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On Bayesian Estimation Methods of the Reliability Function Based on Generalized Exponential Distribution

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Abstract : The purpose of this paper is the derivation of the system reliability formula in the stress resistance (S-S) model when strength X and stress Y are independently random variables, and it follows the Generalized Exponential Distribution of the model; (R_((s,r))=P(┤ at least s of X_1,X_2,…,X_r exceed ├ Y)), when X and Y follow GED, with estimating the reliability of the system stress-strength model via different estimation methods under the name Maximum Likelihood Estimator and Bayesian estimation, Finally comparisons of the proposed estimation methods using the Mote Carlo simulation based on the criteria of mean squared error.

Keywords: Stress-Strength, Generalized Exponential Distribution, Bayesian methods, Monte Carlo, and Mean square error.

Introduction: Assume that X and Y are identical and independent. Let X is a strength random variable exposed to a common stress? Y, then the reliability system contained one component denoted by $R = P(X > Y)$, In the same electrical circuit, when there are K^{th} number of light bulbs, each lamp has durability (strength), generated heat because of its work (stress), we need s of bulbs to work, if the generated heat is less than the durability of lamps, the circuit is working. This system above refers to what is called a multicomponent stress-strength system (s-out of- r). This system is studied by Bhattacharyya and Johnson in (1973)[4], that is mean when the strength of the light bulbs $X_1, X_2, \dots, X_r$ imposed to a common stress Y, and then the reliability system of the multicomponent stress-strength system computed by

$$R_{(s,r)} = \text{P(at least s of } X_1, X_2, \dots, X_r \text{ exceed } Y \text{)}$$

Many researchers have studied the estimated reliability of multicomponent stress strength. In (1974), Bhattacharyya and Johnson studied the reliability of multicomponent stress-strength following Exponential distribution; this system works only if at least s-out of- k open paren s less than k close paren strengths exceed the stress estimated by MVUM [4]. In (1981), Kim and Kang studied the estimation of the multicomponent stress-strength for Weibull distribution with unknown scale parameters and known shape parameters and considered minimum variance unbiased estimator of system reliability [9]. In (1991), Pandey and Uddin estimated the reliability of multi-component stress-strength model based on Burr distribution using the Bayes estimator and the non-Bayes estimators (MLE), and they concluded that the Bayes estimator was the best [11]. On the other hand in (1999), Gupta and Kundu found the Generalized Exponential distribution as a special case of the Generalized Weibull distribution (GWD) when $\beta = 1$, where the cumulative distribution function of (GWD) is given by [10];

$$F(x) = \left(1 - e^{-(\Psi x)^{\beta}}\right)^{\alpha 1} ;x > 0 \text{ and } \alpha, \beta, \Psi > 0$$

The first studied the Generalized Exponential distribution by Gupta and Kundu, in $(2000), (2001), (2003), (2005), (2007)$.

2. The Reliability System $R_{(s,r)}$

Assume that, $\mathrm{X} \sim \mathrm{GED}(\alpha 1, \Psi)$ and $\mathrm{Y} \sim \mathrm{GED}(\alpha 2, \Psi)$ the probability density functions (PDF) for each X and Y are respectively became

$$f(x; \alpha 1, \Psi) = \begin{cases} \alpha 1\, \Psi e^{-\Psi x}(1 - e^{-\Psi x})^{\alpha 1 - 1}; & \text{For } x > 0; \alpha 1, \Psi > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y; \alpha 2, \Psi) = \begin{cases} \alpha 2 \Psi e^{-\Psi y}(1 - e^{-\Psi y})^{\alpha 2 - 1}; & \text{For } y > 0; \alpha 2, \Psi > 0 \\ 0 & \text{otherwise} \end{cases}$$

And, the CDF of X and Y are respectively given as

$$F(x, \alpha 1, \Psi) = (1 - e^{-\Psi x})^{\alpha 1}$$
$$F(y, \alpha 2, \lambda) = (1 - e^{-\Psi y})^{\alpha 2}$$

Assume $X_1, X_2, \dots \dots, X_r$ are strength and subject to common stress Y Then, the reliability system for a multicomponent in stress-strength model $R_{(s,r)}$ will be:
$R_{(s,r)} = P(\text{ at least s of the } X_1, X_2, \dots, X_r \text{ exceed Y})$

$$= \sum_{i=s}^{r} \binom{r}{i} \int_0^\infty (1 - F_x(y))^i (F_x(y))^{r-i} f(y) dy$$

Now, we to finding the reliability of multicomponent system in stress-strength model, $R_{(s,r)}$ for the Generalized Exponential distribution

$R_{(s,r)} = \sum_{i=s}^{r} \binom{r}{i} \int_o^\infty (1 - (1 - e^{-\Psi y})^{\alpha 1})^i ((1 - e^{-\Psi y})^{\alpha 1})^{r-i} \alpha 2 \Psi e^{-\Psi y} (1 - e^{-\Psi y})^{\alpha 2 - 1} dy$

$= \sum_{i=s}^{r} \binom{r}{i} \beta \Psi \int_0^\infty (1 - (1 - e^{-\Psi y})^{\alpha 1})^i (1 - e^{-\Psi y})^{\alpha 1 r - \alpha 1 i + \alpha 2 - 1} e^{-\Psi y} dy$

Let $z = 1 - e^{-\Psi y} \rightarrow y = \frac{-\ln(1-z)}{\Psi} \rightarrow dy = \frac{dz}{\Psi(1-z)}$

$$R_{(s,r)} = \sum_{i=s}^{r} \binom{r}{i} \beta \Psi \int_0^1 (1 - z^{\alpha 1})^i (z)^{\alpha 1 r - \alpha 1 i + \alpha 2 - 1} (1 - z) \frac{dz}{\Psi(1 - z)}$$

$$= \sum_{i=s}^{r} \binom{r}{i} \alpha 2 \int_o^1 (1 - z^{\alpha 1})^i (z)^{\alpha 1 r - \alpha 1 i + \alpha 2 - 1} dz$$

Let $w = z^{\alpha 1} \rightarrow z = w^{\frac{1}{\alpha 1}} \rightarrow dz = \frac{1}{\alpha 1} w^{\frac{1}{\alpha 1} - 1} dw$

$$R_{(s,r)} = \sum_{i=s}^{r} \binom{r}{i} \frac{\alpha 2}{\alpha 1} \int_o^1 (1 - w)^i (w)^{r - i + \frac{\alpha 2}{\alpha 1} - \frac{1}{\alpha 1}} \; w^{\frac{1}{\alpha 1} - 1} dw$$

$$= \sum_{i=s}^{r} \binom{r}{i} \frac{\alpha 2}{\alpha 1} \int_0^1 (1 - w)^i (w)^{r - i + \frac{\alpha 2}{\alpha 1} - 1} dw$$

$$= \sum_{i=s}^{r} \binom{r}{i} \frac{\alpha 2}{\alpha 1} B\left(r - i + \frac{\alpha 2}{\alpha 1}, i + 1\right)$$

$$= \sum_{i=s}^{r} \binom{r}{i} \frac{\alpha 2}{\alpha 1} \frac{\Gamma\left(r - i + \frac{\alpha 2}{\alpha 1}\right) \Gamma(i + 1)}{\Gamma\left(r + \frac{\alpha 2}{\alpha 1} + 1\right)}$$

$$R_{(s,r)} = \frac{\alpha 2}{\alpha 1} \sum_{i=s}^{r} \frac{r!}{(r - i)!} \left[\prod_{j=0}^{i} \left(r + \frac{\alpha 2}{\alpha 1} - j\right)\right]^{-1} \quad \text{; r, i, j are integers}$$

3. Maximum Likelihood Estimator (MLE)

Suppose the strength random sample of size n say $x_1, x_2, \dots \dots, x_n$ follows GED $(\alpha 1, \Psi)$ and $y_1, y_2, \dots \dots, y_m$ be the stress random sample of size m follow GED $(\alpha 2, \Psi)$

The Maximum Likelihood Function of the observed samples obtained by

$l = L(\alpha 1, \alpha 2; x, y) = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} f(y_j)$

$l = \prod_{i=1}^{n} \alpha 1 \Psi e^{-\Psi x_i} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \prod_{j=1}^{m} \Psi \alpha 2 e^{-\lambda \Psi_j} \left(1 - e^{-\Psi y_j}\right)^{\alpha 2 - 1}$

$l = \alpha 1^n \Psi^{(n+m)} e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \alpha 2^m e^{-\Psi \sum_{j=1}^{m} y_j} \prod_{j=1}^{m} \left(-e^{-\Psi y_j}\right)^{\alpha 2 - 1}$

Take Logarithm to both sides, we get:

$\ln(l) = n \ln \alpha 1 + (n + m) \ln \Psi - \Psi \sum_{i=1}^{n} x_i + (\alpha 1 - 1) \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i}) + m \ln \alpha 2 - \Psi \sum_{j=1}^{m} y_j + (\alpha 2 - 1) \sum_{j=1}^{m} \ln\left(1 - e^{-\Psi y_j}\right)$

The partial derivatives for above equation w.r.t $\alpha 1$ and $\alpha 2$ respectively, and equal the result to zero, we conclude:

$$\frac{d \text{Ln}(l)}{d\alpha 1} = \frac{n}{\alpha 1} + \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i}) = 0$$

$$\frac{d \text{Ln}(l)}{d\alpha 2} = \frac{m}{\alpha 2} + \sum_{j=1}^{m} \ln\left(1 - e^{-\Psi y_j}\right) = 0$$

Thus, the Maximum Likelihood estimator for the unknown shape parameters $\alpha 1$ and $\alpha 2$ will be respectively as follows:

$$\hat{\alpha 1}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})}$$

$$\hat{\alpha 2}_{MLE} = \frac{-m}{\sum_{j=1}^{m} \ln\left(1 - e^{-\Psi y_j}\right)}$$

Hence, we get the Maximum Likelihood estimation for $R_{(s,r)}$

$$\hat{R}_{(s,r)_{MLE}} = \frac{\hat{\alpha 2}_{MLE}}{\hat{\alpha 1}_{MLE}} \sum_{i=s}^{r} \frac{r!}{(r-i)!} \left[\prod_{j=0}^{i} \left(r + \frac{\hat{\alpha 2}_{MLE}}{\hat{\alpha 1}_{MLE}} - j \right) \right]^{-1}$$

4. Bayes Method Based on Jeffrey's Prior Information

Let $X \sim \mathrm{GE}(\alpha 1, \Psi)$ and $Y \sim \mathrm{GE}(\alpha 2, \Psi)$ know ,we have to find Bayes estimator for $\alpha 1$ using non-information prior distribution $g(\alpha 1)$ based on modified Extension of Jeffery prior

$$g(\alpha 1) \propto [I(\alpha 1)]^c$$

When $I(\alpha 1)$ is Fisher information, which is obtained by:

$$I(\alpha 1) \quad = -nE\left[\frac{\partial^2 \ln f(x; \alpha 1, \Psi)}{\partial \alpha 1^2}\right] = \frac{n}{\alpha 1^2}$$

Then, $g1(\alpha 1) \propto \left[\frac{n}{\alpha 1^2}\right]^c$ and $g1(\alpha 1) = kn^c \alpha 1^{-2c}$

The Likelihood function $L(x_1, x_2, \dots., x_n; \alpha 1)$ will be:

$$\begin{aligned} L(x_1, x_2, \dots, x_n; \alpha 1) \quad &= \prod_{i=1}^{n} f(x_i, \alpha 1, \Psi) \\ &= \prod_{i=1}^{n} \alpha 1 \Psi e^{-\Psi x_i} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \\ &= \alpha 1^n \Psi^n e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \end{aligned}$$

The joint probability density function $H(x_1, x_2, \dots, ., x_n; \alpha 1)$ is given by $H1(x_1, x_2, \dots, x_n; \alpha 1) = L(x_1, x_2, \dots, x_n; \alpha 1). g1(\alpha 1)$

$$H1(x_1, x_2, \dots., x_n; \alpha 1) = \alpha 1^n \Psi^n e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \cdot kn^c \alpha 1^{-2c}$$

The marginal probability density function of $x_1, x_2, \dots, x_n$ will be

$$P1(x_1, x_2, \dots, . x_n) = \int_0^\infty L(x_1, x_2, \dots, x_n; \alpha 1) \cdot g1(\alpha 1) d\alpha 1$$

$$= \int_0^\infty \alpha 1^n \Psi^n e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} \cdot kn^c \alpha 1^{-2c} d\alpha 1$$

Then, the Posterior distribution $\Pi 1(\alpha 1/x_i)$, $i = 1, \dots, n$, as follow:

$$\begin{aligned} \Pi 1(\alpha 1/x_i) \quad &= \frac{L(x_1, x_2, \dots, x_n; \alpha 1) \cdot g1(\alpha 1)}{\int_0^\infty L(x_1, x_2, \dots, x_n; \alpha 1) \cdot g1(\alpha 1) d\alpha 1} \\ &= \frac{\alpha 1^n \Psi^n e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} . kn^c \alpha 1^{-2c}}{\int_0^\infty \alpha 1^n \Psi^n e^{-\Psi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} . kn^c \alpha 1^{-2c}} \\ &= \frac{\alpha 1^{n-2c} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1}}{\int_0^\infty \alpha 1^{n-2c} \prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} d\alpha 1} \end{aligned}$$

Let $\prod_{i=1}^{n} (1 - e^{-\Psi x_i})^{\alpha 1 - 1} = e^{-\alpha 1 \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1}} e^{-\sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})}$

Then $\Pi 1(\alpha 1/x_i)$ became as

$$\Pi 1(\alpha 1/x_i) = \frac{\alpha 1^{n-2c} e^{-\alpha 1 \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1}}}{\int_{\alpha 1 = 0}^{\infty} \alpha 1^{n-2c} e^{-\alpha \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1}} d\alpha 1}$$

Suppose

$$y = \alpha 1 \sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1} \quad \rightarrow \alpha 1 = \frac{y}{\sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1}}$$

$$\rightarrow d\alpha 1 = \frac{dy}{\sum_{i=1}^{n} \ln(1 - e^{-\Psi x_i})^{-1}}$$

Thus, $\Pi 1(\alpha 1/x_i)$ will be

$$\Pi 1(\alpha 1/x_i)=\frac{\alpha 1^{n-2c}e^{-\alpha 1\sum_{i=1}^{n}\ln\left(1-e^{-\Psi x_i}\right)^{-1}}\left[\sum_{i=1}^{n}\ln\left(1-e^{-\Psi x_i}\right)^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)}$$

5. Bayes Estimator Based on Gamma Prior Information

We take $g2(\alpha 1)=\frac{\beta^{K+4}\alpha 1^{k+3}e^{-\beta\alpha 1}}{\Gamma(k+4)}$

$\alpha 1$~Gamma(k+4, β) and k , β are known constant

Then, the Posterior distribution $\Pi 2(\alpha 1/x_i)$, i = 1, ... , n, as follow:

$$\Pi 2(\alpha 1/x_i)=\frac{L(x_1,x_2,\dots,x_n;\alpha 1)\cdot g2(\alpha 1)}{\int_0^\infty L(x_1,x_2,\dots,x_n;\alpha 1)\cdot g2(\alpha 1)d\alpha 1}$$

$$\int_0^\infty L(x_1,x_2,\dots,x_n;\alpha 1)\cdot g2(\alpha 1)d\alpha 1=\int_0^\infty \Psi^n\alpha^n e^{-\Psi\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(1-e^{-\Psi x_i})^{\alpha 1-1}\frac{\beta^{K+4}\alpha 1^{k+3}e^{-\beta\alpha 1}}{\Gamma(k+4)}d\alpha 1$$

$$=\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}\frac{\beta^{K+4}}{\Gamma(k+4)}\int_0^\infty \alpha 1^{n+k+3}e^{-\beta\alpha 1}e^{-(\alpha 1-1)\sum\ln\left(1-e^{-\Psi x_i}\right)^{-1}}d\alpha 1$$

$$=\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}e^{-\sum ln\left(1-e^{-\Psi x_i}\right)}\frac{\beta^{K+4}}{\Gamma(k+4)}\int_0^\infty \alpha 1^{n+k+3}e^{-\alpha 1\left(\beta+\sum\ln\left(1-e^{-\Psi x_i}\right)^{-1}\right)}d\alpha 1$$

$$=\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}e^{-\sum ln\left(1-e^{-\Psi x_i}\right)}\frac{\beta^{K+4}}{\Gamma(k+4)}\int_0^\infty \alpha 1^{n+k+3}e^{-\alpha 1(\beta+T)}d\alpha 1$$

Let $Y=\alpha 1(\beta+T)\ \rightarrow\ \frac{Y}{\beta+T}=\alpha 1\ \rightarrow\ \frac{dY}{(\beta+T)}=d\alpha 1$

$$\int_0^\infty L(x_1,x_2,\dots,x_n;\alpha 1)\cdot g2(\alpha 1)d\alpha 1=\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}e^{-\sum\ln\left(1-e^{-\Psi x_i}\right)}\beta^{K+4}\frac{\Gamma(n+k+4)}{(\beta+T)^{n+K+4}\Gamma(k+4)}$$

$$\Pi 2(\alpha 1/x_i)=\frac{\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}e^{-\sum\ln\left(1-e^{-\Psi x_i}\right)}\frac{\beta^{K+4}}{\Gamma(k+4)}\alpha 1^{n+k+3}e^{-\alpha 1(\beta+T)}}{\Psi^n e^{-\Psi\sum_{i=1}^{n}x_i}e^{-\sum\ln\left(1-e^{-\Psi x_i}\right)}\beta^{K+4}\frac{\Gamma(n+k+4)}{(\beta+T)^{n+K+4}\Gamma(k+4)}}$$

$$\Pi 2(\alpha 1/x_i)=\frac{(\beta+T)^{n+K+4}e^{-\alpha 1(\beta+T)}\alpha 1^{n+k+3}}{\Gamma(n+k+4)}$$

$\Pi 2(\alpha 1/x_i)\sim$ Gamma$(n+k+4,\beta+T)$

6. Bayes Estimator Under Squared Error Loss Function

6.1. The Case of Jeffrey's Prior Information

using square loss function which is given by $\mathrm{L}(\hat{\alpha 1},\alpha 1)=\mathrm{k}(\hat{\alpha 1}-\alpha 1)^2$

Then the Risk function $R1(\hat{\alpha 1},\alpha 1)$ will be

$$R1(\hat{\alpha 1},\alpha 1)=\mathrm{E}[\mathrm{L}(\hat{\alpha 1},\alpha 1)]$$

$R1(\hat{\alpha 1},\alpha 1)=\int_0^\infty \mathrm{k}(\hat{\alpha 1}-\alpha 1)^2\cdot\Pi 1(\alpha 1/x_i)d\alpha 1$

$$=\int_0^\infty \mathrm{k}(\hat{\alpha 1}-\alpha 1)^2\cdot\frac{\alpha 1^{n-2c}e^{-\alpha 1\sum_{i=1}^{n}\ln\left(1-e^{-(\Psi x_i)^2}\right)^{-1}}\left[\sum_{i=1}^{n}\ln\left(1-e^{-(\Psi x_i)^2}\right)^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)}d\alpha 1$$

The partial derivatives of the equation above with respect to $\hat{\alpha 1}$ and equating the results to the zero:

$$\frac{\partial R1}{\partial\hat{\alpha 1}}=2k\hat{\alpha 1}-\frac{2k(n-2c+1)}{\sum_{i=1}^{n}\ln(1-e^{-\Psi x_i})^{-1}}$$

$$0=2k\hat{\alpha 1}-\frac{2k(n-2c+1)}{\sum_{i=1}^{n}\ln(1-e^{-\Psi x_i})^{-1}}$$

Then the Bayes estimator under Jeffrey's Prior Information of $\alpha 1$ will be

$$\hat{\alpha 1}_{Bjs}=\frac{(n-2c+1)}{\sum_{i=1}^{n}\ln(1-e^{-\Psi x_i})^{-1}}$$

In this work, we assume c = 2, Therefore, the Bayes estimator under Jeffrey's Prior Information of $\alpha 1$ will be:

$$\hat{\alpha 1}_{Bjs}=\frac{(n-3)}{\sum_{i=1}^{n}\ln(1-e^{-\Psi x_i})^{-1}}$$

And by the same way assume the stress Y random sample follows GRD with two parameter $(\alpha 2,\Psi)$ with size m to find the Bayes estimator under Jeffrey's Prior Information of $\alpha 2$ will be

$$\hat{\alpha 2}_{Bjs} = \frac{(m-3)}{\sum_{j=1}^{m} \ln(1-e^{-\Psi x_i})^{-1}}$$

the Bayes estimator under Jeffrey's Prior Information for $R1_{(s,r)}$ will be

$$\hat{R}1_{(s,r)_{Bjs}} = \frac{\hat{\alpha 2}_{Bjs}}{\hat{\alpha 1}_{Bjs}} \sum_{i=s}^{r} \frac{r!}{(r-i)!} \left[\prod_{j=0}^{i} \left(r + \frac{\hat{\alpha 2}_{Bjs}}{\hat{\alpha 1}_{Bjs}} - j \right) \right]^{-1}$$

6.2. The Case of Gamma Prior Information

The Risk function $R2(\hat{\alpha 1}, \alpha 1)$ will be

$R2(\hat{\alpha 1}, \alpha 1) = \int_0^\infty \mathrm{k1}(\hat{\alpha 1} - \alpha 1)^2 \cdot \Pi 2(\alpha 1 / x_i) d\alpha 1$

$$= \int_0^\infty \mathrm{k1}(\hat{\alpha 1} - \alpha 1)^2 \cdot \frac{1}{\Gamma(n+k+4)} \alpha^{n+k+3} e^{-\alpha(\beta+T)} (\beta+T)^{n+k+4} d\alpha 1$$

Where $\mathrm{T} = \sum \ln(1-e^{-\Psi x_i})^{-1}$

The partial derivatives of the equation above with respect to $\hat{\alpha 1}$ and equating the results to the zero:

$$\frac{\partial R2}{\partial \hat{\alpha 1}} = 2k1\hat{\alpha 1} - 2k1 \frac{n+k+4}{(\beta+T)}$$

Then, the Bayes estimator under Gamma Prior Information of $\alpha 1$ will be

$$\hat{\alpha 1}_{Bgs} = \frac{n+k+4}{(\beta + \sum ln(1-e^{-\Psi x_i})^{-1})}$$

And by the same way assume the stress Y random sample follows GRD with two parameter $(\alpha 2, \Psi)$ with size m to find the Bayes estimator under Gamma Prior Information of $\alpha 2$ will be

$$\hat{\alpha 2}_{Bgs} = \frac{m+k+4}{(\beta + \sum ln\left(1-e^{-\Psi y_j}\right)^{-1})}$$

the Bayes estimator under Gamma Prior Information for $R1_{(s,r)}$ will be

$$\hat{R}1_{(s,r)_{Bgs}} = \frac{\hat{\alpha 2}_{Bgs}}{\hat{\alpha 1}_{Bgs}} \sum_{i=s}^{r} \frac{r!}{(r-i)!} \left[\prod_{j=0}^{i} \left(r + \frac{\hat{\alpha 2}_{Bgs}}{\hat{\alpha 1}_{Bgs}} - j \right) \right]^{-1}$$

7. Bayes Estimator Under logarithmic Error Loss Function

7.1. The Case of Jeffrey's Prior Information

using logarithmic Error Loss Function, which is given by $E((\ln \alpha 1) \mid x_i) = \int_0^\infty \ln \alpha 1 \;\; \Pi 1(\alpha 1 \mid x_i) d\alpha 1$

$$= \int_0^\infty \ln \alpha 1 . \frac{\alpha 1^{n-2c} e^{-\alpha 1 \sum_{i=1}^{n} ln\left(1-e^{-(\Psi x_i)^2}\right)^{-1}} \left[\sum_{i=1}^{n} ln\left(1-e^{-(\Psi x_i)^2}\right)^{-1}\right]^{n-2c+1}}{\Gamma(n-2c+1)} d\alpha 1$$

$= \int_0^\infty \ln \alpha 1 . \frac{\alpha 1^{n-2c} e^{-\alpha 1 T} T^{n-2c+1}}{\Gamma(n-2c+1)} d\alpha 1$ Where $\mathrm{T} = \sum \ln(1-e^{-\Psi x_i})^{-1}$

Let $y = \alpha 1 T \;\; \rightarrow \;\; \alpha 1 = \frac{y}{T} \rightarrow \;\; d\alpha 1 = \frac{dy}{T}$

$E(\ln \alpha 1 \mid x_i) = \int_0^\infty \ln \frac{y}{T} . \frac{(\frac{y}{T})^{n-2c} \;\; e^{-y} \;\; T^{n-2c+1}}{\Gamma(n-2c+1)} d\alpha 1$

$= \frac{\Gamma'(n-2c+1)}{\Gamma(n-2c+1)} - \frac{\mathrm{Ln}(T)}{\Gamma(n-2c+1)} \Gamma(n-2c+1)$

Therefore $E(\ln \alpha 1 \mid x_i) = \Psi 1(n-2c+1) - \mathrm{Ln}(T)$

Where $\Psi 1(n-2c+1) = \frac{\Gamma'(n-2c+1)}{\Gamma(n-2c+1)}$

$$\hat{\alpha 1}_{Bjl} = e^{\Psi 1(n-2c+1) - \ln(T)}$$

$$\hat{\alpha 1}_{Bjl} = \frac{e^{\Psi 1(n-2c+1)}}{T}$$

$$\hat{\alpha 1}_{Bjl} = \frac{e^{\Psi 1(n-2c+1)}}{\sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}}$$

In this work, we assume $\mathrm{c} = 2$, therefor, the Bayes estimator under Jeffrey's Prior Information of $\alpha 1$ will be: Then the Bayes estimator of $\alpha 1$ will be

$$\hat{\alpha 1}_{Bjl} = \frac{e^{\Psi 1(n-3)}}{(\sum ln(1-e^{-\Psi x_i})^{-1})}$$

And by same way assume the stress Y random sample follows GRD with two parameter $(\alpha 2, \Psi)$ with size m to find the Bayes estimator under Jeffrey's Prior Information of $\alpha 2$ will be

$$\hat{\alpha 2}_{Bjl} = \frac{e^{\Psi 1(m-3)}}{(\sum ln(1-e^{-\Psi y_j})^{-1})}$$

the Bayes estimator under Jeffrey's Prior Information for $R1_{(s,r)}$ will be

$$\hat{R}1_{(s,r)_{Bjl}} = \frac{\hat{\alpha 2}_{Bjl}}{\hat{\alpha 1}_{Bjl}} \sum_{i=s}^{r} \frac{r!}{(r-i)!}\left[\prod_{j=0}^{i}\left(r+\frac{\hat{\alpha 2}_{Bjl}}{\hat{\alpha 1}_{Bjl}}-j\right)\right]^{-1}$$

7.2. The Case of Gamma Prior Information

$\mathrm{L}(\alpha 1, \hat{\alpha}1) = \left(\ln\frac{\hat{\alpha}1}{\alpha 1}\right)^2 = (\ln\hat{\alpha 1} - \ln\alpha 1)^2$

$$E[\mathrm{L}(\alpha 1, \hat{\alpha 1} \mid x_i)] = (\ln\hat{\alpha 1})^2 - 2\ln\hat{\alpha 1}E(\ln\alpha 1 \mid x_i) + E((\ln\alpha 1)^2 \mid x_i)$$

The partial derivatives of the equation above with respect to $\hat{\alpha 1}$ and equating the results to the zero:

$$\frac{\partial R2}{\partial\hat{\alpha 1}} = 0 = \frac{2(\ln\hat{\alpha 1})}{\hat{\alpha 1}} - \frac{2E(\ln\alpha 1 \mid x_i)}{\hat{\alpha}1}$$

$$\hat{\alpha}_{Bgl} = e^{E(\ln\alpha 1 | x_i)}$$

$$E((\ln\alpha 1) \mid x_i) = \int_0^\infty \ln\alpha 1 \;\; \Pi 2(\alpha 1 \mid x_i) d\alpha 1$$

$$= \int_0^\infty \ln\alpha 1 \frac{(\beta+T)^{n+k+4} e^{-\alpha 1(\beta+T)} \alpha^{n+k+3}}{\Gamma(n+k+4)} d\alpha 1$$

$$= \int_0^\infty \frac{1}{\Gamma(n+k+4)}\left(\ln\left(\frac{Y}{\beta+T}\right)\right)\left(\frac{Y}{\beta+T}\right)^{n+k+3} e^{-Y}(\beta+T)^{n+k+4}\frac{dy}{\beta+T}$$

$$= \int_0^\infty \frac{1}{\Gamma(n+k+4)}\left(\ln\left(\frac{Y}{\beta+T}\right)\right)\left(\frac{Y}{\beta+T}\right)^{n+k+3} e^{-Y}(\beta+T)^{n+k+4}\frac{dy}{\beta+T}$$

$$= \int_0^\infty \frac{1}{\Gamma(n+k+4)}\left(\ln\left(\frac{Y}{\beta+T}\right)\right)\left(\frac{Y}{\beta+T}\right)^{n+k+3} e^{-Y}(\beta+T)^{n+k+4}\frac{dy}{\beta+T}$$

$$= \int_0^\infty \frac{1}{\Gamma(n+k+4)}\left(\ln\left(\frac{Y}{\beta+T}\right)\right) Y^{n+k+3} e^{-Y} dY$$

$$= \frac{1}{\Gamma(n+k+4)}\int_0^\infty (\mathrm{Ln}\, Y) Y^{\,n+k+3} e^{-Y} dY - \frac{\mathrm{Ln}(\beta+T)}{\Gamma(n+k+4)}\int_0^\infty Y^{n+k+3} e^{-Y} dY$$

$$= \frac{\Gamma'(n+k+4)}{\Gamma(n+k+4)} - \frac{\mathrm{Ln}(\beta+T)}{\Gamma(n+k+4)}\Gamma(n+k+4)$$

Therefore $\quad E(\ln\alpha 1 \mid x_i) = \Psi 1(n+k+4) - \mathrm{Ln}(\beta+T)$

$$\hat{\alpha 1}_{Bgl} = e^{\Psi 1(n+k+4) - \ln(\beta+T)}$$

$$\hat{\alpha 1}_{Bgl} = \frac{e^{\Psi 1(n+k+4)}}{\beta+T}$$

$$\hat{\alpha 1}_{Bgl} = \frac{e^{\Psi 1(n+k+4)}}{\beta + \sum_{i=1}^{n}\ln(1-e^{-\lambda x_i})^{-1}}$$

Then the Bayes estimator under gamma Prior Information of $\alpha 1$ will be

$$\hat{\alpha 1}_{Bgl} = \frac{e^{\Psi 1(n+k+4)}}{(\beta + \sum ln(1-e^{-\Psi x_i})^{-1})}$$

And by same way assume the stress Y random sample follows GRD with two parameter $(\alpha 2, \Psi)$ with size m to find the Bayes estimator under Gamma Prior Information of $\alpha 2$ will be

$$\hat{\alpha 2}_{Bgl} = \frac{e^{\Psi 1(m+k+4)}}{(\beta + \sum ln(1-e^{-\Psi y_j})^{-1})}$$

the Bayes estimator under Gamma Prior Information for $R1_{(s,r)}$ will be

$$\hat{R}1_{(s,r)_{Bgl}} = \frac{\hat{\alpha 2}_{Bgl}}{\hat{\alpha 1}_{Bgl}} \sum_{i=s}^{r} \frac{r!}{(r-i)!}\left[\prod_{j=0}^{i}\left(r+\frac{\hat{\alpha 2}_{Bgl}}{\hat{\alpha 1}_{Bgl}}-j\right)\right]^{-1}$$

8. Simulation study

In this paper, we mainly perform some simulation experiments to observe the behavior of the five methods of estimation for different sample sizes and for different parameter values. Not that the generation of generalized exponential Distribution $(\alpha 1, \Psi)$ is simply as: if u follows uniform distribution in [0,1], then we get

$F(x) = (1 - e^{-\Psi x})^{\alpha 1}$

$U_i = (1 - e^{-\Psi x_i})^{\alpha 1}$

$x_i = \frac{-\ln\left(1 - U_i^{\left(\frac{1}{\alpha 1}\right)}\right)}{\Psi}$ follows generalized exponential Distribution $(\alpha 1, \Psi)$

The estimation of $R_{(s,r)}$ is made by five different estimation methods: the maximum likelihood, the Bayes Method Based on Jeffrey's Prior Information(two types), and the Bayes Estimator Based on Gamma Prior Information(two types), for each one of the reliability functions and using simulation study with (r=10000) replication. Monte Carlo simulation is applied for different sample sizes from very small to large (25, ,50,75), and different parameter values of different models suggested to each reliability functions by using MATLAB program and comparing between the estimators of each one by mean square errors (MSE'S)

Table.1 Shows the estimation value of $R1_{(s,k)} = 0.75000$, $R_{(s,r)} = (1,3)$ when $\alpha = \beta = 2$, $\lambda = 2$, k1=5 & B11=4

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$
(25,25)	0.74604	0.71099	0.71015	0.71099	0.71015
(25,50)	0.74896	0.72677	0.76065	0.72477	0.75970
(25,75)	0.74945	0.73625			
(50,25)	0.74593		0.78063	0.73367	0.77938
(50,50)	0.74824	0.70029			
(75,25)	0.74462		0.66028	0.70242	0.66144
(75,50)	0.74801	0.71703			
(75,75)	0.74843		0.71674	0.71703	0.71674
		0.69024			
			0.63493	0.69307	0.63661
		0.70166			
			0.68774	0.70232	0.68819
		0.71536			
			0.71518	0.71536	0.71518

Table. 2 Shown MSE value of $R1_{(s,k)} = 0.75000, (\text{s}, \text{k}) = (1,3)$ when $\alpha = \beta = 2 \& \lambda = 2$.

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$	Best
(25,25)	0.00283	0.03818	0.03905	0.03818	0.03905	mle
(25,50)	0.00206	0.03529	0.03205	0.03557	0.03212	
(25,75)			0.02989			mle
(50,25)	0.00188	0.03319	0.04725	0.03350	0.02995	
(50,50)			0.03401			mle
(75,25)	0.00219	0.03806	0.05334	0.03768	0.04696	
(75,50)			0.03787			mle
(75,75)	0.00142	0.03370	0.03163	0.03370	0.03401	
						mle
	0.00195	0.03930		0.03874	0.05286	
						mle
	0.00119	0.03512		0.03500	0.03778	
						mle
	0.00093	0.03146		0.03146	0.03163	
						mle

Table.3 Shown estimation value of $R1_{(s,k)} = 0.50000$, $R_{(s,r)} = (2,3)$ when $\alpha = \beta = 2 \& \lambda = 2$.

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$
(25,25)	0.49804	0.49547	0.49536	0.49547	0.49536
(25,50)	0.50062	0.51143	0.56064	0.50864	0.55925
(25,75)	0.50359				
(50,25)	0.49570	0.52402	0.59028	0.52036	0.58835
(50,50)	0.49963				

(75,25)	0.49572	0.47048	0.41986	0.47331	0.42130
(75,50)	0.49825				
(75,75)	0.49936	0.49320	0.49317	0.49320	0.49317
		0.45788	0.40975	0.46159	0.41174
		0.48063	0.46250	0.48151	0.46310
		0.48968	0.48964	0.48968	0.48964

Table. 4 Shown MSE value of $R1_{(s,k)} = 0.50000, (\text{ s}, \text{k}) = (2,3)$ when $\alpha = \beta = 2 \& \lambda = 2.$

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$	Best
(25,25)	0.00667	0.05716	0.05827	0.05716	0.05827	mle
(25,50)	0.00493	0.05612	0.05903	0.05611	0.05893	mle
(25,75)			0.05965			mle
(50,25)	0.00444	0.05386	0.06226	0.05378	0.05941	mle
(50,50)			0.05313			mle
(75,25)	0.00493	0.05618	0.06561	0.05600	0.06206	mle
(75,50)			0.05201			mle
(75,75)	0.00326	0.05271	0.04736	0.05271	0.05313	mle
	0.00466	0.05594		0.05562	0.06533	
	0.00278	0.05059		0.05055	0.05196	
	0.00224	0.04712		0.04712	0.04736	

Table.1 Shown estimation value of $R1_{(s,k)} = \ 0.60000,\ \ R_{(s,r)} = (2,4)$ when $\alpha = \beta = 2 \&\ \lambda = 2.$

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$
(25,25)	0.59625	0.57246	0.57191	0.57246	0.57191
(25,50)	0.59992	0.59300	0.63894	0.59034	0.63765
(25,75)	0.60050				
(50,25)	0.59565	0.60493	0.66583	0.60149	0.66408
(50,50)	0.59844				
(75,25)	0.59361	0.55707	0.50675	0.55982	0.50820
(75,50)	0.59796				
(75,75)	0.59838	0.57775	0.57757	0.57775	0.57757
		0.54357	0.47504	0.54720	0.47709
		0.55684	0.53898	0.55769	0.53957
		0.57364	0.57352	0.57364	0.57352

Table. 2 Shown MSE value of $R1_{(s,k)} = \ 0.60000, (\text{ s}, \text{k}) = (2,4)$ when $\alpha = \beta = 2 \& \lambda = 2.$

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$	Best
(25,25)	0.00568	0.05596	0.05710	0.05596	0.05710	mle
(25,50)	0.00421	0.05217	0.05130	0.05236	0.05130	mle
(25,75)			0.05076			mle
(50,25)	0.00386	0.05023	0.06410	0.05041	0.05068	mle
(50,50)			0.05076			mle
(75,25)	0.00440	0.05479	0.07074	0.05443	0.06380	mle
(75,50)			0.05316			mle
(75,75)	0.00386	0.05023	0.04694	0.05041	0.05068	mle
	0.00392	0.05595		0.05541	0.07022	
	0.00243	0.05035		0.05025	0.05307	
	0.00191	0.04670		0.04670	0.04694	

Table.1 Shown estimation value of $R1_{(s,k)} = 0.56321,\ \ R_{(s,r)} = (2,4)$ when $\alpha = 2.5\ ,\ \beta = 4 \ \&\ \lambda = 2.$

(n,m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$
(25,25)	0.56167	0.59992	0.59988	0.59992	0.59988
(25,50)	0.56488	0.58902	0.59301	0.58642	0.59176
(25,75)	0.56466				
(50,25)	0.55941	0.59526	0.60295	0.59191	0.60127
(50,50)	0.56228				

(75,25)	0.55806	0.58397	0.53621	0.58665	0.53763
(75,50)	0.56062				
(75,75)	0.56241	0.59060	0.59059	0.59060	0.59059
		0.57000	0.50400	0.57357	0.50605
		0.59555	0.57876	0.59638	0.57933
		0.60806	0.60806	0.60806	0.60806

Table. 2 Shown MSE value of $R1_{(s,k)} = 0.75000, (\text{s}, \text{k}) = (2,4)$ when $\alpha = 2.5$, $\beta = 4$ &$\lambda = 2$.

(n, m)	$R_{(s,r)mle}$	$R_{(s,r)Bjs}$	$R_{(s,r)Bgs}$	$R_{(s,r)Bjl}$	$R_{(s,r)Bgl}$	Best
(25,25)	0.00624	0.05419	0.05473	0.05419	0.05473	mle
(25,50)	0.00455	0.05250	0.05634	0.05239	0.05620	mle
(25,75)			0.05917			mle
(50,25)	0.00412	0.05214	0.05610	0.05191	0.05890	mle
(50,50)			0.05772			mle
(75,25)	0.00459	0.05309	0.05772	0.05304	0.05597	mle
(75,50)			0.04821			mle
(75,75)	0.00433	0.05151	0.04632	0.05138	0.05744	mle
	0.00433	0.05151		0.05138	0.05744	
	0.00257	0.04791		0.04792	0.04820	
	0.00204	0.04620		0.04620	0.04632	

Conclusions

in the multi-component S-S models of estimate the system reliability the tables (1-8) illustrated the simulation results of the proposed estimation methods, with the set parameters for this model $(\alpha 1, \alpha 2) = (2,2)$ and $(2.5,4)$when the shape parameter $\Psi = 2$ for all cases. Tables (2,4,6 and 8showed that the order rank of the proposed estimators as follows; $R_{(s,r)mle}$, $R_{(s,r)Bjs}$, $R_{(s,r)Bgs}$, $R_{(s,r)Bjl}$ and $R_{(s,r)Bgl}$ on MSE of the Generalized exponential distribution. On the other hand, $R_{(s,r)mle}$ is better than others.

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