δ^* - connectedness in tritopological space

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ABSTRACT :-

In this work , I introduce a new definition for δ^*-connectedness in tritopological space . Also i study the basic specifications for the new definition of δ^*-connectedness and .

INTRODUCTION :-

Throughout this paper I adopt the notations and terminology of [1] , [2] , [3] , [4] and [5] , X and Y are finite sets . And let (X,T,Ω,ρ) be a Tritopological space , a subset A of X is said to be δ^*-open set iff $A \subset T-\text{int}(\Omega-\text{cl}(\rho-\text{int}(A)))$. [١] , and the family of all δ^*-open sets is denoted by δ* .O(X) . The complement of δ^*-open set is called a δ^*-closed set , δ* .C(X) denoted to The family of all δ* - closed sets of X .The relative tritopological space for Y is denoted by (Y,T_y,Ω_y,ρ_y) such that $T_y =\{ G \cap Y : G\in T \}$, $\Omega_y =\{ G \cap Y : G\in \Omega \}$ and $\rho_y =\{ G \cap Y : G\in \rho \}$ then (Y,T_y,Ω_y,ρ_y) is called the subspace of tritopological space (X,T,Ω,ρ) . And the relative tritopological space for Y with respect to δ^*-open sets is the collection $\delta^*.O(X)_y$ given by ;

$\delta^*.O(X)_y = \{ G \cap Y : G \in \delta^*.O(X) \}$. And let $A \subset X$. A point $x \in X$ is said to be δ^*- interior point of A iff there exists an δ^*-open set G such that $x \in G \subset A$, and the set of all δ^*-interior points of A denoted by $\delta^* - \mathrm{int}(A)$. And a point $x \in X$ is said to be δ^*-exterior point of A iff it is an δ^*-interior point of the complement of A , that is , iff there exists an δ^*-open set G such that $x \in G \subset A^c$ or equivalently $x \in G$ and $G \cap A = \varphi$. And the set of all δ^*-exterior points of A denoted by δ^* - ext(A) . And the intersection of all δ^*-closed sets containing A is called δ^*-closure of A , and denoted by $\delta^* - \mathrm{cl}(A)$. And the point x of a tritopological space (X,T,Ω,ρ) is said to be a δ^*-frontier point (or δ^*-boundary point) of a subset A of X iff it is neither δ^*-interior nor δ^*-exterior point of A .The set of all δ^*-frontier points of A , denoted by δ^*-fr(A) .

1- δ^*- separated sets in tritopological space .

1.1 Definition :

Let (X,T,Ω,ρ) be a tritopological space , Tow non empty subsets A and B of X are said to be δ^*-separated iff $A \cap \delta^*\text{-}\mathrm{cl}(B) = \varphi$ and $\delta^*\text{-}\mathrm{cl}(A) \cap B = \varphi$. These two conditions are equivalent to the single condition $[A \cap \delta^*\text{-}\mathrm{cl}(B)] \cup [\delta^*\text{-}\mathrm{cl}(A) \cap B] = \varphi$

1.2 Example :

Let $X = \{a,b,c,d\}$, $T = \{ X , \varphi , \{b,c,d\} \}$
, $\Omega = \{ X , \varphi , \{a\}\}$
and $\rho = \{ X , \varphi , \{a\} , \{c,d\} , \{c\} , \{b,c\} , \{a,c,d\} \{a,c\} , \{a,b,c\} , \{b,c,d\}\}$.

(X,T) , (X,Ω) , (X,ρ) are three topological spaces on X , then (X,T,Ω,ρ) is a tritopological space , such that :

$\delta^*.O(X) = \{ X , \varphi , \{a\},\{c\} , \{a,b\} , \{a,c\},\{c,d\} ,\{a,d\}, \{b,c\}, \{a,c,d\} , \{a,b,c\} , \{a,b,d\} , \{b,c,d\} \}$

If we take $A = \{a\}$ and $B = \{c\}$

Then $[A\cap\delta^*\text{- cl}(B)]\cup[\delta^*\text{- cl}(A) \cap B] = (\{a\}\cap\{c\})\cup(\{a\}\cap\{c\}) = \varphi$

Hence $\{a\}$ and $\{c\}$ are δ^*-separated sets .

Also ($\{a,b\}$ and $\{c,d\}$) , ($\{a,d\}$ and $\{b,c\}$) etc. are δ^*-separated sets .

1.3 Theorem :

Let $(Y,T_y ,\Omega_y ,\rho_y)$ be a subspace of a tritopological space (X,T,Ω,ρ) and let A , B be two subsets of Y , then A , B are δ^*-separated iff they are δ_y^*-separated .

Proof :

$\delta_y^*\text{-cl}(A) = \delta^*\text{-cl}(A) \cap Y$ and $\delta_y^*\text{-cl}(B) = \delta^*\text{-cl}(B) \cap Y$ [1]

Now $[\delta_y^*\text{-cl}(A) \cap B] \cup [A \cap \delta_y^*\text{-cl}(B)]$

$= [\delta^*\text{-cl}(A) \cap Y \cap B] \cup [A \cap \delta^*\text{-cl}(B) \cap Y]$

$= [\delta^*\text{-cl}(A) \cap B] \cup [A \cap \delta^*\text{-cl}(B)]$ [because $A , B \subset Y$]

Hence $[\delta_y^*\text{-cl}(A) \cap B] \cup [A \cap \delta_y^*\text{-cl}(B)] = \varphi$ iff

$[\delta^*\text{-cl}(A) \cap B] \cup [A \cap \delta^*\text{-cl}(B)] = \varphi$

It follows that A , B are δ^*-separated iff they are δ_y^*-separated .

1.4 Example :

Let $X = \{a,b,c,d\}$, $T = \{ X , \varphi \}$

, $\Omega = \{ X , \varphi , \{b,c,d\} , \{a,c,d\}, \{c,d\} \}$

and $\rho = \{ X , \varphi , \{c\} , \{d\} , \{c,d\} \}$

(X,T) , (X,Ω) , (X,ρ) are three topological spaces then (X,T,Ω,ρ) is a tritopological space , such that :

$\delta^*.O(X) = \{ X , \varphi , \{c\} , \{d\} , \{c,d\} , \{b,c\} , \{a,c,d\} , \{a,b,c\} , \{a,c\} , \{a,d\} , \{b,d\} , \{a,b,d\} , \{b,c,d\} \}$.

And let $Y = \{a,c,d\}$, then $Ty = \{ Y , \varphi \}$

, $\Omega y = \{ Y , \varphi , \{c,d\}\}$

and $\rho y = \{ Y , \varphi , \{c\},\{d\},\{c,d\} \}$

(Y,Ty) , $(Y,\Omega y)$, $(Y,\rho y)$ are three topological spaces then $(Y,Ty,\Omega y,\rho y)$ is a subspace of a tritopological space (X,T,Ω,ρ) , such that :

$\delta^*.O(X)y = \{ Y, \varphi , \{c\} , \{d\} , \{a,c\}, \{a,d\}, \{c,d\} \}$.

If we take $A = \{c\} \subset Y$ and $B = \{d\} \subset Y$

Then $[A \cap \delta^*\text{- cl}(B)] \cup [\delta^*\text{- cl}(A) \cap B] = (\{c\} \cap \{d\}) \cup (\{c\} \cap \{d\}) = \varphi$

Hence $\{c\}$ and $\{d\}$ are δ^*-separated sets . And it is clearly that $\{c\}$ and $\{d\}$ are δ_y^* -separated sets .

And conversely ; it is clear that $\{c\}$ and $\{d\}$ are δ_y^* -separated sets then $\{c\}$ and $\{d\}$ are δ^*-separated sets .

1.5 Theorem :

If A and B are δ^*-separated subsets of a tritopological space (X,T,Ω,ρ) and $C \subset A$ and $D \subset B$, then C and D are also δ^*-separated .

Proof :

We are given that $A \cap \delta^*\text{- cl}(B) = \varphi$ and $\delta^*\text{- cl}(A) \cap B = \varphi$ ……..(1)

Also $C \subset A \Rightarrow \delta^*\text{- cl}(C) \subset \delta^*\text{- cl}(A)$ and

$D \subset B \Rightarrow \delta^*\text{- cl}(D) \subset \delta^*\text{- cl}(B)$ ……….(2)

It follows from (1) and (2) that $C \cap \delta^*\text{- cl}(D) = \varphi$ and $\delta^*\text{- cl}(C) \cap D = \varphi$

Hence C and D are δ^*-separated .

1.6 Theorem :

Two δ^*-closed (δ^*-open) subsets A , B of a tritopological space (X,T,Ω,ρ) are δ^*-separated iff they are disjoint .

Proof :

Since any two δ^*-separated sets are disjoint , we need only prove that two disjoint δ^*-closed (δ^*-open) sets are δ^*-separated .

If A and B are both disjoint and δ^*-closed , then :

$A \cap B = C$, $\delta^*\text{- cl}(A) = A$ and $\delta^*\text{- cl}(B) = B$ [1]

So that $\delta^*\text{- cl}(A) \cap B = \varphi$ and $A \cap \delta^*\text{- cl}(B) = \varphi$

Showing that A and B are δ^*-separated .

If A and B are both disjoint and δ^*-open , then A^c and B^c are both δ^*-closed so that :

$\delta^*\text{- cl}(A^c) = A^c$ and $\delta^*\text{- cl}(B^c) = B^c$

Also $A \cap B = \varphi \Rightarrow A \subset B^c$ and $B^c \subset A$

$\Rightarrow \delta^*\text{- cl}(A) \subset \delta^*\text{- cl}(B^c) = B^c$ and $\delta^*\text{- cl}(B) \subset \delta^*\text{- cl}(A^c) = A^c$

$\Rightarrow \delta^*\text{- cl}(A) \cap B = \varphi$ and $A \cap \delta^*\text{- cl}(B) = \varphi$

Hence $\Rightarrow$ A and B are δ^*-separated .

2- δ^*- connected and δ^*- disconnected sets in tritopological space .

2.1 Definition :

Let (X,T,Ω,ρ) be a tritopological space , a subset A of X is said to be δ^*-disconnected iff it is the union of two non empty δ^*-separated sets . That is , iff there exist two non empty sets C and D such that $C \cap \delta^*\text{- cl}(D) = \varphi$, $\delta^*\text{- cl}(C) \cap D = \varphi$ and $A = C \cup D$, A is said to be δ^*- connected iff it is not δ^*-disconnected .

2.2 Example :

Let $X = \{a,b,c\}$, $T = \{ X , \varphi , \{b\} , \{a,b\} \}$

, $\Omega = \{ X , \varphi , \{c\}\}$

and $\rho = \{ X , \varphi , \{a\} , \{b\} , \{a,b\} , \{a,c\}\}$

(X,T) , (X,Ω) , (X,ρ) are three topological spaces on X , then (X,T,Ω,ρ) is a tritopological space , such that :

$\delta^*.O(X) = \{ X , \varphi , \{a\},\{b\} , \{a,b\} , \{a,c\}, \{b,c\} \}$

If we take $A = \{a,b\}$, $C = \{a\}$ and $D = \{b\}$, then :

$A = C \cup D$ and $\{a\} \cap \{b\} = \varphi$

And $C \cap \delta^*\text{- cl}(D) = \{a\} \cap \{b\} = \varphi$, $\delta^*\text{- cl}(C) \cap D = \{a\} \cap \{b\} = \varphi$

Then $\{a\}$ and $\{b\}$ are δ^*-separated sets .

Hence the set $\{a,b\}$ is δ^*-disconnected .

But $\{a,c\}$, $\{b,c\}$, $\{b\}$, $\{a\}$ are δ^*-connected sets .

2.3 Remark :

(i) The empty set in $\delta^*.O(X)$ is trivially δ^*-connected .

(ii) every singleton set in $\delta^*.O(X)$ is δ^*-connected since it cannot be expressed as a union of two non empty δ^*-separated sets .

2.4 Definition :

Two points a and b of a tritopological space X are said to be δ^*-connected iff they are contained in a δ^*- connected subset of X .

2.5 Example :

In the last example the points a and c are δ^*- connected because they are contained in {a,c} which is δ^*- connected subset of X .

2.6 Theorem :

Let $(Y,T_y ,\Omega_y ,\rho_y)$ be a subspace of a tritopological space (X,T,Ω,ρ) and $A \subset Y$, then A is δ^*-disconnected iff it is $\delta_y{}^*$- disconnected . equivalently , A is δ^*- connected iff it is $\delta_y{}^*$-connected .

Proof :

By theorem (1.3) , two non empty subsets of Y are δ^*-separated iff they are $\delta_y{}^*$-separated . Therefore A is the union of two δ^*-separated sets iff it is the union of two $\delta_y{}^*$-separated sets . Hence the result .

2.7 Example :

Tt is clear in example (1.4) , the set {c,d} is δ^*-disconnected then it is $\delta_y{}^*$- disconnected , and the converse is true .

And the set {a,c} and {a,d} are δ^*-connected then it is $\delta_y{}^*$- connected , and the converse is true .

2.8 Theorem :

A tritopological space (X,T,Ω,ρ) is δ^*-disconnected iff there exists a non empty proper subset of X which is both δ^*-open and δ^*-closed in X .

Proof :

Let A be a non empty proper subset of X which is both δ^*-open and δ^*-closed in X . We have to show that X is δ^*-disconnected :

Let $B = A^c$. Then B is non empty since A is a proper subset of X .

Moreover , $A \cup B = X$ and $A \cap B = \varphi$, since A is both δ^*-open and δ^*-closed , B is also both δ^*-open and δ^*-closed .

Hence δ^*- cl(A) = A and δ^*- cl(B) = B it follows that $\delta^*\text{- cl}(A) \cap B = \varphi$ and $A \cap \delta^*\text{- cl}(B) = \varphi$. Thus X has been expressed as a union of two δ^*-separated sets and so X is δ^*-disconnected .

Conversely ; Let X is δ^*-disconnected . Then there exist non empty subsets A and B of X such that $\delta^*\text{- cl}(A) \cap B = \varphi$ and $A \cap \delta^*\text{- cl}(B) = \varphi$ and $A \cup B = X$. Since $\delta^*\text{- cl}(A) = A$ and $\delta^*\text{- cl}(B) = B \Rightarrow A \cap B = \varphi$.

Hence $A = B^c$ and B is non empty , A is a proper subset of X . Now $A \cup \delta^*\text{- cl}(B) = X$. [since $A \cup B = X$ and $B \subset \delta^*\text{- cl}(B) \Rightarrow X \subset A \cup \delta^*\text{- cl}(B)$ but $A \cup \delta^*\text{- cl}(B) \subset X$ always]

Also $A \cap \delta^*\text{- cl}(B) = C \Rightarrow A = (\delta^*\text{- cl}(B))^c$ and similarly $B = (\delta^*\text{- cl}(A))^c$.

Since δ^*- cl(A) and δ^*- cl(B) are δ^*-closed sets , it follows that A and B are δ^*-open sets , and since $A = B^c$, A is also δ^*-closed . Thus A is non empty proper subset of X which is both δ^*-open and δ^*-closed .

In the same way we can show that B is also non empty proper subset of X which is both δ^*-open and δ^*-closed .

2.9 Corollary :

A tritopological space (X,T,Ω,ρ) is δ^*-connected iff the only non empty subset of X which is both δ^*-open and δ^*-closed in X is X itself .

Proof :

Easy to prove by using above theorem .

2.10 Theorem :

A tritopological space (X,T,Ω,ρ) is δ^*-disconnected iff any one of the following statements holds :

(i) X is the union of two non empty disjoint δ^*-open sets .

(ii) X is the union of two non empty disjoint δ^*-closed sets .

Proof :

Let X be a δ^*-disconnected . Then there exists a non empty proper subset A of X which is both δ^*-open and δ^*-closed . Then A^c is also both δ^*-open and δ^*-closed also $A \cup A^c = X$. Hence the sets A and A^c satisfy the requirements of (i) and (ii) .

Conversely ; let $X = A \cup B$ and $A \cap B = \varphi$, where A , B are non empty δ^*-open sets . It follows that $A = B^c$ so that A is δ^*-closed .

Since B is non empty , A is a proper subset of X . Thus A is a non empty proper subset of X which is both δ^*-open and δ^*-closed . Hence by the theorem (2.7) , X is δ^*-disconnected .

Again , let $X = C \cup D$ and $C \cap D = \varphi$, where C , D are non empty δ^*-closed sets . then $C = D^c$ so that C is δ^*-open . Since D is non empty , C is a proper subset of X which is both δ^*-open and δ^*-closed . Hence X is δ^*-disconnected by the theorem .

Thus it is shown that if any one of the conditions (i) and (ii) holds , then X is δ^*-disconnected .

2.11 Corollary :

A subset Y of a tritopological space X is δ^*-disconnected iff Y is the union of two non empty disjoint sets both δ^*-open (δ^*-closed) in Y .

Proof :

Easy to prove .

2.12 Corollary :

Let (X,T,Ω,ρ) be a tritopological space and let Y be a subset of X . Then Y is δ^*-disconnected iff there exist non empty sets G and H both δ^*-open (δ^*-closed) in X such that : $G\cap H \neq \varphi$, $H\cap G \neq \varphi$,$Y \subset G\cup H$ and $G\cap H \subset X\text{-}Y$.

Proof :

By corollary (2.10) , Y is δ^*-disconnected iff there exist non empty sets G and H both δ^*-open (δ^*-closed) in X such that : $G\cap H \neq \varphi$, $H\cap G \neq \varphi$, $(G\cap Y) \cap (H\cap Y) = \varphi$

And $(G\cap Y) \cup (H\cap Y) = Y$

Now $(G\cap Y) \cap (H\cap Y) = \varphi$

$\Leftrightarrow$ $(G\cap H) \cap Y = \varphi$

$\Leftrightarrow$ $(G\cap H) \subset X\text{-}Y$

And $(G\cap Y) \cup (H\cap Y) = Y$

$\Leftrightarrow$ $(G\cup H)\cap Y = Y$

$\Leftrightarrow$ $Y \subset G\cup H$. Hence the result .

2.13 Theorem :

A tritopological space (X,T,Ω,ρ) is δ^*-connected iff every non empty proper subset of X has a non empty δ^*- frontier .

Proof :

Let every non empty proper subset of X have a non empty δ^*- frontier . , to show that X is δ^*-connected . Suppose ; if possible ; X is δ^*-disconnected , then there exist non empty disjoint sets G and H both δ^*-open (δ^*-closed) in X . Such that $X = G \cup H$ .Therefore $G = \delta^*\text{-int}(G) = \delta^*\text{-cl}(G)$. But $\delta^*\text{-fr}(G) = \delta^*\text{-cl}(G) - \delta^*\text{-int}(G)$ [1]

Hence $\delta^*\text{-fr}(G) = G - G = \varphi$. Which is contrary to our hypothesis .

Hence X must be δ^*-connected

Conversely , let X be δ^*-connected and suppose , if possible , there exists a non empty proper subset A of X such that $\delta^*\text{-fr}(A) = \varphi$

Now $\delta^*\text{-cl}(A) = \delta^*\text{-int}(A) \cup \delta^*\text{-fr}(A) = A \cup \delta^*\text{-fr}(\delta^*\text{-cl}(A))$ [1]

Hence $\delta^*\text{-cl}(A) = \delta^*\text{-int}(A) = A$ showing that A is both δ^*-open and δ^*-closed in X and therefore X is δ^*-disconnected by theorem (2.7) .

But this is a contradiction . Hence every non empty proper subset of X must have a non empty δ^*- frontier .

References :

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الخلاصة :

في هذا العمل قدمت تعريف جديد للإتصال في الفضاء الثلاثي التبولوجي، وقد اطلقت عليه اسم (δ^*- **الإتصال**) كذلك درست المواصفات الأساسية لهذا التعريف الجديد في الفضاء الثلاثي التبولوجي .