A note on equality of MINQUE and simple estimator in the multivariate linear model

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Abstract:

Under the multivariate linear model $\{Y, X\beta, \sum \otimes V\}$**, equality of MINQUE and so called "simple" estimator** $(1/f)Y^T MY$**, with M=I-X(X\`X)$^+$X^T,** $f = rank(X:V) - rank(X)$ **for Σ is considered. It is revealed that this equality holds if and only if the quadratic form** $\sum^{-1} Y'MY$ **admits Wishart-distribution under multivariate normality of Y.**

Keywords: multivariate linear model, MINQUE, BLUE, OLSE, Wishart - distribution

1. Introduction:

Consider the general multivariate linear model $\{Y, X\beta, \sum \otimes V\}$

Where, $E(y) = X\beta$, $D(y) = \sum \otimes V$, **E and D being expectation and dispersion operators. X and V are known matrices of orders** $n \times p$ **and** $n \times n$ **respectively, and** $\otimes$ **denotes the Kronecker Product of matrices (see Rao 1973 p.29) both possibly deficient in rank. Assume that the model is consistent, i.e.**

$Y \in R(X:V)$,

Where $R(A)$ **stands for the range of a matrix A and (A: B) denotes the partitioned matrix with A and B placed next to each other.**

Suppose we wish to estimate the unknown positive definite Σ by using the competing estimators

$$(1/f)Y^T M(MVM)^+ MY \qquad (1)$$

And

$$(1/f)Y^T MY \qquad (2)$$

Where $f = rank(X:V) - rank(X)$ **is assumed to be positive and $M = I - X(X^T X)^+$. Here A^+ stands for the unique Moore-Penrose inverse of a matrix A(see Rao 1973 p. 26) and A^T denotes the transpose of A.**

Formula (1) provides the MINQUE (Minimum Norm Quadrate Unbiased Estimator) for Σ, cf. Theorem 3.4(a) in Rao (1974). It can also be written as

$$(1/f)Y^T(MVM)^+ Y \qquad (3)$$

Since we have $(MVM)^+ = M(MVM)^+$ **in view of** $R[(MVM)^+] \subseteq R(M)$, **and then** $(MVM)^+ = (MVM)^+ M$ **in view of the symmetry of** $(MVM)^+$. **According to Theorem 3.4 in Rao (1974), the MINQUE can be represented in further different forms. For example under the weakly singular multivariate linear model, i.e. in case** $R(X) \subseteq R(V)$, **the estimator (1) can be written as**

$$(1/f)Y^T\left(V^+ - V^+ X(X^T V^+ X)^+ X^T V^+\right)Y \qquad (4)$$

Where $f = rank(V) - rank(X)$. **This may also be derived from the results in Baksalary et al. (1990).**

Formula (2) will be called "simple" estimator and of course coincides with (1) when V=I. However , in the general case the simple estimator (2) need not even be unbiased since we do not necessarily have trace(MV)=f. On the other hand, we always have trace $= rank(MV) = f$ **, which ensures unbiasedness of (3). (see e.g. Theorem 5 in Marsaglia and Styan (1974) for rank(MV)=f, and note that trace** $[(MVM)^+ V] = rank[(MVM)^+ V]$ **since** $[(MVM)^+ V]$ **is idempotent Needless to say that the MINQUE is unbiased by definition.**

In the following section we investigate equality of MINQUE and simple estimator when X and V can be deficient in rank.

2. Equality of the estimators:

By using the above mentioned identity

$$(MVM)^+ = M(MVM)^+ = (MVM)^+ M,$$

It is easy to see that (1), which equals (3), and (2) coincide for all values $Y \in R(X:V)$ **if and only if**

$$Z^T V(MVM)^+ VZ = Z^T VMVZ \qquad \textbf{(5)}$$

for all $n \times 1$ **vectors z. This is known to be satisfied if and only if the matrix** $V(MVM)^+ V - VMV$ **is skew-symmetric. But since the matrix is also symmetric it must be zero. Hence (5) is equivalent to**

$$V(MVM)^+ V = VMV \qquad \textbf{(6)}$$

Let L denote the unique non-negative definite square root of V. Then (6) can be written as

$$LZZ^+ L = LZZ^+ L \qquad \textbf{(7)}$$

With $Z = LM$ **, by using** $Z^+ = (Z^T Z)^+ Z^T$ **and** $LZ = VM$ **. Since** $L^+ L = LL^+, L^+ LZZ^+ = ZZ^+ = ZZ^+ LL^+$ **and** $L^+ LZZ^T = ZZ^T = ZZ^T LL^+$**, it is easily seen that (7) is equivalent to**

$$ZZ^+ = ZZ^T \qquad \textbf{(8)}$$

Obviously (8) is satisfied if and only if ZZ^T **is idempotent, which reads**

$$LMVML = LML,$$

or equivalently

$$VMVMV = VMV \qquad \textbf{(9)}$$

Further, in the special case rank(X:V)=n we obtain rank(MV)=n-rank(X), cf. Theorem 5 in Marsaglia and Styan (1974). Since always n-rank(X)=rank(M) we have rank(MV)=rank(M), which means R(MV)=R(M). with K=VM, Eq. (9) reads

$$KMVMK^T = KMK^T \qquad \textbf{(10)}$$

Since we have $K^+KM = MV$ and $K^+KM = M$ when $R(K^T) = R(M)$,(10)is equivalent to MVM=M (11)
when rank(X:V)=n. Observe that (11) may also be obtained directly from the equality of (3) and (2), which holds for all $n \times 1$ vectors if and only if $(MVM)^+$=M. But since M= M^+, this gives (11). Observe on the other hand that (11) implies rank(X:V)=n in view of rank(X:V)=rank(X)+ rank(MVM).Hence VMVMV= VMV,
$\mathrm{rank}(X \;:\; V) = n \Leftrightarrow MVM = M$.
Our derivations may be comprised in the following
<u>Proposition(1)</u>

Under the general multivariate linear model $\{Y, X\beta, \sum \otimes V\}$,MINQUE for Σ Coincides with simple estimator for Σ for all $Y \in R(X : V)$ if and only if VMVMV= VMV, where M=I-X(X^TX) $^+$. Moreover $R(X : V)$ is the whole space together with the above coincidence if and only if MVM=M.
Note that when Y in the multivariate linear model is multivariate normally distributed then the condition VMVMV= VMV is necessary and sufficient for $\sum^{-1} y^T My$ to have a Wishart-distribution i.e W(k,δ), cf. Theorem 9.2.1 in Rao and Mitra (1971), in which case k=trace(MV)=f and δ=0. Hence the question of equality of MINQUE and simple estimator forΣ in the multivariate linear model is equivalent to the question of $\Sigma^{-1} Y^T MY$ having a Wishart distribution W(k,δ) in the multivariate linear model with normally distributed Y.Observe that in general we do not have k = trace(MV)= rank(MV)= f. However, since (10) is equivalent to Z^+=Z^T,i.e. LMVM=LM, we see that the condition VMVMV= VMV is equivalent to VMVM= VM, which means that VM and hence MV is idempotent. But for idempotent matrices rank and trace coincide. The former equivalence has also been observed by Bhimasankaram and Majumdar (1980)who trace it back to Mitra (1968).

By adapting a table from Rao and Mitra (1971,p.161) to the situation under model $\left\{Y, X\beta, \sum \otimes V\right\}$ with normally distribution Y one may obtain a general representation of V being necessary and sufficient for $(1/\sigma^2)Y^T MY$ to have $\chi^2(trac(M),0)$ -distribution . From this representation one immediately observes MVM= M, showing that rank(X:V)=n is implicitly comprised therein. Complementing the table, for non- singular however , Chikuse (1981)concludes that $\chi^2(trace(M),0)$ -ness of $(1/\sigma^2)Y^T MY$ is equivalent to equality of$Y^T MY$ and $Y^T(MVM)^+MY$. General characterizations of the class of all matrices satisfying the identity VMVMV= VMV(or VMVM= VM) may be derived from theorem 4.4 in Bhimasankaram and Majumdar(1980) or theorem 2 in Basksalary et al. (1980). A general non- negative definite solution to MVM= M with respect toV can also be obtained from Baksalary (1984), whose result is claimed to be advantageous over that derived by Khatri and Mitra(1976,Lemma2.1).

3.Relationship with known results

It is well known that one representation of the BLUE(Best Linear Unbiased Estimator)for Xβ is given by

$$X(X^T T^+ X)^+ X^T T^+ Y \qquad (12)$$

with $T=V+XX^T$, whereas the OLSE olse (Ordinary Least Square Estimator) for Xβ reads

$$XX^+Y \qquad (13)$$

Conditions for equality of (12) and (13) for all $Y \in R(X:V)$ are well known in the literature. One of them being $XX^+ V= V XX^+$,c f. Puntanen and Styan (1989). Trivially the latter condition is equivalent to the symmetry of the matrix MV.

Now taking the results of the previous section into account, we can state that coincidence of MINQUE and simple estimator for Σ for all $Y \in R(X:V)$ holds together with coincidence of BLUE and OLSE for Xβ for all $y \in R(X:V)$ if and only if MV is idempotent and symmetric, or in other words, MV is an orthogonal projector.

As mentioned before we have $R(MV)=R(M)$ when rank(X:V)=n.This means that under rank(X:V)=n the matrix MV is an orthogonal projector if and only MV= M. On the other hand , MV= M entails rank(X:V)=n. Hence we may state

<u>Proposition2:</u>

Under the multivariate linear model $\{Y, X\beta, \sum \otimes V\}$, Coincides of MINQUE and simple estimator forΣ for all $Y \in R(X:V)$ holds together with Coincidencs of BLUE and OLSE for Xβ for all $Y \in R(X:V)$ if and only if MV is an orthogonal projector, where $M=I-X(X^TX)^+X^T$. Moreover $R(X:V)$ is the whole space together with the above coincidences if and only if MV= M.

As mentioned above , the condition MV M = M as well as the condition MV= M entails rank(X:V)=n, which may equivalently be expressed as

rank(V)=n- rank(X)+dim $[R(X) \cap R(V)]$ (14)

cf. Marsagli and Styan (1974). Hence under each of these two conditions, V is non-singular if and only if rank(X)=dim $[R(X) \cap R(V)]$,which in turn is equivalent to $R(X) \subseteq R(V)$.

Eventually note that when MVis symmetric, i.e. M and V commute , then M and V can be simultaneously diagonalized by some orthogonal matrix, cf. Rao (1973,p.41).Hence we have V M= M if and only if there exists an orthogonal matrix U such that

$M = U\begin{bmatrix} I_S & 0 \\ 0 & 0 \end{bmatrix}U^T$ **and** $V = U\begin{bmatrix} I_S & 0 \\ 0 & 0 \end{bmatrix}U^T$

where s=n- rank(X) and D is a $(n-s)\times(n-s)$ non-negative definite diagonal matrix.

References:

[1] Baksalary, J.K,1984.Nonnegative definite and positive definite solutions to the matrix equation AXA^*=B.Linear and Multilinear Algebra16,133-139.

[2] Baksalary, J.K,Hauk, J.,Kala,R.,1980. Nonnegative definite and positive definite solutions to some matrix equations occuring in distribution theory of quadratic forms.Sankhya¯,ser. A42,283-291.

[3] Baksalary, J.K.,Puntanen,S.,Styan, G.P.H.,1990.A property of the dispersion matrix of the best linear unbiased estimator in the general Gauss-Markov model. Sankhya¯,ser.A 52,279-296.

[4] Bhimasankaram,p.,Majumder,D.,1980.Hermitian and nonnegative definite solutions of some matrix equations connected with distribution of quadratic forms. Sankhya¯,ser.A 42,272-282.

[5] Chikuse, Y.,1981.Representations of the covariance matrix for the robustness in the Gauss-Markov model. Comm.Statist. Theory Methods 10,1997-204.

[6] Groβ , J.(1997).A note on equality of MINQUE and simple estimator in the general Gauss-Markov model.Statistics probability letters 35,335-339.

[7] Khatri,C.G.,Mitra, S.K., 1976.Hermitian and Nonnegative definite solutions of linear matrix equation . SIAM J.Appl. Math.31,579-585.

[8] Marsaglia, G.,Styan ,G.P.H., 1974.Equalities and inequalities for ranks of matrices Linear and Multilinear Algebra 2,269-292

[9] Mitra ,S.K.,1968.On a generalized inverse of a matrix and applications.Sankhya¯ ser.A 30,107-114.

[10] Puntanen,S.,Styan,G.P.H., 1989.The equality of the ordinary least squares estimator and the best linear unbiased estimator.Amer.Statist.43,153-164.

[11] Rao,C.R,1973.Linear Statistical Inference and Its A applications.2nd ed .Wiley , New York.

[12] Rao,C.R,1974.Projectors, generalized Inference and the BLUEs' J.Ray.Statist Soc.Ser B36,442-448.

[13] Rao,C.R.,Mitra, s.k.,1971.Generalized Inverse of Matrices and it Applications.Wiley,NewYork.