b)$

(ii) $\mu(a^{-1}) = \mu(a)$ , for all $a, b \in G$.

**Theorem 1.6 [4]**

If $\mu$ is a fuzzy subset of G, then $\mu$ is a fuzzy subgroup of G, if and only if, $\mu$ satisfies the following conditions:

(i) $\mu \circ \mu \subseteq \mu$

(ii) $\mu^{-1} = \mu$

where $\mu^{-1}(x) = \mu(x)$, $\forall$ x $\in$G.

**Proposition 1.7 [7]**

Let $\mu$ be a fuzzy group. Then $\mu(a) \leq \mu(e) \quad \forall a \in G$.

**Definition 1.8 [8]**

If $\mu$ is a fuzzy subgroup of G, then $\mu$ is said to be abelian ,if $\forall\ x, y \in G$, $\mu(x) > 0, \mu(y) > 0$, then $\mu(xy) = \mu(yx)$.

**Definition 1.9 [9] , [10]**

A fuzzy subgroup $\mu$ of G is said to be normal fuzzy subgroup if $\mu(x * y) = \mu(y * x)$ , $\forall x, y \in G$.

**Definition 1.10 [ 11 ]**

Let $\lambda$ and $\mu$ be two fuzzy subsets of G. The commutator of $\lambda$ and $\mu$ is the fuzzy subgroup $[\lambda, \mu]$of G generated by the fuzzy subset $(\lambda, \mu)$ of G which is defined as follows, for any x $\in$ G:

$$(\lambda, \mu)(x) = \begin{cases} \sup \{\lambda(a) \wedge \mu(b)\} & \text{if } x \text{ is a commutator} \\ x = [a, b] & \end{cases}$$

Now, we introduce the following theorems about the commutator of two fuzzy subsets of a group which are needed in the next section :

**Theorem 1. 11[ 11 ]**

If A, B are subsets of G, then$[\chi_A, \chi_B] = \chi_{[A,B]}$

where for all x $\in$ G: $\chi_A(a) = \begin{cases} 1, & if\ a \in A \\ 0, & if\ a \notin A \end{cases}$

**Theorem 1.12[ 11]**

If $\lambda, \mu, \beta$ and $\delta$ are fuzzy subsets of G such that $\lambda \subseteq \mu$ and $\beta \subseteq \delta$, then $[\lambda, \beta] \subseteq [\mu, \delta]$.

**Definition 1.13 [12]**

A fuzzy subgroup $\mu$ of G is said to be normal fuzzy subgroup if $\mu(x * y) = \mu(y * x)$ , $\forall x, y \in G$.

**Corollary 1.14[ 11 ]**

If $\lambda, \beta$ are normal fuzzy subgroups of $\mu$ and $\delta$, respectively. Then $[\lambda, \beta]$ is a normal fuzzy subgroup of $[\mu, \delta]$.

**Proposition 1.15 [3]**

Let λ be a fuzzy subgroup of a fuzzy group $\mu$, then $\lambda$ is a normal fuzzy subgroup in $\mu$ if and only if $\lambda_t$ is normal subgroup in $\mu_t, \forall t \in (0,1]$, where $\lambda(e) = \mu(e)$.

Now, we introduce an important concept about the fuzzy subset.

**Definition 1.16[ 11 ]**

Let $\lambda$ be a fuzzy subset of G. Then the tip of $\lambda$ is the supremum of the set $\{\lambda(x) | x \in G\}$.

**Theorem 1. 17[ 11 ]**

Let $\lambda and \mu$ be fuzzy subsets of G. Then the tip of $[\lambda, \mu]$ is the minimum of tip of $\lambda$ and tip of $\mu$.

Now, we are ready to define the concept of derived chain, which is of great importance in the next section :

**Definition 1. 18[ 11 ]**

Let $\lambda$ be a fuzzy subgroup of G. We call the chain

$$\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq ...... \supseteq \lambda^{(n)} \supseteq ....$$

of fuzzy subgroups of G the derived chain of $\lambda$.

## 2. Solvable Fuzzy Subgroups of A Group

In this section, we propose an alternative definition of a solvable fuzzy group and study some of its properties :

Definition 2.1

Let $\lambda$ be a fuzzy subgroup of G with tip $\alpha$. If the derived chain $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq ............ \supseteq \lambda^{(n)} \supseteq ......$ of $\lambda$ terminates finitely to $(e)_\alpha$, then we call $\lambda$ a solvable fuzzy subgroup of G. If $k$ is the least nonnegative integer such that $\lambda^{(k)} = (e)_\alpha$, then we call the series $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} ......... \supseteq \lambda^{(k)} = (e)_\alpha$ the derived series of $\lambda$.

Now, we introduce another definition of solvable fuzzy groups.

**Definition 2.2**

Let $\lambda$ be a fuzzy subgroup of G with tip $\alpha$. We call a series $\lambda = \lambda_0 \supseteq \lambda_1 \supseteq ............ \supseteq \lambda_n = (e)_\alpha$ of fuzzy subgroups of G a solvable series for $\lambda$ if for $0 \le i < n$, $[\lambda_i, \lambda_i] \subseteq \lambda_{i+1}$. This is equivalent to saying that :

min $\{\lambda_i(a),\lambda_i(b)\}\le \lambda_{i+1}([a,b]),\quad for\ 0\le i<n.$

To prove the equivalence between definitions (2.1) and (2.2), we need firstly the following proposition .

**Proposition 2.3**

If $\lambda=\lambda^{(0)}\supseteq\lambda^{(1)}\supseteq \ldots\ldots\ldots\supseteq\lambda^{(n)}=(e)_\alpha$ be a derived series and $\lambda=\lambda_0\supseteq\lambda_1\supseteq\ldots\ldots\supseteq\lambda_n=(e)_\alpha$ be a solvable series , then : $\lambda^{(i)}\subseteq\lambda_i,\ i=0,1\ldots\ldots,n.$

**Proof**

We show by induction that $\lambda^{(i)}\subseteq\lambda_i, i=0,1,\ldots.n$ .we have $\lambda^{(0)}=\lambda=\lambda_0$, and we have $\lambda^{(1)}=[\lambda^{(0)},\lambda^{(0)}]=[\lambda_0,\lambda_0]\subseteq\lambda_1$.

Therefore, the result holds for $n=1$.

Now, let $\lambda^{(i)}\subseteq\lambda_i$ for some $i=0,1,\ldots.n-1$ then by theorem (1.12), $\lambda^{(i+1)}=[\lambda^{(i)},\lambda^{(i)}]\subseteq[\lambda_i,\lambda_i]\subseteq\lambda_{i+1}$

Hence, $\lambda^{(i)}\subseteq\lambda_i$ for $i=0,1,\ldots..n$ .

Now, we are ready to prove the equivalence between definitions (2.1) and (2.2)

**Proposition 2.4**

Definition ( 2.1 ) $\Leftrightarrow$ Definition ( 2.2 ).

**Proof**

Let $\lambda$ be a fuzzy subgroup of G with tip $\alpha$ . First, suppose definition (2.1) is hold then the derived series of $\lambda$ is a solvable series for $\lambda$ .That is definition (2.2) hold .

Conversely, suppose definition (2.2) is hold that is, $\lambda$ have a solvable series $\lambda=\lambda_0\supseteq\lambda_1\supseteq\ldots\ldots\ldots\supseteq\lambda_n=(e)_\alpha$ such that $[\lambda_i,\lambda_i]\subseteq\lambda_{i+1}$ for some $0\le i<n$ .

Form proposition (2.3), $\lambda^{(i)}\subseteq\lambda_i$, for 0 ≤ i ≤ n .

Therefore, we get $(e)_\alpha\subseteq\lambda^{(n)}\subseteq\lambda_n=(e)_\alpha$ . Consequently, $\lambda$ is solvable.

That is, definition (2.1) is hold.

Now ,we introduce a nontrivial example of a solvable fuzzy subgroup of the group $S_4$ ( the group of all permutations on the set $\{1,2,3,4\}$ ) .

**Example 2.5**

Let $D_4=\{(1),(12)(34),(13)(24),(14)(23),(24),(1234),(1432),(13)\}$ .Which is a dihedral subgroup of $S_4$ with center $C=\{(1),(13)(24)\}$.

Let $\lambda$ be the fuzzy subset of $S_4$ defined by :

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in C = \{(1),(13)(24)\} \\ ½ & \text{if } x \in \{\langle(1234)\rangle \setminus C\} \\ ¼ & \text{if } x \in D_4 \setminus \langle(1234)\rangle \\ 0 & \text{if } x \in S_4 \setminus D_4 \end{cases}$$

$$\begin{cases} 1 & \text{if } x = (1) \\ ¼ & \text{if } x = (13)(24) \\ 0 & \text{otherwise} \quad , x \in S_4 \end{cases}$$

Clearly, $\lambda$ is a fuzzy subgroup of $S_4$ .The fuzzy subgroup $\lambda^{(1)}$ has the following definition:

$\lambda^{(1)}(x) =$

And ,

$$\lambda^{(2)}(x) = \begin{cases} 1 & \text{if } x = (1) \\ 0 & \text{otherwise} \quad , x \in S_4 \end{cases}$$

Thus, we have $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \lambda^{(2)} = (e)_1$

Hence, $\lambda$ is a solvable fuzzy subgroup of $S_4$ .

We will give the following remark :

**Remark 2.6**

1) If G is a solvable group, then $1_G$ is a solvable fuzzy subgroup of G.

**Proof**

Let G be a solvable group, then $G = G^{(0)} \supseteq G^{(1)} \supseteq .......... \supseteq G^{(n)} = (e)$. Now :

$$1_G = 1_{G^{(0)}} \supseteq 1_{G^{(1)}} \supseteq .......... \supseteq 1_{G^{(n)}} \supseteq 1_{\{(e)\}} = (e)_1$$

Hence, from definition (2.1), $1_G$ is a solvable fuzzy subgroup of G.

2) If H is a subgroup of G, then for all n ≥ 1, $(\chi_H)^{(n)} = \chi_{H^{(n)}}$. [2]

Now, we can give the following example :

**Example 2.7**

$(S_3,\circ)$ is solvable group with $S_3 \supseteq A_3 \supseteq \{e\}$. Now ,let $\lambda(x) = 1_{S_3}$ that is $\lambda(x) = 1$ for all $x \in S_3$ :

$\lambda^{(1)}(x) = 1_{A_3}$ that is $\lambda^{(1)}(x) = \begin{cases} 1 & if\ x \in A_3 \\ 0 & otherwise \end{cases}$

and

$\lambda^{(2)}(x) = 1_{\{e\}}$ that is $\lambda^{(2)}(x) = \begin{cases} 1 & if\ x = \{e\} \\ 0 & otherwise \end{cases}$

then $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \lambda^{(2)} = (e)_1$

That is $1_{S_3} \supseteq 1_{A_3} \supseteq 1_{\{e\}} = (e)_1$

Thus , $1_{S_3}$ is a solvable fuzzy subgroup of $S_3$.

Now we introduce the following theorem :

**Theorem 2.8**

A subgroup A of G is solvable if and only if ,$\chi_A$ is a solvable fuzzy subgroup of G

Proof

Let $A$ be any subgroup of G. Consider the derived chain of $A$

$A = A^{(0)} \supseteq A^{(1)} \supseteq \ldots\ldots\ldots\ldots \supseteq A^{(n)} \supseteq \ldots\ldots\ldots\ldots$

From Remark (2.6) , $(\chi_A)^{(n)} = \chi_{A^{(n)}}$ for each $n \geq 0$ and therefore, $(\chi_A)^{(n)} = (e)_1$ if and only if, $A^{(n)} = (e)$. The result now is clear .

Now, we have the following result :

**Theorem 2.9**

If $\lambda and \mu$ are two solvable fuzzy subgroups of G. Then $[\lambda, \mu]$ is a solvable fuzzy subgroup of G.

**Proof**

Let $\lambda$ is solvable fuzzy subgroup of G with tip $\alpha_1$ , then :

$\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq \ldots\ldots\ldots \supseteq \lambda^{(n)} = (e)_{\alpha_1}$

Also, $\mu$ is solvable fuzzy subgroup of G with tip $\alpha_2$ , then :

$\mu = \mu^{(0)} \supseteq \mu^{(1)} \supseteq ......... \supseteq \mu^{(n)} = (e)_{\alpha_2}$

From theorem (1.17 ), the tip of $[\lambda, \mu] = \alpha_1 \wedge \alpha_2$. Therefore ,

$[\lambda, \mu] = [\lambda^{(0)}, \mu^{(0)}] \supseteq [\lambda^{(1)}, \mu^{(1)}] \supseteq ............ \supseteq [\lambda^{(n)}, \mu^{(n)}] = (e)_{\alpha_1 \wedge \alpha_2}$

Then $[\lambda, \mu]$ is solvable fuzzy subgroup of G.

Next, we shall state the following theorem :

**Theorem 2.10**

Let $\lambda$ be a solvable fuzzy subgroup of G. Then $\lambda_t$ is a solvable, $\forall t \in (0,1]$.

**Proof**

Suppose $\lambda$ is a solvable fuzzy subgroup of G with tip $\alpha$. Then $\lambda$ has a solvable series $\lambda = \lambda^{(0)} \supseteq \lambda^{(1)} \supseteq .......... \supseteq \lambda^{(n)} = (e)_\alpha$

Now,

$\lambda_t = (\lambda^{(0)})_t \supseteq (\lambda^{(1)})_t \supseteq ............. \supseteq (\lambda^{(n)})_t = ((e)_\alpha)_t = \{e\}$

Hence, $\lambda_t$ is a solvable subgroup of G.

Now, we will give the following interesting theorem :

**Theorem 2.11**

Every fuzzy subgroup of a solvable group is solvable.

**Proof**

Let G be a solvable group. Then :

$G = G^{(0)} \supseteq G^{(1)} \supseteq ............. \supseteq G^{(n)} = (e)$

Let $\lambda$ be any fuzzy subgroup of G with tip $\alpha$ *for* $0 \le i \le n$. Define $\lambda_i$ by

$$\lambda_i(x) = \begin{cases} \lambda(x) & \text{if } x \in G^{(i)} \\ 0 & \text{otherwise} \end{cases} \quad , x \in G$$

Then $\lambda = \lambda_0 \supseteq \lambda_1 \supseteq .......... \supseteq \lambda_n$ is a finite chain of fuzzy subgroups of G, such that $S(\lambda_i) \subseteq G^{(i)}$ for each $i$ , clearly $\lambda_n = (e)_\alpha$

Let $a, b \in G$ and $o \le i \le n-1$ ,if $a \notin G^{(i)}$ or $b \notin G^{(i)}$ then :

$\min\{\lambda_i(a), \lambda_i(b)\} = 0 \le \lambda_{i+1}([a,b])$ . on the other hand, if $a, b \in G^{(i)}$ , then $[a,b] \in G^{(i+1)}$. And therefore, $\min\{\lambda_i(a), \lambda_i(b)\} = \min\{\lambda(a), \lambda(b)\}$

$$\leq \lambda([a,b])$$
$$= \lambda_{i+1}([a,b])$$

Thus ,

$\lambda = \lambda_0 \supseteq \lambda_1 \supseteq .......... \supseteq \lambda_n = (e)_\alpha$ is a solvable series for $\lambda$, and $\lambda$ is solvable.

Now, we can obtain the following result :

<u>Corollary 2.12</u>

If $\lambda, \mu$ are fuzzy subgroups of a solvable group G, then $[\lambda, \mu]$ is solvable

**Proof**

$\lambda, \mu$ are solvable fuzzy subgroups of G from theorem (2.11).

Then $[\lambda, \mu]$ is solvable fuzzy subgroups of G from theorem (2.9).

**<u>Theorem 2.13</u>**

If $\lambda, \mu$ are fuzzy subsets of a solvable group G. Then $[\lambda, \mu]$is solvable.

**Proof**

Let G be a solvable group, then :

$G = G^{(0)} \supseteq G^{(1)} \supseteq ............. \supseteq G^{(n)} = (e)$.

Now, if $\gamma, \beta$ are the tips of $\lambda, \mu$, respectively, then from theorem (1.17 ) $\alpha = \gamma \wedge \beta$ is the tip of $[\lambda, \mu]$.

Define $[\lambda_i, \mu_i]$ as follows:

$$[\lambda_i, \mu_i](x) = \begin{cases} [\lambda, \mu]\ (x) & \text{if } x \in G^{(i)} \end{cases}$$

Then $[\lambda, \mu] = [\lambda_0, \mu_0] \supseteq [\lambda_1, \mu_1] \supseteq .......... \supseteq [\lambda_n, \mu_n]$ is a finite chain of fuzzy subgroups of G, such that $S([\lambda_i, \mu_i]) \subseteq G^{(i)}$ for each $i$ .

Clearly $[\lambda_n, \mu_n] = (e)_\alpha$

Let $a,b \in G$ and $0 \le i \le n-1$ , If $a \notin G^{(i)}$ or $b \notin G^{(i)}$ , then $\min\{[\lambda_i,\mu_i](a),[\lambda_i,\mu_i](b)\} = 0 \le [\lambda_{i+1},\mu_{i+1}]([a,b])$ on the other hand, if $a,b \in G^{(i)}$, then $[a,b] \in G^{(i+1)}$, and therefore,

$$\begin{aligned}\min\{[\lambda_i,\mu_i](a),[\lambda_i,\mu_i](b)\} &= \min\{[\lambda,\mu](a),[\lambda,\mu](b)\}\\ &\le [\lambda,\mu]([a,b])\\ &= [\lambda_{i+1},\mu_{i+1}]([a,b])\end{aligned}$$

Thus ,

$$[\lambda,\mu] = [\lambda_0,\mu_0] \supseteq [\lambda_1,\mu_1] \supseteq .......... \supseteq [\lambda_n,\mu_n] = (e)_\alpha$$

is a solvable series for $[\lambda,\mu]$, and $[\lambda,\mu]$is solvable.

Now, we shall state and prove the following theorem :

**Theorem 2.14**

Every fuzzy subgroup of solvable fuzzy group is solvable.

Proof

Let $\lambda$ be a solvable fuzzy subgroup of a group G with tip $\alpha$ and let $\mu$ be a fuzzy subgroup of G, such that $\mu \subseteq \lambda$ in view. By theorem (1.12 ), we get $\mu^{(n)} \subseteq \lambda^{(n)}$ for all $n \ge 0$ . Also, $\lambda$ is a solvable then $\lambda^{(n)} = (e)_\alpha$.

Thus we have $(e)_\alpha \subseteq \mu^{(n)} \subseteq \lambda^{(n)} = (e)_\alpha$, then $\mu^{(n)} = (e)_\alpha$

Hence, $\mu$ is a solvable fuzzy subgroup.

Now, we introduce the following theorem :

**Theorem 2.15**

Let $\lambda$ be a fuzzy subgroup of G, and $t^* = \inf\{\lambda(x)/x \in G\}$, suppose that $t^* > 0$. Then the following are equivalent:

(i) G is solvable

(ii) $\lambda$ is solvable

(iii) $\lambda_t$ is solvable $\forall t \in (0,1]$ .

**Proof**

Now, G is solvable if and only if, $1_G$ is solvable, but $\lambda \subseteq 1_G$ then from theorem (2.11) . $\lambda$ is solvable then (i) $\Rightarrow$ (ii)

by theorem (2.10), (ii) $\Rightarrow$ (iii).

Now, suppose (iii) holds, $G = \lambda_{t^*}$

Thus G is solvable, that is (iii) $\Rightarrow$ (i).

In the following proposition we obtain a sufficient condition for truth of the converse of theorem (2.14).

**Proposition 2.16**

Let $\lambda, \mu$ be fuzzy subgroups of G, such that $S(\lambda) = S(\mu), \mu \subseteq \lambda$ *and* $\mu$ is solvable. Then $\lambda$ is solvable.

**Proof**

From theorem (2.10), $\mu_t$ is solvable, $\forall$ t $\in$ (0, 1]

Since $S(\mu) \subseteq \mu_t$, then by theorem (2.14), $S(\mu)$ is solvable and $S(\lambda)$ is solvable.

And from theorem (2.15), G is solvable .

Consequently, $\lambda$ is solvable.

**Theorem 2.17**

Let $\lambda and \mu$ be fuzzy subgroups of G, such that $\mu$ is a normal fuzzy in $\lambda$ . If $\lambda$ is a solvable fuzzy group, then $(\lambda/\mu)^{(t)}$ is solvable fuzzy group, $\forall t \in (0,1]$.

**Proof**

Since $\lambda$ is solvable fuzzy group, then from theorem (2.10), $\lambda_t$ is solvable for all $t \in (0,1]$ .

Also, by proposition (1.15 ), $\mu_t$ is normal in $\lambda_t$ $\forall t \in (0,1]$.

Then $(\lambda_t/\mu_t)$ is solvable . But , $(\lambda_t/\mu_t) \cong (\lambda/\mu)^{(t)}$

Then $(\lambda/\mu)^{(t)}$ is solvable for all $t \in (0,1]$.

Also, we have the following theorem :

**Theorem 2.18**

Let $\lambda and \mu$ be fuzzy subgroups of G, such that $\mu$ is a normal fuzzy in $\lambda$. If $\lambda$ is a solvable fuzzy group, then $(\lambda/\mu)$ is a solvable fuzzy semi-group.

**Proof**

Since $\lambda$ is solvable fuzzy group. Then from theorem (2.10), $\lambda_t$ is solvable for all $t \in (0,1]$ .

Also, by proposition (1.15 ), $\mu_t$ is normal fuzzy subgroup in $\lambda_t$ for all $t \in (0,1]$.

Then $(\lambda_t/\mu_t)$ is solvable, But $(\lambda_t/\mu_t) \cong (\lambda/\mu)_t$ .

Hence $(\lambda/\mu)$ is solvable fuzzy semi-group.

Next , we introduce some important propositions :

**Proposition 2.19**

Let $\lambda, \mu and \gamma$ be fuzzy subgroups of G such that $\gamma$ is normal fuzzy in $\lambda and \mu$. If $\lambda and \mu$ are solvable fuzzy then $[\lambda/\gamma , \mu/\gamma]$is solvable.

**Proof**

Since $\lambda, \mu$ are solvable fuzzy subgroups of G and $\gamma$ normal in $\lambda and \mu$, then from theorem (2.18), $\lambda/\gamma$ *and* $\mu/\gamma$ are solvable fuzzy semi-group.

Also, by theorem (2.9). $[\lambda/\gamma , \mu/\gamma]$ . Is solvable.

**Proposition 2.20**

Let $\alpha$ be a normal fuzzy subgroup of $\lambda and \beta$ be a normal fuzzy subgroup of $\mu$. If $\lambda, \mu$ are solvable fuzzy subgroups of G, then $([\lambda,\mu] \;/\; [\beta,\alpha])$is solvable fuzzy semi-group.

Proof

Let $\lambda, \mu$ two solvable fuzzy subgroups of G, then from theorem (2.9), $[\lambda, \mu]$ is solvable fuzzy subgroup of G.

And since $\beta,\alpha$ are normal fuzzy in $\lambda,\mu$ respectively then by corollary (1.14), $[\beta,\alpha]$is normal fuzzy in $[\lambda,\mu]$

Thus from theorem (2.18), $([\lambda,\mu] \;/\; [\beta,\alpha])$ is solvable fuzzy semi-group.

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**بعض النتائج حول الزمر الضبابية القابلة للحل**

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**قسم الرياضيات – كلية التربية أبن الهيثم – جامعة بغداد**

**<u>الخلاصة :-</u>**

يتضمن البحث تعريف الزمرة الضبابية القابلة للحل بأكثر من صيغة ثم أثبات تكافئ الصيغ المختلفة للتعريف و دراسة خواصها وتقديم البراهين المهمة حول المفهوم .