A CHARACTERIZATION OF THE BEST AND BEST ONE – SIDED ALGEBRAIC APPROXIMATION IN L_p [0,1] (0 < p < ∞)

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Abstract

In this paper, I characterized the order of the best one- sided algebraic approximation to bounded measurable Function in terms of the average modulus of smoothness of that function. For obtaining direct statements, I constract smooth functions close to the original function and after that Approximate one-sidedly the smoothing functions. As far as we know there are no simple definitions of operators for one-sided approximation which works for any measurable functions in L_p metric for each ($0 < p < \infty$) This was one of the reasons Leading me to investigations of operator for one – sided approximation.

1- Introduction

We shall consider bounded and measurable real valued function f defined on [0,1]. Let L_∞[0,1] be the space of all functions f such thah $\sup\{ |f(\chi)| . \chi \in [0,1] \} < \infty$, and L_p [o,1] be the space of all bounded measurable functions f for which $\| f \|_p < \infty$, it mean $(\int_0^1 | f |^p)^{1/p} < \infty$

Let P_n be the set of all algebraic polynomials of degree not greater than n.Then the best one - sided approximation in L_p metric for each ($0 < p \le \infty$) is

(1.1) $\widetilde{E}(f)_p = \inf\{\| g^+ - g^- \|_p : g^\pm \in \mathbf{P}_n ; g^- \le f \le g^+ \}$.

For $f \in \mathbf{L}_p$ we define the best approximation of f by the functions from P_n as

(1.2) $E(f)_p = \inf\{ \| f - \mathbf{P}_n \|_p : P \in P_n \}$.

For a characterization of the structural properties for a given function f from $\mathbf{L}_p$ or $\mathbf{L}_\infty$ we shall use the following moduli

(1.3) $\omega_k(f, \chi, \delta) = \sup\{ | \Delta_h^k f(y) | ; y, y + kh \in [\chi - k\delta/2, \chi + k\delta/2] \}$,

(1.4) $\gamma_k(f, \delta)_p = \| \omega_k(f, ., \delta) \|_p$,

where

$$\Delta_h^k f(y) = \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(y + ih), y, h \in R,$$

and $\delta \ge 0$.

Henceforth k and p are fixed numbers, k natural standing for the order of the moduli and $0 < p < \infty$. With C we denote positive constants and with C (A, B, ...)

-- constants depending only on the marked parameters. These constants may differ at each occurrence.

For $\chi \in [0, 1]$ we set

$$(1.5)\ h_{\sqrt{n}}(\chi) = \frac{\sqrt{\chi(1-\chi)}}{\sqrt{n} + \frac{1}{2}n}.$$

Using Bernstein polynomial $B_n(f\,;\chi)$, we define our one – sided operator which mapping any bounded measurable function f to on algebraic polynomial in p_n ; as follows

$$(1.6)\ \Lambda(f;\chi) = B_n(f;\chi) \pm \sum_{i=0}^{n} \binom{n}{i} \left| \Delta^{k}_{h(i/n)} f(i/n) \right| \chi^{i} (1-\chi)^{n-i}, \text{ where } \chi \subset [0,1]$$

2 . Assertions

Now we shall introduce some results and theorems which we make use of them in our research.

Lemma 1. (E . Bhaya 'a, [1])
If $f \subset L_p$ and $0 \leq \delta' \leq \delta$, then
$(2.7)\ \gamma_k(f, \delta')_p \leq \gamma_k(f, \delta)_p$.

Lemma 2. (E . Bhaya 'a, [1])
Let $f \subset L_p$ and δ, λ be two positive real numbers then for any 0 $0 < P \leq 1$, we have

$(2.8)\ \gamma_k(f, \lambda\delta)_p \leq (2[\chi])^{k+1/p} \gamma_k(f, \delta)_p$,
wher $[\chi] = \min\{u : u \geq \chi;\ u \text{ is an integer}\}$

Lemma 3. (K. Ivanov. [2])
If $\lambda\,\chi > 0$, then for each $\chi, y \in [-1, 1]$ and $|\chi - y| \leq \chi\ \Delta_n(\chi)$ we have
$(2.9)\ (4\lambda + 2)^{-1} h_n(\chi) \leq h_n(y)\,(2\lambda + 3/2)\,h_n(\chi)$.

Lemma 4. (*B.* Sendov and V. Popov, [3])
Let $f \in L_p$ and δ, χ be two positive real numbers then for any $1 \leq p < \infty$. We have
$(2.10)\ \gamma_k(f, \chi\delta)_p \leq (2\chi)^{k+1} \gamma_k(f, \delta)_p$.

Lemma 5. (G. Tachev,
[4])For every sequence $\{a_k\}$ with the property
$a_k \geq 0$, $k = 0, 1, \ldots n$
$a_k = 0$, otherwise, we have for each $0 < p \leq 1$, that

$$(2.11) \int_0^1 \left| \sum_{k=0}^{n} a_k \binom{n}{k} \chi^k (1-\chi)^{n-k} \right|^p d\chi \leq c(p) \frac{1}{n} \sum_{k=0}^{n} \max_{|j/n - k/n| \leq 6h_{\sqrt{n}}(k/n)} |a_j|^p;$$

where $\chi \in [0, 1]$ and $j = 0, 1, \ldots, n$.

3. The Main Results.

In this article I proved direct inequalities concerning the best one sided algebraic approximations and best algebraic approximations of the bounded measurable functions in Lp – spaces (0<p<∞) .

Theorem I.

For any bounded and measurable function f in [0, 1] and ($0 < p < \infty$), we have

(3.12) $A^{\pm}(f) \in p_n$;

(3.13) $A^{-}(f) \le f \le A^{+}(f)$ for any $\chi \in [0, 1]$;

(3.14) $\left\| A^{+}(f) - A^{-}(f) \right\|_p \le c(p)\gamma_k(f, \frac{1}{\sqrt{n}})_p$; and

$$\widetilde{E}(f)_p \le c(p)\gamma_k(f, \frac{1}{\sqrt{n}})_p .$$

Theorem II.

If $f \in L_p[0, 1]$, then for any $0 < p < \infty$, we have

$$E(f)_p \le c(p)\gamma_k(f, \frac{1}{\sqrt{n}})_p.$$

Proof of theorem I.

Since $B_n(f,x)$ and $x^i(1-x)^{n-i}$ are algebraic polynomials of degree not greater than n, so that $A^{\pm} \subset P_n$ the positively of $x^i(1-x)^{n-i}, x \in [0,1]$ and $\left|\Delta_{h_{\sqrt{n}}}(i/n)f(i/n)\right|$ give (3.13). Then for $0 < p \le 1$, using (1.3) and (2.11) we have

$$\left\|A^{+}(f,x) - A^{-}(f,x)\right\|_p^p \le c(p)\int_0^1\left|\sum_{i=0}^{n}\omega_k\left(f,i/n,h_{\sqrt{n}}(i/n)\right)\binom{n}{i}x^i(1-x)^{n-i}\right|^p dx$$

$$\le c(p)1/n\sum_{i:(|i/n-j/n|\le 6h_{\sqrt{n}}(i/n))}^{n}\max \omega_k^p\left(f.j/n,6h_{\sqrt{n}}(j/n)\right), j=0,\dots,n .$$

(2.9) implies

$$\left\|A^{+}(f,x) - A^{-}(f,x)\right\|_p^p \le c(p)1/n\sum_{i=0}^{n}\omega_k^p\left(f,i/n,ch_{\sqrt{n}}(i/n)\right), j=0,\dots,n .$$

$$\le c(p)\tau_k^p\left(f,1/\sqrt{n}\right)_p^p.$$

Now whenever $1 < p < \infty$, (1.3), Jenssens inequality, (2.9) and (1.4) give

$$\left\|A^{+}(f,x) - A^{-}(f,x)\right\|_p^p \le 2\left(\sum_{i=0}^{n}\omega_k^p\left(f,i/n,h_{\sqrt{n}}(i/n)\right)\int_0^1\binom{n}{i}x^i(1-x)^{n-i}dx\right)^{1/p}$$

$$\le 2\tau_k\left(f,ch_{\sqrt{n}}(x)\right)_p .$$

At the end, from (2.7) and (2.10) we get

$$\left\|A^{+}(f,x) - A^{-}(f,x)\right\|_p^p \le c(p)\tau_k^p\left(f,1/\sqrt{n}\right)_p^p.$$

(1.1) implies

$$\widetilde{E}(f)_p \le c\,\tau_k\left(f,1/\sqrt{n}\right)_p .$$

Proof of theorem II

Using (1.2), (1.1) and theoremI we obtain

$$\begin{aligned} E(f)_p &\le \widetilde{E}(f)_p \\ &\le \left\| \Lambda^+(f,x) - \Lambda^-(f,x) \right\|_p^p \\ &\le c\,\tau_k\left(f,1/\sqrt{n}\right)_p . \end{aligned}$$

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