Necessary and Sufficient Optimality Conditions in Nonlinear Programming Involving B-locally Connected Function

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الخلاصة:
في هذا البحث B-locally connected function عرفت. البرمجة الخطية المتضمنة لهذه الدوال درست. الشرط الكافي نوع كوهن-تكر أعطي تحت شرط المشتقة من جهة اليمين بالنسبة للقوس عند نقطة الامثلية موجودة .كذلك الشرط الضروري نوع فرتز-جوهن أعطي تحت شرط دالة المشتقة لدالة الهدف ودالة المشتقة للقيود المصفرة هي arc wise connected عند نقطة المثلية.ومستمرة عند القيود الغير مصفرة.

Abstract :

B-locally connected function is defined. Nonlinear programming problem is considered involving this function. Kuhn-Tucker type sufficient optimally conditions are given under the hypotheses that the right differential with respect to an arc at an optimal point exists.
Also Fritz –John type necessary optimality criteria under the hypothesis that the right differentials, at an optimal point, of the objective and the active constraint functions are arc wise connected and inactive constraint functions are continuous.

1. Introduction

The convexity notion plays an important role in the mathematical programming field. Various generalizations of convex function have appeared in literature. Ortega and Rheingold [6] extended the concept of convex functions by defining arc wise connected functions on arc wise connected sets .for which points lying on continuous arcs, instead of line segments satisfy certain inequalities. Avriel and Zang [1] expanded the class of quasiconvex functions and pseudo convex functions by defining arc wise Q-connected and arc wise P-connected functions. Kaul, lyall and kaur [3] introduced locally connected and locally Q-connected functions defined on locally connected sets which generalize arc wise connected functions [6] .Kaul and Lyall [4] defined the directional derivative (with respect to an arc) of a real valued function, called the right differential, at a point of a locally connected set .they [4] also defined locally P-connected functions in terms of their right differentials and

obtained a number of sufficient optimality criteria for a nonlinear programming involving these functions

In this paper b-locally connected functions, b-locally Q-connected functions and b-locally P-connected functions have defined. Necessary and sufficient optimality conditions were given .

For $b(x,x^*,\lambda)=1$ class of b-locally connected functions ,b-locally Q-connected functions and b-locally P-connected functions reduce to locally connected functions. Locally Q- connected functions and locally P-connected functions respectively.

2.<u>Definitions and Preliminaries</u>

Definition 2.1 [2] :Aset $S \subseteq R^n$ is said to be an arc wise connected (AC) set if for each pair of points $x^*, x \in S$,there exists a continuous vector valued function

$H_{x^*,x}(\lambda) \in S, \qquad \lambda \in [0,1]$ **, such that**

$H_{x^*,x}(0) = x^* \qquad H_{x^*,x}(1) = x$

Definition 2.2 [2]: A real value function f is said to be arc-wise connected function if for each pair of points $x^*, x \in S$,there exists an arc $H_{x^*,x}(\lambda) \in S,$ satisfying

$f(H_{x^*,x}(\lambda)) \le (1-\lambda) f(x^*) + \lambda f(x) \qquad$ **0<λ<1**

Definition 2.3[3]: Aset $S \subseteq R^n$ is said to be a locally connected (LC) set if for each pair of points $x^*, x \in S$, there exists a maximum positive number $a(x^*,x) \le 1$ and a vector valued function $H_{x^*,x}(\lambda) \in S, \quad 0 < \lambda < a(x^*,x)$

$H_{x^*,x}(\lambda)$ is continuous in interval]0, $a(x^*,x)$ [and

$H_{x^*,x}(0) = x^* \qquad H_{x^*,x}(1) = x$

Let $S \subseteq R^n$ be a locally connected set satisfying the above conditions and f be a real valued function defined on S

Definition 2.4[3]: *f* is said to be a locally connected (LCN) function if for each points $x^*, x \in S$,there exists a positive number $d(x^*,x) \le a(x^*,x)$ satisfying

$f(H_{x^*,x}(\lambda)) \le (1-\lambda) f(x^*) + \lambda f(x), \qquad 0 < \lambda < d(x^*,x)$

Definition 2.5:*f* is said to be b-locally connected (BLCN) function if for each pair of points $x^*, x \in S$,there exists a positive number $d(x^*,x) \le a(x^*,x)$ and a function $b: S \times S \times [0,1] \to R_+$ Such that

$$f(H_{x^*,x}(\lambda)) \le (1-\lambda b(x,x^*,\lambda)) f(x^*) + \lambda b(x,x^*,\lambda) f(x),$$

$$0 < \lambda < d(x^*,x), \lambda b(x,x^*,\lambda) \le 1$$

If *f* is b-locally connected for each $x^* \in S$ for the same b, then *f* is said to be b-locally connected on S

Definition 2.6: ***f*** **is said to be b-locally Q-connected (BLQCN) function if for each pair of points** $x^*, x \in S$ **there exists a positive number** $d(x^*, x) \le a(x^*, x)$ **and a function** $b: S \times S \times [0,1] \to R_+$ **Such that**

$$f(x) \le f(x^*) \Rightarrow b(x, x^*, \lambda) f(H_{x^*,x}(\lambda)) \le b(x, x^*, \lambda) f(x^*), \quad 0 < \lambda < d(x^*, x) \text{ and}$$

$$\lambda b(x, x^*, \lambda) \le 1.$$

Definition 2.7 [4]: The right differential of $f: S \to R$ **at** x^* **with respect to the arc** $H_{x^*,x}(\lambda)$ **is given by**

$$df^+(x^*, H_{x^*,x}(0^+)) = \lim_{\lambda \to 0^+} \frac{f(H_{x^*,x}(\lambda)) - f(x^*)}{\lambda}$$

Provide that the limit exists

Definition 2.8: If the right differential of ***f*** **with respect to** $H_{x^*,x}(\lambda)$ **at** x^* **exists and further for each** $x \in S$

$$df^+(x^*, H_{x^*,x}(0^+)) \ge 0 \Rightarrow b(x, x^*, \lambda) f(x) \ge b(x, x^*, \lambda) f(x^*)$$

then ***f*** **is said to be b-locally P-connected (BLPCN) at** x^***.**

If ***f*** **is b-locally P-connected at each** $x^* \in S$ **for the same b, then** ***f*** **is said to be b-locally P-connected on S**

For $b(x, x^*, \lambda) = 1$ **b-locally connected function, b-locally Q-connected function reduce to locally connected function and locally Q-connected function respectively.**

However, there are functions which are b-locally connected functions but not locally connected function for example

$$f: S \to R, \quad S = (0, \frac{\pi}{2}), \quad f(x) = \sin(x)$$

$$b(x, x^*, \lambda) = \begin{cases} 2 & x > x^* \quad \lambda \in [0, \frac{1}{2}] \\ 0 & x < x^* \quad \lambda \in [0, \frac{1}{2}] \end{cases}$$

$$H_{x^*,x}(\lambda) = ((1-\lambda) x^{*2} + \lambda x^2)^{1/2}$$

S is locally connected set, $a(x^*, x) = 1$

f **is not locally connected function because for** $x = \frac{\pi}{3}, x^* = \frac{\pi}{6}$ **there doesn't exists any positive number** $d(x^*, x)$ **such that**

$f(H_{x^*,x}(\lambda)) \le (1 - \lambda f(x^*) + \lambda f(x),$ **for** $0 < \lambda < d(x^*, x)$ **but** ***f*** **is b-locally connected function at** $x = \frac{\pi}{3}, x^* = \frac{\pi}{6}$**,** $0 < \lambda < \frac{1}{2}, d(x^*, x) = \frac{1}{2}$**.**

Class of BLCN is larger than class of LCN. So this work consider a generalization of LCN

Definition 2.9: If the right differential of *f* with respect to $H_{x^*,x}(\lambda)$ **at** x^* **exists and further for each** $,x \in S$

$$\bar{b}(x,x^*)df^+(x^*,H_{x^*,x}(0^+)) \geq 0 \Rightarrow f(x) \geq f(x^*)$$

where $\bar{b}(x,x^*) = \lim_{\lambda \to 0^+} b(x,x^*,\lambda)$

Then *f* is said to be b-locally strongly P-connected (BLSPCN) at x^*

If *f* is b-locally strongly P-connected at each $x^* \in S$ **for the same b, then *f* is said to be b-locally strongly P-connected on S.**

Theorem 2.1: let $f : S \to R$ **if the right differential of *f* with respect to** $H_{x^*,x}(\lambda)$ **at** x^* **exists and further *f* is b-locally connected (BLCN) at** x^* **then**

$$\bar{b}(x,x^*)[f(x) - f(x^*) \geq df^+(x^*,H_{x^*,x}(0^+))$$

where

$$\bar{b}(x,x^*) = \lim_{\lambda \to 0^+} b(x,x^*,\lambda) \quad \textbf{and} \quad \lambda b(x,x^*,\lambda) \leq 1$$

Proof: Suppose *f* is b-locally connected a $x^* \in S$**, i.e. for each** $,x \in S$ **there exists a positive number** $d(x^*,x) \leq a(x^*,x)$ **and a function** $b : S \times S \times [0,1] \to R_+$ **such that**

$$f(H_{x^*,x}(\lambda)) \leq (1 - \lambda b(x,x^*,\lambda))f(x^*) + \lambda b(x,x^*,\lambda)f(x),$$

$$0 < \lambda < d(x^*,x), \ \lambda b(x,x^*,\lambda) \leq 1.$$

Therefore

$$\frac{f(H_{x^*,x}(\lambda)) - f(x^*)}{\lambda} \leq b(x,x^*,\lambda)[f(x) - f(x^*)]$$

$$0 < \lambda < d(x^*,x), \ \lambda b(x,x^*,\lambda) \leq 1$$

Taking the limit as $\lambda \to 0^+$**,**

$$\lim_{\lambda \to 0^+} \frac{f(H_{x^*,x}(\lambda)) - f(x^*)}{\lambda} \leq \lim_{\lambda \to 0^+} b(x,x^*,\lambda)[f(x) - f(x^*)],$$

$$df^+(x^*,H_{x^*,x}(0^+)) \leq \bar{b}(x,x^*)[f(x) - f(x^*)].$$

The proof is complete.

Theorem 2.2: Let $f : S \to R$ **If the right differential of *f* with respect to** $H_{x^*,x}(\lambda)$ **at** x^* **exists and further *f* is b-locally Q-connected (BLQCN) at** x^* **then**

$$f(x) \leq f(x^*) \Rightarrow \bar{b}(x,x^*)df^+(x^*,H_{x^*,x}(0^+)) \leq 0$$

where

$\bar{b}(x,x^*)=\lim_{\lambda\to 0^+} b(x,x^*,\lambda)$ **and** $\lambda b(x,x^*,\lambda)\le 1$

Proof: *f* is b-locally Q-connected at x^***, therefore for each pair of points** $x^*, x\in S$**, there exists a positive number** $d(x^*,x)\le a(x^*,x)$ **and a function** $b:S\times S\times[0,1]\to R_+$ **such that**

$$f(x)\le f(x^*)\Rightarrow b(x,x^*,\lambda)f(H_{x^*,x}(\lambda))\le b(x,x^*,\lambda)f(x^*),$$

$$0<\lambda<d(x^*,x),\qquad \lambda b(x,x^*,\lambda)\le 1$$

i.e.

$b(x,x^*,\lambda)[f(H_{x^*,x}(\lambda)-f(x^*)]\le 0$

Dividing by $\lambda>0$ **and taking the limit as** $\lambda\to 0^+$ **,we get**

$\bar{b}(x,x^*)df^+(x^*,H_{x^*,x}(0^+))\le 0$ **the proof is complete**

3. Sufficient Optimally Criteria

Consider the nonlinear programming problem

(NP) $\begin{cases} Minimize & f(x) \\ subject..to & g(x)\le 0,.x\in S\end{cases}$

where $S\subseteq R^n$ **is a nonempty locally connected set and** $f:S\to R, g:S\to R^m$ **such that the right differentials at** x^* **exists with respect to the arc** $H_{x^*,x}(\lambda)$

Let $S^0=\{x\in S/g(x)\le 0\}$ **be the set of all feasible solution to (NP)**

Let $N_\varepsilon(x^*)=\{x\in R^n/\|x-x^*\|<\varepsilon\}$

Definition 3.1:

(a) x^* **is said to be a local minimum solution to problem (NP) if** $x^*\in S^0$ **and there exists** $\varepsilon>0$ **such that** $x\in N_\varepsilon(x^*)\cap S^0\Rightarrow f(x)\ge f(x^*)$

(b) x^* **is said to be the minimum solution to problem (NP) if** $x^*\in S^0$ **and**

$$f(x^*)=\min_{x\in S^0} f(x)$$

The next theorem gives sufficient optimality criteria.

Theorem 3.1: let $x^*\in S$ **and let *f* be** b_1**- locally connected function at** x^* **and g be** b_2**- locally connected function at** x^***.If there exists** $\bar{u}\in R^m$ **such that** $(x^*,\bar{u})$ **satisfy the following relations:**

$$df^+(x^*,H_{x^*,x}(0^+))+\bar{u}^T dg^+(x^*,H_{x^*,x}(0^+))\ge 0\qquad \forall x\in S^0 \qquad \textbf{(3.1)}$$

$$\bar{u}^T g(x^*)=0 \qquad \textbf{(3.2)}$$

$$g(x^*)\le 0 \qquad \textbf{(3.3)}$$

$$\bar{u}\ge 0 \qquad \textbf{(3.4)}$$

with $\bar{b}_1(x,x^*)=\lim_{\lambda\to 0^+} b_1(x,x^*,\lambda)$ **then** x^* **is an optimal solution to problem (NP)**

Proof: $g(x^*)\le 0$ **this implies to** $x^*\in S^0$**,so** x^* **is feasible solution to problem (NP).**

Also *f* is b_1**- locally connected function at** x^***.Therefore for any** $x\in S^0$**, Theorem (2.1) yields (using (3.1) and that g is** b_2**- locally connected at** x^* **.):**

$$\bar{b}_1(x,x^*)[f(x)-f(x^*)\geq df^+(x^*,H_{x^*,x}(0^+))$$
$$\geq -\bar{u}^T dg^+(x^*,H_{x^*,x}(0^+))$$
$$\geq -\bar{u}^T\bar{b}_2(x,x^*)[g(x)-g(x^*)]$$

As $\bar{u}\geq 0$ **,** $g(x\ )\leq 0$ **and** $\bar{b}_2(x,x^*)\geq 0$**, this implies to**

$$\bar{b}_1(x,x^*)[f(x)-f(x^*)\geq -\bar{u}^T\bar{b}_2(x,x^*)g(x)\geq 0$$

Thus, it follows that as $\bar{b}_1(x,x^*)\geq 0$

$f(x)\geq f(x^*)\qquad \forall x\in S^0$ **. Hence** x^* **is an optimal solution of (NP).**

4. <u>Necessary Optimality Criteria</u>

The following alternative theorem stated by Jeyakumar [2].

Theorem 4.1: let S be a nonempty arc wise connected set in R^n **and let** $f:S\to R^k$ **be an arc wise connected function on S .then either**

$f(x)<0$ **has a solution** $x\in S$

or

$\lambda^T f(x)\geq 0$ **for all** $x\in S$ **, for some** $\lambda\in R^k$ **,** $\lambda\geq 0$

But both alternatives are never true simultaneously.

Lemma 4.1: Let $x^*\in S^0$ **be a local minimum solution for (NP). We assume that g is continuous at** x^***,and that the right differential of *f* and** g **exists at** x^* **with respect to** $H_{x^*,x}(\lambda)$**.then the system**

$$\begin{cases} df^+(x^*,H_{x^*,x}(0^+))<0 \\ dg_i^+(x^*,H_{x^*,x}(0^+))<0 \end{cases} \qquad i\in I \qquad \textbf{(4.1)}$$

has no solution $x\in S^0$**, where** $I=I(x^*)=\{i/g_i(x^*)=0\}$ **and** $J=J(x^*)=\{i/g_i(x^*)\leq 0\}$

Proof: Let $x\in S^0$ **be a solution of the system (4.1).Since the right differentials of *f* and** $g_i, i\in I$ **at** x^* **exists with respect to the arc** $H_{x^*,x}(\lambda)$

Therefore

$$f(H_{x^*,x}(\lambda))=f(x^*)+\lambda df^+(x^*,H_{x^*,x}(0^+))+\lambda\alpha(\lambda) \qquad \textbf{(4.2)}$$

and

$$g_i(H_{x^*,x}(\lambda))=g_i(x^*)+\lambda dg_i^+(x^*,H_{x^*,x}(0^+))+\lambda\alpha_i(\lambda) \qquad i\in I \qquad \textbf{(4.3)}$$

where

$$\alpha:[0,1]\to R \qquad \lim_{\lambda\to 0^+}\alpha(\lambda)=0 \qquad \textbf{(4.4)}$$
$$\alpha_i:[0,1]\to R \qquad \lim_{\lambda\to 0^+}\alpha_i(\lambda)=0 \qquad \textbf{(4.5)}$$

Using (4.1),(4.4) and (4.5) we get ,for small enough λ **say** $0<\lambda<\lambda_0$

$$df^{+}(x^{*},H_{x^{*},x}(0^{+}))+\alpha(\lambda)<0$$

and

$$dg_{i}^{+}(x^{*},H_{x^{*},x}(0^{+}))+\alpha_{i}(\lambda)<0 \qquad \textbf{For } i\in I$$

Hence it follows by using the relation (4.2),(4.3) that for $0<\lambda<\lambda_{0}$

$$f(H_{x^{*},x}(\lambda))-f(x^{*})<0 \qquad \textbf{(4.6)}$$

$$g_{i}(H_{x^{*},x}(\lambda))-g_{i}(x^{*})<0 \qquad i\in I \qquad \textbf{(4.7)}$$

Now $g_{i},i\in J$ **is continuous at** x^{*} **and** $H_{x^{*},x}(\lambda)$ **is a continuous function of** λ **,therefore**

$$\lim_{\lambda\to 0^{+}} g_{i}(H_{x^{*},x}(\lambda))=g_{i}(x^{*})<0 \qquad i\in I$$

Which implies that there exists λ^{*}_{i} **,** $0<\lambda^{*}_{i}<a(x^{*},x)$ **(** $i\in I$ **) such that**

$$g_{i}(H_{x^{*},x}(\lambda))<0 \qquad \textbf{For } 0<\lambda<\lambda^{*}_{i} \qquad \textbf{(4.8)}$$

Let $\lambda^{*}=\min(\lambda_{0},\lambda_{i}^{*})$ **. Then from (4.6) to (4.8) it follows that for** $0<\lambda<\lambda^{*}$ $H_{x^{*},x}(\lambda)\in S^{0}$ **and** $f(H_{x^{*},x}(\lambda))<f(x^{*})$ **which is a contradiction as** x^{*} **is an optimal solution of (NP).Hence the system (4.1) has no solution** $x\in S$ **.**

Theorem (4.2): (Necessary Optimality Criteria): Let x^{*} **be an optimal solution of (NP).**

If $df^{+}(x^{*},H_{x^{*},x}(0^{+}))$ **and** $dg_{I}^{+}(x^{*},H_{x^{*},x}(0^{+}))$ **are arc wise connect functions of x,** g_{i} **,** $i\in J$ **is a continuous at** x^{*} **with S arc wise connected set ,then there exists** $r_{0}^{*}\in R$ $r^{*}\in R^{m}$ **Such that**

$$r_{0}^{*}df^{+}(x^{*},H_{x^{*},x}(0^{+}))+r^{*T}dg^{+}(x^{*},H_{x^{*},x}(0^{+}))\geq 0 \qquad \forall x\in S \qquad \textbf{(4.9)}$$

$$r^{*T}g(x^{*})=0 \qquad \textbf{(4.10)}$$

$$(r_{0}^{*},r^{*})\geq 0 \qquad \textbf{(4.11)}$$

where $I=I(x^{*})=\{i/g_{i}(x^{*})=0\}$ **and** $J=J(x^{*})=\{i/g_{i}(x^{*})\leq 0\}$

Proof: According to theorem (4.1), exactly one of the following two systems has a solution

$$df^{+}(x^{*},H_{x^{*},x}(0^{+}))+dg_{i}^{*}(x^{*},H_{x^{*},x}(0^{+}))<0$$

or there exists $r_{0}^{*},r_{i}^{*}\in R$ **,** $i\in I$ **Such that**

$$r_{0}^{*}df^{+}(x^{*},H_{x^{*},x}(0^{+}))+r_{I}^{*T}dg_{I}^{+}(x^{*},H_{x^{*},x}(0^{+}))\geq 0 \qquad \forall x\in S \qquad \textbf{(4.12)}$$

$$(r_{0}^{*},r_{I}^{*})\geq 0$$

Since the assumption of the lemma is satisfied, so the first has no solution. Hence the second system has a solution

Defining $r_{j}^{*}=0$ **for** $j\in J$ **,by (4.12) we get (4.9).Because for** $i\in I$ **,** $g_{i}(x^{*})=0$ **,this implies for** $r^{*}=(r_{i}^{*},r_{j}^{*})$ **we have** $r^{*T}g(x^{*})=0$ **which is the relation (4.10).The proof is complete.**

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