OPERATORS FOR BEST APPROXIMATION AND BEST ONE SIDED APPROXIMATION IN L_p METRIC FOR EACH $0<p\le 1$

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Abstract

We define operators mpping any periodic bounded measurable function to trigonometric polynomials. These operators realize the degree of best approximation and best onesided approximation.

1. Notation and Definitions

We shall consider 2π periodic bounded measurable function defined on X, where $X=[0,2\pi]$ or $X=[-\pi,\pi]$. Let $L_\infty(X)$ be the space of all such functions f such that $\|f\|_\infty<\infty$ and $L_p(X)$ be the space of all 2π periodic bounded measurable functions for which $\|f\|_p<\infty$.

Let $\Delta_h^k f(x)$ denote the kth difference of a function f at the point x. We denote by

(1.1) $\omega_k(f,x,\delta)=\sup_{|h|\le\delta}\left\{\left|\Delta_h^k f(x)\right|, t,t+kh\in[x-k\delta/2,x+k\delta/2]\cap X\right\}$

the local modulus of a function f $k\in N,\delta>0$

the average modulus of smoothness is

(1.2) $\tau_k(f,\delta)_p=\|\omega_k(f,.,\delta)\|_p$.

some properties of ω_k,τ_k are given in[1].

By T_m we denote the set of all trigonometric polynomials of degree npt greater than m and by $\widetilde{T}_m$ we denote the set of the conjugate of any T_m.

Let $E(f)_p$ be the degree of best approximation in $L_p, p<1$ by element from $Y(Y=T_m or Y=\widetilde{T}_m$, i.e.

(1.3) $E(f)_p=\inf_{T\in Y}\|f-T\|_p$.

The best onesided approximation in $L_p, p<1$ by element from $Y(Y=T_m or Y=\widetilde{T}_m$ is denote by $\widetilde{E}(f)_p$ where

$$\widetilde{E}(f)_p=\inf_{\substack{T^{\pm}\in Y,\\ T^-\le f\le T^+}}\|f-T\|_p$$

Now let us define the operator

(1.5) $Z(f,x)=\int_X\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}f(x+jv)c_1(n,p)\left(\frac{\sin\left(\left[n^{1/p+1}\right]v/2\right)}{\sin v/2}\right)^{2\ell+2}dv$

where n,k,ℓ are positive integers. The number $[u]=\min\{u;u\geq n;u\ \ is\ \ integer\}$ and $c_1(n,p)$ is chosen such that $\int_X c_1(n,p)\left(\frac{\sin([n^{1/p+1}]v/2)}{\sin v/2}\right)^{2\ell+2}=1$. Notice that

(1.6) $Z(f)\in T_{(\ell+1)([n^{1/p+1}]-1)}$

By using the same lines as in [1] p.16-20) we can prove that $Z(f)$ is a polynomial which realize the order of the best approximation :

(1.7) $\|f-Z\|_p\leq c(p)\tau_k(f,1/n)_p,p\leq 1$, where p is a constant depending on p.

Using $Z(f)$ we construct our onesided operators mapping any 2π periodic bounded measurable function to trigonometric polynomial to prove inequalities imply characterizations of the orders of the best onesided approximation as follow:

$$Z^{\pm}(f,x)=Z(f,x)\pm\sum_{i=0}^{n-1}\omega_k(f,i/n,\delta(i/n))\left(\frac{\sin^4(\pi/2n)\sin^2(i)}{\sin^2(i/2n)}\right)^{1/p},$$

where $\delta(i/n)=\sqrt{(1-i/n)^2/2/(\sqrt{n}+1/2n)^2}$.

(1.6) imply that $Z^{\pm}(f)\in T_{(\ell+1)([n^{1/p+1}]-1)}$. From (1.7) and positivity of ω_k and $\left(\frac{\sin^4(\pi/2n)\sin^2(i)}{\sin^2(i/2n)}\right)^{1/p}$ we get $Z^{-}(f,x)\leq f(x)\leq Z^{+}(f,x)$ for any x.

As an intermediate approximating tool let as define the following transform

1.9 $\tilde{g}(x)=nc_2(n,p)\int_0^{1/n}\tilde{f}(x+u)\left(\frac{\sin(n[n^{1/p}]u/2)}{\sin u/2}\right)^4 du$, where $\tilde{f}$ is the conjugate function to f and $c_2(n,p)$ chosen such that

$$\int_0^{1/n}c_2(n,p)\left(\frac{\sin(n[n^{1/p}]u/2)}{\sin u/2}\right)^4 du=1$$

2. Assertions

Lemma 1. [3] For any $T\in T_m$ and $0<p_2\leq p_1\leq\infty$ we have

(2.11) $\|T\|_{p_1}\leq c(p_1,p_2)n^{1/p_2-1/p_1}\|T\|_{p_2}$.

Lemma 2. [2] If $f\in L_s[a,b]$ then for any $0<r\leq s$ and $a,b\in R$ we have

(2,12) $\|f\|_r\leq(b-a)^{1/r-1/s}\|f\|_s$.

3. The main results

Inequalities between the quantities $\tilde{E}(f)_{p_1}$ and $\tilde{E}(f)_{p_2}$ $0<p_2\leq p_1\leq 1,\tilde{E}(f)_p$ and $n^{-1/p}$ and $\tilde{E}(f)_p$ and $\tau_1(\tilde{f},1/n)_p$ are obtained in this article. These inequalities .

Theorem 1. If $f\in L_{p_1}(X)$ and $0<p_2\leq p_1\leq 1$ then

$\widetilde{E}(f)_{p_1} \le c(p_1, p_2)\, n^{-1/p_1}\, \widetilde{E}(f)_{p_2}$.

Theorem 2. if $f \in L_p(X), p < 1$ we have

$\widetilde{E}(f)_p \le c(p)\, n^{-1/p}$.

Theorem 3. If $\widetilde{f} \in L_p(X), p < 1$ for any $n \ge 3$ we have

$\widetilde{E}(f)_p \le c(p)\, \tau_1(\widetilde{f}, 1/n)_p$.

We also need the following lemmas

Lemma3

(3.13) $c_2(n,p) \approx 1/n^3 \left[n^{1/p}\right]^4$

proof. (1.10) imply

$$c_2^{-1}(n,p) = \int_0^{1/n} \left(\frac{\sin\left(n\left[n^{1/p}\right]u/2\right)}{\sin u/2} \right)^4 du.$$

Now since $(2n/(2n+1)\pi)^4 \le \sin^4 u \le u^4$ for any $u \in [n\pi, (2n+1)\pi/2]$ so that

$$c_2^{-1}(n,p) \approx \left(n\left[n^{1/p}\right]\right)^3 \int_0^{1/n} \left(\frac{\sin(u/2)}{u/2} \right)^4 du \approx \left(n^3 \left[n^{1/p}\right]^4 \right) \circ$$

Lemma 4. Let f be a bounded measurable function, then for each $n \ge 3$, we have

(3.14) $|\widetilde{g}| \le |\widetilde{f}|$.

Proof. From (1.9) we have

$$|\widetilde{g}(x)| = \left| nc_2(n,p) \int_0^{1/n} \widetilde{f}(x+u) \left(\frac{\sin\left(n\left[n^{1/p}\right]u/2\right)}{\sin u/2} \right)^4 du \right|$$

$$\le \left| nc_2(n,p) \int_0^{1/n} \sup_{|u| \le 1/n} \widetilde{f}(x+u) \left(\frac{\sin\left(n\left[n^{1/p}\right]u/2\right)}{\sin u/2} \right)^4 du \right|$$

$$\le \pi^4 |\widetilde{f}| / 3 \left[n^{1/p}\right]^4 \le |\widetilde{f}| \circ$$

Lemma 5. Let f be a bounded measurable function then for each $n \ge 3$ we have

(3.15) $|\widetilde{g} - \widetilde{f}| \le c_1 \omega_1(\widetilde{f}, x + 1/2n, 1/n)$

where c_1 is a constant.

Proof. Using (3.14), (1.9) and (1.10) we have

$|\widetilde{g} - \widetilde{f}| \le |\widetilde{g} - n\widetilde{f}|$

$$\le nc_2(n,p) \int_0^{1/n} \sup_{|u| \le 1/n} \left| \widetilde{f}(x+u) - \widetilde{f}(x) \right| \left(\frac{\sin\left(n\left[n^{1/p}\right]u/2\right)}{\sin u/2} \right)^4 du$$

(1.1) , $\sin u/2 \ge u/\pi$ and (3.13) imply

$$|\widetilde{g}(x) - \widetilde{f}(x)| \le c_1 \omega_1(\widetilde{f}, x + 1/2n, 1/n) \circ$$

Lemma 6. Let f be a bounded measurable function then for each $n \ge 3$, we have

(3.16) $\left|2\tilde{g}(x)-\tilde{f}(x)\right| \le c_2\omega_1\left(\tilde{f},x+1/2n,1/n\right)$, where c_2 is a constant

Proof.

$\left|2\tilde{g}(x)-\tilde{f}(x)\right| \le \left|2\tilde{g}(x)-2n\tilde{f}(x)\right|$.

Then by the same lines used in Lemma 5, we can get the result directly ⊛

Proof of theorem 1.

Be bounded operators in $T_{(\ell+1)([n^{1/p}]-1)}$ such that $\left\|Z^+-Z^-\right\|_p \le \left\|T^+-T^-\right\|_p$ and $\left\|T^+-T^-\right\|_p = \tilde{E}(f)_p$ for $p<1$.

Then applying (1.4), (2.11)and (2.12) to have

$\tilde{E}(f)_p \le \left\|Z^+-Z^-\right\|_{p_1}$,where

$$Z^{\pm}(f,x)=Z(f,x)\pm\sum_{i=0}^{n-1}\omega_k(f,i/n,\delta(i/n))\left(\frac{\sin^4(\pi/2n)\sin^2(i)}{\sin^2(i/2n)}\right)^{1/p_2},\text{then}$$

$$\begin{aligned}\tilde{E}(f)_{p_1} &\le c_1(p_1,p_2)n^{1/p_2-1/p_1}\left\|Z^+-Z^-\right\|_{p_2}\\ &\le c_2(p_1,p_2)n^{1/p_2-1/p_1}\left\|Z^+-Z^-\right\|_1\\ &\le c_3(p_1,p_2)n^{-1/p_1}\left\|Z^+-Z^-\right\|_{1/2}.\end{aligned}$$

Whenever $p_2 \ge 1/2$ we have from (2.12) and our hypothesis that

$\tilde{E}(f)_{p_1} \le c_4(p_1,p_2)n^{-1/p_1}\ \tilde{E}(f)_{p_2}$.

And when $p_2 < 1/2$ we have

$$\tilde{E}(f)_{p_1} \le c_3(p_1,p_2)n^{-1/p_1}\left(\int_X\left|T^+-T^-\right|^{p_2+q}\right)^{1/2},q\subset(0,1).$$

By our hypothesis we have

$$\begin{aligned}\tilde{E}(f)_{p_1} &\le c_5(p_1,p_2)n^{-1/p_1}\left(\int_X\left|T^+-T^-\right|^{p_2}\right)^{1/p_2-m},m\in(0,\infty)\\ &\le c_6(p_1,p_2)n^{-1/p_1}\left(\int_X\left|T^+-T^-\right|^{p_2}\right)^{1/p_2}.\end{aligned}$$

That's complete the proof.

Proof of theorem 2

We have by (1.4), (2.11) that

$\tilde{E}(f)_p \le \left\|Z^+-Z^-\right\|_p$ where

$$Z^{\pm}(f,x)=Z(f,x)\pm\sum_{i=0}^{n-1}\omega_k(f,i/n,\delta(i/n))\left(\frac{\sin^4(\pi/2n)\sin^2(i)}{\sin^2(i/2n)}\right)^{1/p_2}\text{ and}$$

$0<p_2\le p\ \le 1$. Then

$$\widetilde{E}(f)_p \le c_1(p,p_2)n^{-1/p-1/p_1}\left\|Z^+ - Z^-\right\|_{p_2}.$$

Using the inequality $\sin u \le u$ to have complete the proof ⊙

Proof of theorem 3

(1.3) implies

$E(\widetilde{f})_p^p \le \left\|\widetilde{f} - \widetilde{T}(\widetilde{f})\right\|_p^p$, where $\widetilde{T} \in \widetilde{T}_n$ and it is a bounded operator of .Then

$$E(\widetilde{f})_p^p \le \left\|\widetilde{f} - \widetilde{g}\right\|_p^p + \left\|\widetilde{T} - \widetilde{g}\right\|_p^p$$

$$\le \left\|\widetilde{f} - \widetilde{g}\right\|_p^p + \left\|\widetilde{g}\right\|_p^p + \left\|\widetilde{T}(\widetilde{g})\right\|_p^p + \left\|\widetilde{T}(\widetilde{g})\right\|_p^p + \left\|\widetilde{T}(\widetilde{g}) - \widetilde{T}(\widetilde{f})\right\|_p^p.$$

Then since $\widetilde{T}(\widetilde{f})$ is a bounded linear operator we get

$$E(\widetilde{f})_p^p \le c_1(p)\left(\left\|\widetilde{f} - \widetilde{g}\right\|_p^p + \left\|\widetilde{g}\right\|_p^p\right)$$

$$= c_1(p)\left(\left\|\widetilde{f} - \widetilde{g}\right\|_p^p + \left\|\widetilde{g} - \widetilde{f} + \widetilde{f} + \widetilde{g} - \widetilde{g}\right\|_p^p\right)$$

$$\le c_2(p)\left(\left\|\widetilde{f} - \widetilde{g}\right\|_p^p + \left\|2\widetilde{g} - \widetilde{f}\right\|_p^p\right)$$

Using (3.15),(3.16) periodicity of f and (1.2) we get our result ⊙

References

[1] E. S. Bhaya, 2000, A study on the approximation of bounded measurable functions with some discrete series in Lp spaces for p<1. M.Sc. Thesis, Baghdad University.

[2] N. B. Hasser and J. A. Sullivan, 1970, Real analysis. New York.

[3] A. Zygmund, 1988, Trigonometric series. Cambridge.

مؤثرات لاحسن اقتراب و أحسن أقتراب جانبي في الفضاءات L_p لكل $0 < p \le 1$

الخلاصة

قمنا في هذا البحث بإيجاد بعض المؤثرات المثلثية التي بواسطتها تمكنا من إيجاد درجة أحسن تقريب و أحسن تقريب جانبي للدوال f و مرافقاتها، من تلك المتعددات التي قمنا بالتوصل إليها و بعض المتعددات المثلثية في الفضاءات L_p عندما $0 < p \le 1$.