# Some Types of Stability For Nonlinear Systems Using Generalized Lyapunov-Like Function

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***Abstract:***

**By using the definition of generalized Lyapunov –Like function and some types of stability , we give a new bounds for Lyapunov function , which is lead to getting some types of stability depending on some hypothesis which is depend on this function . we proved that those bounds and hypothesis on Lyapunov function satisfies the conditions where the using nonlinear systems will be stable ,also the type of stability of the system depends on those bounds and hypothesis . We got four types of stability which are (globally stable, globally asymptotically stable, exponentially stable and uniformly exponentially stable**).

**بعض أنواع ألأستقراريه للأنظمة اللاخطيه باستخدام**
**(The Generalized Lyapunov-Like Function)**

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**المستخلص :**

**بالاعتماد على مجموعة من التعاريف عن مفهوم (The generalized lyapunov- like function ) وبعض أنواع الاستقرارية ، قمنا بوضع بعض القيود على دالة ليابنوف والتي يمكن بواسطتها الحصول على هذه الأنواع من الاستقراية بالاعتماد على بعض الفرضيات التي تعتمد اساساً على هذه الدالة . وقد برهنا في مجموعة من النظريات بأن هذه القيود والفرضيات على دالة ليابنوف تؤدي الى تحقق الشروط التي تجعل النظام المستخدم مستقراً وبأنواع مختلفة باختلاف تلك القيود والفرضيات وقد حصلنا على أربعة أنواع من الأستقرارية هي :**
**Globally stable , globally asymptotically stable , exponentially stable and)**
**uniformly exponentially stable ).**

## *1.Introduction:*

**Among many forms of performance specifications used in design ,the most imp- ortant requirement is that the system be stable .An unstable system is generally conside- red to be useless. When all types of system are considered (linear,nonlinear,time- invariant, and time-varying) the definition of stability can be given in many different forms [5].**

**It is recognized that the Lyapunov function method serves as a main technique to reduce a given complicated system into a relatively simpler system and provides useful applications to control theory [6,9,11,12,18]. There have been a number of interesting development in searching the stability criteria for nonlinear differential system[1,4,7,13].**

**The use of Lyapuvov functinals is certainly the main approach for deriving sufficient conditions for the asymptotic stability[17].**

**The result of A.M. Lyapunov has come into widespread usage in many topics of mathematics. In particular, it continues to be of great importance in modern treatments of the asymptotic behaviour of the solutions of differential system. In the language of dynamical systems ,one wishes to determine the nature of the global attractor. In doing this ,one often uses special properties of functions in the equations or,indeed, of the nature of the model itself [2,14].**

**A fundamental notion in the stability analysis of dynamical systems is that of global asymptotic stability (GAS) which characterizes systems for which all trajectories converge to some equilibrium in a reasonable manner. When considering differential equations, there are two equivalent definitions of the GAS property. The more common definition is that a system is GAS if it is both (locally) stable and satisfies an attractivity property [3].**

**The exponential stability for nonlinear system ,in general may not be easly verified. Only a few investigations have with exponential stability conditions for nonlinear time-varying system[15,16].**

## *2.Basic facts:*

**Consider the nonlinear system described by the time-varying differential equation:**

$$x'(t) = f(x(t),t), t \geq 0$$
$$x(t_0) = x_0, t_0 \geq 0 \qquad (1)$$

**Where** $x(t) \in R^n$ ($R^n$ **is the n-dimensional Euclidean vector space),** $f(x,t): R^n \times R^+ \to R^n$

**is a given nonlinear function satisfying** $f(0,t) = 0$ **for all** $t \in R^+$ ($R^+$ **is the set of all non-negative real numbers).we shall assume that the conditions are imposed on system (1) such that the existence of its solutions is guaranteed.**

***Definition(2.1):***[10]

**A function $V(x,t): R^n \times R^+ \to R$ is Lipschitzian in $x$ (uniformly in $t \in R^+$) if there is a number $L > 0$ such that for all $t \in R^+$,**

$$|V(x_1,t) - V(x_2,t)| \le L\|x_1 - x_2\|, \forall (x_1,x_2) \in R^n \times R^n$$

**Where $\| \ \|$ is the Euclidean norm.**

***Definition(2.2):***[10]

**A function $V(x,t): R^n \times R^+ \to R$ is called a generalized Lyapunov-Like function for system(1) if $V(x,t)$ is continuous in $t \in R^+$ and Lipschitzian in $x \in R^n$ (uniformly in $t$)**
**and there exist positive functions $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ and there exist positive numbers $k, p, q, r, \delta$ such that :**

$$\lambda_1(t)\|x\|^p \le V(x,t) \le \lambda_2(t)\|x\|^q, \forall (x,t) \in R^n \times R^+$$
$$V'(x,t) \le -\lambda_3(t)\|x\|^r - ke^{-\delta t}, \forall t \ge 0, x \in R^n / \{0\} \quad \textbf{(2)}$$

***3.Globally and globally asymptotically stable:***

***Definition(3.1):***[8]

**Let the origin be a solution of the system (1) , the system (1) is said to be :**

a. **globally stable if there exists a function $\alpha$ such that for each $x_0 \in R^n$ all the solutions $x(x_0,t)$ are defined on $[0,\infty)$ and satisfy:**
$\|x(x_0,t)\| \le \alpha(\|x_0\|), \forall t \ge 0$
b. **globally asymptotically stable if there exists a function $\alpha$ such that for each**
**$x_0 \in R^n$ all the solutions $x(x_0,t)$ are defined on $[0,\infty)$ and satisfy :**
$\|x(x_0,t)\| \le \alpha(\|x_0\|, t_0)$

***Theorem(3.2):***

**The system (1) is a globally stable if it admits a generalized Lyapunov-Like function**
**which is bounded as:** $V(x,t)$

$$\theta_1 e^{-m(t-t_0)}\|x\|^p \le V(x,t) \le \theta_2 e^{-m(t-t_0)}\|x\|^q \quad \textbf{(3.2.1)}$$

$$V'(x,t) \le -\theta_3 e^{-m(t-t_0)}\|x\|^r - ke^{\delta t}.e^{-m(t-t_0)} \quad \textbf{(3.2.2)}$$

**Where $\theta_1, \theta_2, \theta_3, p, q, r, \delta$ and $k$ are positive constants.**
**And the following condition is hold:**

$$\textbf{(3.2.3)}\ \exists \gamma > 0 : V(x,t) - V^{r/q}(x,t)e^{(\frac{mr}{q} - m)(t-t_0)} \le \gamma e^{-\delta t}$$

**Where** $\delta > m = \dfrac{\theta_3}{\theta_2^{\,r/q}}$

**For all** $(x,t) \in R^n \times R^+$

***Proof:***

**Let** $Q(x,t) = V(x,t)e^{m(t-t_0)}$

$$Q'(x,t) = V'(x,t)e^{m(t-t_0)} + mV(x,t)e^{m(t-t_0)}$$

**From (3.2.2)we obtain:**

$$Q'(x,t) \le -\theta_3 \|x\|^r - ke^{\delta t} + mV(x,t)e^{m(t-t_0)}$$

**From the right hand of(3.2.1),we obtain:**

$$-\|x\|^r \le -\left[\frac{V(x,t)}{\theta_2}\right]^{r/q} e^{\frac{mr}{q}(t-t_0)}$$

**Hence:**

$$Q'(x,t) \le -\left[\frac{\theta_3}{\theta_2^{\,r/q}}\right] V^{r/q}(x,t)e^{\frac{mr}{q}(t-t_0)} - ke^{\delta t} + mV(x,t)e^{m(t-t_0)}$$

$$\le -mV^{r/q}(x,t)e^{\frac{mr}{q}(t-t_0)} - ke^{\delta t} + mV(x,t)e^{m(t-t_0)}$$

$$\le m\left[V(x,t) - V^{r/q}(x,t)e^{(\frac{mr}{q}-m)(t-t_0)}\right]e^{m(t-t_0)} - ke^{\delta t}$$

**From the condition (3.2.3),we obtain:**

$$Q'(x,t) \le m\gamma e^{-\delta t} e^{m(t-t_0)} - ke^{\delta t}$$

**Since** $ke^{\delta t} > 0$**, then:**

$$Q'(x,t) \le m\gamma e^{-\delta t} e^{m(t-t_0)}$$

**Multiplying by** $e^{\delta t_0} > 0$ **,we obtain:**

$$Q'(x,t) \le m\gamma e^{(m-\delta)(t-t_0)}$$

$$Q(x,t) - Q(x_0,t_0) \le \int_{t_0}^{t} m\gamma e^{(m-\delta)(s-t_0)} ds$$

$$Q(x,t) \le Q(x_0,t_0) + \frac{m\gamma}{m-\delta}\left[e^{(m-\delta)(t-t_0)} - 1\right]$$

$$\le Q(x_0,t_0) + \frac{m\gamma}{\delta - m} - \frac{m\gamma}{\delta - m} e^{(m-\delta)(t-t_0)}$$

**Since** $Q(x_0,t_0) = V(x_0,t_0)$ **and** $\dfrac{m\gamma}{\delta - m} e^{(m-\delta)(t-t_0)} > 0$ **, then:**

$$Q(x,t) \le V(x_0,t_0) + \frac{m\gamma}{\delta - m}$$

**From the right hand of (3.2.1),we obtain:**

$$Q(x,t) \le \theta_2 \|x_0\|^q + \frac{m\gamma}{\delta - m}$$

**Setting:** $\theta_2 \|x_0\|^q + \dfrac{m\gamma}{\delta - m} = \alpha(\|x_0\|) \ge 0$

**Hence :**

$$Q(x,t) \le \alpha(\|x_0\|)$$

**From the left hand of (3.2.1),we obtain:**

$$\|x\| \le \left[ \frac{V(x,t)}{\theta_1} e^{m(t-t_0)} \right]^{1/p}$$

$$\|x\| \le \left[ \frac{Q(x,t)}{\theta_1} \right]^{1/p}$$

$$\|x\| \le \left[ \frac{\alpha(\|x_0\|)}{\theta_1} \right]^{1/p}$$

**Therefore the system (1) is a globally stable**.

## *Lemma (3.3):*

**In theorem (3.2) if the left hand of (3.2.1) is** $\theta_1 e^{-mt} \|x\|^p$ **and** $Q(x,t) = V(x,t)e^{mt}$**, then the system (1) is a globally asymptotically stable.**

***Proof:***

$$Q(x,t) = V(x,t)e^{mt}$$

$$Q'(x,t) = V'(x,t)e^{mt} + mV(x,t)e^{mt}$$

**From (3.2.2),we obtain:**

$$Q'(x,t) \le -\theta_3 \|x\|^r e^{mt_0} - ke^{\delta t} e^{mt_0} + mV(x,t)e^{mt}$$

**From the right hand of (3.2.1),we have:**

$$-\|x\|^r \le -\left[ \frac{V(x,t)}{\theta_2} \right]^{r/q} e^{\frac{mr}{q}(t-t_0)}$$

**Hence:**

$$Q'(x,t) \le -\frac{\theta_3}{\theta_2^{\,r/q}} V^{r/q}(x,t) e^{\frac{mr}{q}(t-t_0)} e^{mt_0} - ke^{\delta t} e^{mt_0} + mV(x,t)e^{mt_0}$$

**Since** $ke^{\delta t} e^{mt_0} > 0$ **,then:**

$$Q'(x,t) \le -mV^{r/q}(x,t) e^{\frac{mrt}{q}} e^{(m-\frac{mr}{q})t_0} + mV(x,t)e^{mt}$$

$$\le m\left[V(x,t)-V^{r/q}(x,t)e^{(\frac{mr}{q}-m)t}e^{(m-\frac{mr}{q})t_0}\right]e^{mt}$$

$$\le m\left[V(x,t)-V^{r/q}(x,t)e^{(\frac{mr}{q}-m)(t-t_0)}\right]e^{mt}$$

**From (3.2.3) we obtain:**

$$Q'(x,t)\le m\gamma e^{(m-\delta)t}$$

$$Q(x,t)-Q(x_0,t_0)\le \int_{t_0}^{t} m\gamma e^{(m-\delta)s}ds$$

$$Q(x,t)\le Q(x_0,t_0)+\frac{m\gamma}{\delta-m}e^{(m-\delta)t_0}-\frac{m\gamma}{\delta-m}e^{(m-\delta)t}$$

**Since** $\frac{m\gamma}{\delta-m}e^{(m-\delta)t}>0$ **and** $Q(x_0,t_0)=V(x_0,t_0)e^{mt_0}$ **, then:**

$$Q(x,t)\le V(x_0,t_0)e^{mt_0}+\frac{m\gamma}{\delta-m}e^{(m-\delta)t_0}$$

**From the right hand of (3.2.1) , we obtain:**

$$Q(x,t)\le \theta_2 e^{mt}\|x_0\|^q+\frac{m\gamma}{\delta-m}e^{(m-\delta)t_0}$$

Setting : $\theta_2 e^{mt}\|x_0\|^q+\frac{m\gamma}{\delta-m}e^{(m-\delta)t_0}=\alpha(\|x_0\|,t_0)\ge 0$

From the new left hand of (3.2.1),we obtain:

$$\|x\|\le\left[\frac{V(x,t)}{\theta_1}\right]^{1/p}e^{\frac{mt}{p}}$$

$$\le\left[\frac{Q(x,t)}{\theta_1}\right]^{1/p}$$

$$\le\left[\frac{\alpha(\|x_0\|,t_0)}{\theta_1}\right]^{1/p}$$

**Therefore the system(1) is a globally asymptotically stable.**

## *4. Exponentially and uniformly exponentially stable:*

*Definition (4.1)* **[10]**

**The zero solution of system (1) is exponentially stable if any solution** $x(x_0,t)$ **of the system(1) satisfies:**

$$\|x(x_0,t)\|\le\alpha(\|x_0\|,t_0)e^{-\delta(t-t_0)},\forall t\ge t_0$$

**Where** $\alpha(h,t):R^+\times R^+\to R^+$ **is a non-negative function increasing in** $h\in R^+$**, and** $\delta$ **is a** positive constant.

**If the function $\alpha(.)$ in the above definition does not depend on $t_0$, then the zero solution is called uniformly exponentially stable.**

***Theorem (4.2):***

**The system (1) is exponentially stable if it admits a generalized Lyapunov-Like function $V(x,t)$ which is bounded as:**

$$\varepsilon_1 e^{m(t-t_0)}\|x\|^p \le V(x,t) \le \varepsilon_2 e^{m(t-t_0)}\|x\|^q \qquad \textbf{(4.2.1)}$$

$$V'(x,t) \le -\varepsilon_3\|x\|^r - ke^{-\delta t} \qquad \textbf{(4.2.2)}$$

**Where $\varepsilon_1,\varepsilon_2,\varepsilon_3,p,q,r,\delta$ ,and $k$ are positive constant.**
**And the following two conditions are hold:**

**a.** $\exists\gamma>0: V(x,t)-V^{r/q}(x,t)e^{\frac{-mr}{q}(t-t_0)} \le \gamma e^{-\delta t}$ **, where** $\delta > \dfrac{\varepsilon_3}{\varepsilon_2^{\ r/q}}$

**b.** $m\gamma \ge k$ **, where** $m = \dfrac{\varepsilon_3}{\varepsilon_2^{\ r/q}}$

**for all** $(x,t)\in R^n \times R^+$

***Proof :***

**Let** $Q(x,t) = V(x,t)e^{mt}$

$$Q'(x,t) = V'(x,t)e^{mt} + mV(x,t)e^{mt}$$

**From (4.2.2) we obtain:**

$$Q'(x,t) \le -\varepsilon_3\|x\|^r e^{mt} - ke^{-\delta t}e^{mt} + mV(x,t)e^{mt}$$

**From the right hand of (4.2.1) we have:**

$$-\|x\|^r \le -\left[\frac{V(x,t)}{\varepsilon_2}\right]^{r/q} e^{\frac{-mr}{q}(t-t_0)}$$

**Hence:**

$$Q'(x,t) \le -\left[\frac{\varepsilon_3}{\varepsilon_2^{\ r/q}}\right]V^{r/q}(x,t)e^{mt}e^{\frac{-mr}{q}(t-t_0)} - ke^{(m-\delta)t} + mV(x,t)e^{mt}$$

$$\le -mV^{r/q}(x,t)e^{mt}e^{\frac{-mr}{q}(t-t_0)} - ke^{(m-\delta)t} + mV(x,t)e^{mt}$$

$$\le m\left[V(x,t) - V^{r/q}(x,t)e^{\frac{-mr}{q}(t-t_0)}\right]e^{mt} - ke^{(m-\delta)t}$$

**From the condition (a) we obtain:**

$$Q'(x,t) \le m\gamma e^{-\delta t}e^{mt} - ke^{(m-\delta)t}$$

$$\le (m\gamma - k)e^{(m-\delta)t}$$

$$Q(x,t) - Q(x_0,t_0) \le \int_{t_0}^{t}(m\gamma-k)e^{(m-\delta)s}ds$$

$$\leq \frac{m\gamma-k}{\delta-m}e^{(m-\delta)t_0}-\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t}$$

**Since** $\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t}\geq 0$ **, then:**

$$Q(x,t)\leq Q(x_0,t_0)+\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t_0}$$

**Since** $Q(x_0,t_0)=V(x_0,t_0)e^{mt_0}$ **, then:**

$$Q(x,t)\leq V(x_0,t_0)e^{mt_0}+\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t_0}$$

**From the right hand of (4.2.1) we have:**

$$V(x_0,t_0)\leq \varepsilon_2\|x_0\|^q$$

**Hence:**

$$Q(x,t)\leq \varepsilon_2\|x_0\|^q e^{mt_0}+\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t_0}$$

**Setting:** $\varepsilon_2\|x_0\|^q e^{mt_0}+\frac{m\gamma-k}{\delta-m}e^{(m-\delta)t_0}=\alpha(\|x_0\|,t_0)\geq 0$ **, then:**

$$Q(x,t)\leq \alpha(\|x_0\|,t_0)$$

**From the left hand of (4.2.1) we have:**

$$\|x\|\leq\left[\frac{V(x,t)}{\varepsilon_1}\right]^{1/p}e^{\frac{-m}{p}(t-t_0)}$$

$$\|x\|\leq\left[\frac{Q(x,t)e^{-mt}}{\varepsilon_1}\right]^{1/p}e^{\frac{-m}{p}(t-t_0)}$$

$$\|x\|\leq\left[\frac{\alpha(\|x_0\|,t_0)}{\varepsilon_1}\right]^{1/p}e^{\frac{-m}{p}(2t-t_0)}$$

**Therefore the system (1) is exponentially stable.**

***<u>Lemma (4.3):</u>***

**In theorem (4.2) if** $Q(x,t)=V(x,t)e^{m(t-t_0)}$ **, then the system (1) is uniformly exponentially stable.**

***<u>Proof:</u>***

$$Q(x,t)=V(x,t)e^{m(t-t_0)}$$

$$Q'(x,t)=V'(x,t)e^{m(t-t_0)}+mV(x,t)e^{m(t-t_0)}$$

**From (4.2.2) we obtain:**

$$Q'(x,t)\leq -\varepsilon_3\|x\|^r e^{m(t-t_0)}-ke^{-\delta t}e^{m(t-t_0)}+mV(x,t)e^{m(t-t_0)}$$

**From the right hand of (4.2.1), we obtain:**

$$Q'(x,t) \le -\left[\frac{\varepsilon_3}{\varepsilon_2^{\,r/q}}\right] V^{r/q}(x,t)e^{m(t-t_0)}e^{\frac{-mr}{q}(t-t_0)} - ke^{-\delta t}e^{m(t-t_0)} + mV(x,t)e^{m(t-t_0)}$$

$$\le m\left[V(x,t) - V^{r/q}(x,t)e^{\frac{-mr}{q}(t-t_0)}\right]e^{m(t-t_0)} - ke^{-\delta t}e^{m(t-t_0)}$$

**From the condition (a) and multiplying by** $e^{\delta t_0} > 0$**, we obtain:**

$$Q'(x,t) \le (m\gamma - k)e^{(m-\delta)(t-t_0)}$$

$$Q(x,t) - Q(x_0,t_0) \le \int_{t_0}^{t} (m\gamma - k)e^{(m-\delta)(s-t_0)}ds$$

$$Q(x,t) \le Q(x_0,t_0) + \frac{m\gamma - k}{\delta - m} - \frac{m\gamma - k}{\delta - m}e^{(m-\delta)(t-t_0)}$$

$$Q(x,t) \le V(x_0,t_0) + \frac{m\gamma - k}{\delta - m}$$

$$\le \varepsilon_2\|x_0\|^q + \frac{m\gamma - k}{\delta - m}$$

**Setting :** $\varepsilon_2\|x_0\|^q + \dfrac{m\gamma - k}{\delta - m} = \alpha(\|x_0\|) \ge 0$**, then:**

$$Q(x,t) \le \alpha(\|x_0\|)$$

**From the left hand of (4.2.1),we obtain:**

$$\|x\| \le \left[\frac{V(x,t)}{\varepsilon_1}\right]^{1/p} e^{\frac{-m}{p}(t-t_0)}$$

$$\le \left[\frac{Q(x,t)e^{-m(t-t_0)}}{\varepsilon_1}\right]^{1/p} e^{\frac{-m}{p}(t-t_0)}$$

$$\le \left[\frac{\alpha(\|x_0\|)}{\varepsilon_1}\right]^{1/p} e^{\frac{-2m}{p}(t-t_0)}$$

**Therefore the system (1) is uniformly exponentially stable.**

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