Fuzzy Quotient Rings and

Fuzzy Isomorphism Theorem

Areeg Tawfeek Hameed

College of Education for women , Dept.of Mathematics, University of Kufa

ABSTRACT

The concept of fuzzy sets was introduced by (Zadeh L.A.) in 1965 and in 1982 (Liu W.J.) introduced the concept of fuzzy rings, fuzzy ideals and Since that time many papers were introduced in different mathematical scopes of theoretical and practical applications.

A characterization of rings which are both homomorphism and isomorphism is obtained. We are prove to the fuzzy rings whose fuzzy ideals assume to obtain it. Kernal fuzzy ring have been defined and conditions have been obtained under which we need

حلقات القسمة الضبابية

ونظرية التماثل الضبابية

أريج توفيق حميد (مدرس)

جامعة الكوفة ،كلية التربية للبنات ، قسم الرياضيات

المستخلص

عام و١٩٨٢ قدم ليو مفهوم الحلقات الضبابية عام ١٩٦٥ (fuzzy sets)مفهوم المجموعات الضبابية بعد أن قدم زاده(fuzzy ring) و المثالي الضبابي (fuzzy ideal) ومنذ ذلك الحين أُجريت العديد من البحوث في مختلف المجالات الرياضية النظرية والتطبيقية حول هذا الموضوع.

لقد هدف البحث الى دراسة الحلقة isomorphism, homomorphism وكيفية الحصول عليها بإستخدام الحلقة الضبابية والمثالي الضبابي.

اما النواة الضبابية (kernel fuzzy ring) والتي عرفت وذكرت شروطها التي يمكن استخدامها للحصول على ما نريد من homomorphism.

SECTION ONE **BASIC CONCEPTS**

This section contains some definitions and properties of fuzzy subset, fuzzy ring, fuzzy ideal which we will be used in the next chapter.

1.1 PRELIMINARY CONCEPTS

Let (R,+,·) be a ring. A fuzzy subset of R is a function from R into [0,1], in my paper , (R,+,·) be a commutative ring with identity, ([1],[2]) .

Let A and B be fuzzy subset of R. We write A ⊆ B if A(x) ≤ B(x) for all $x \in$ R. If A ⊆ B and there exists x ∈ R such that A(x) < B(x), then we write A ⊂ B and we say that A is a proper fuzzy subset of B, (1). Note that A = B if and only if A(x) = B(x), for all $x \in$ R, ([1],[3]).

For each t ∈ [0,1], the set $A_t = \{ x \in R \mid A(x) \geq t \}$ is called a level subset of R and the set $A_* = \{ x \in R \mid A(x) = A(0) \}$, ([1],[4]).

Let A and B be a fuzzy subsets of R, the product A∘B define by $A \circ B\ (x) = \sup \{A(y), B(z)\} \mid x = y \cdot z, y, z \in R$, for all $x \in$ R, [2] .

We let ϕ denote $\phi(x) = 0$ for all $x \in$ R, the empty fuzzy subset of R, ([3],[5]) .

Let A be a non empty fuzzy subset of R, A is called a fuzzy subgroup of R if for all $x, y \in R$, $A(x + y) \geq \min \{A(x), A(y)\}$ and $A(x) = A(-x)$, [5] .

A is a non empty fuzzy subset of R, A is called a fuzzy ring of R if and only if for all $x, y \in R$, then $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \min \{A(x), A(y)\}$, ([3],[5]) .

A non empty fuzzy subset A of R is called a fuzzy ideal of R if and only if for all $x, y \in R$, then $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \max \{A(x), A(y)\}$, ([1],[5]) .

It is clear that every fuzzy ideal of R is a fuzzy ring of R, but the converse is not true.

Let X be a fuzzy ring of R and A be a fuzzy ideal of R such that A ⊆ X. Then A is a fuzzy ideal of the fuzzy ring X, [6] .

If A is a fuzzy ideal of R, then A_* and A_t are ideals of R, ([1],[4]) .

1.2 HOMOMORPHISM FUNCTIONS

In this section, we shall give some concepts about homomorphism and isomorphism functions. Also, we give some properties of these concepts.

DEFINITION 1.2.1 [7] :

A mappimg f from a ring R to a ring R′ is called ring homomorphism if it satisfies the following properties for all $a, b \in R$:

1. **$f(a + b) = f(a) +' f(b)$**
2. **$f(a \cdot b) = f(a) \cdot' f(b)$**

PROPOSITION 1.2.2 :

If f : R $\rightarrow$ R$'$, g : R$'$ $\rightarrow$ R$''$ are homomorphism between the fuzzy rings A, B and C, then f $\circ$ g is a homomorphism between A and C.

PROOF: **A:R $\rightarrow$ [0,1], B : R$'$ $\rightarrow$ [0,1] and C:R$''$ $\rightarrow$ [0,1] are fuzzy rings, f and g are homomorphism such that** R $\xrightarrow{g\circ f}$ R$''$

R$'$

To prove that f $\circ$ g is a homomorphism, let a, b $\in$ R, then

1. **g $\circ$ f (a + b) = g (f (a + b)) = g (f (a) +$'$ f(b)) = g (f (a)) +$''$ g (f(b)) = g $\circ$ f (a) +$''$ g $\circ$ f (b)**
2. **g $\circ$ f ($a \cdot b$) = g (f ($a \cdot b$)) = g (f (a) $\cdot'$ f(b)) = g (f (a)) $\cdot''$ g (f(b)) = g $\circ$ f (a) $\cdot''$ g $\circ$ f (b)**

Hence g $\circ$ f is a homomorphism between A and C.

Note that : If f and g are isomorphism, then g $\circ$ f is an isomorphism since f and g are 1 – 1 and onto implies that g $\circ$ f is 1 – 1 and onto.

PROPOSITION 1.2.3 :

Let X : R $\rightarrow$ [0,1], Y : R$'$ $\rightarrow$ [0,1] are fuzzy rings f : R $\rightarrow$ R$'$ be homomorphism between them and A : R $\rightarrow$ [0,1] a fuzzy ideal of X, B : R$'$ $\rightarrow$ [0,1] a fuzzy ideal of Y, then

1. **f (A) is a fuzzy ideal of Y.**
2. **f^{-1}(B) is a fuzzy ideal of X.**

PROOF: **(1) Let $x, y \in R'$ such that f (a) = x, f (b) = y, where $a, b \in R$**

$$\begin{aligned} f(A)(x-y) &= \sup\{\min\{A(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \\ &\geq \sup\{\min\{X(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \\ &= \sup\{\min\{Y(x), f(A)(y)\} \mid x = f(a), y = f(b)\} \\ &\geq \sup\{\min\{f(A)(x), f(A)(y)\} \mid x = f(a), y = f(b)\} \ (9) \\ &\geq \min\{f(A)(x), f(A)(y)\} \end{aligned}$$

$$\begin{aligned} f(A)(x \cdot y) &= \sup\{\min\{A(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \\ &\geq \sup\{\min\{X(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \\ &= \sup\{\min\{Y(x), f(A)(y)\} \mid x = f(a), y = f(b)\} \geq \min\{Y(x), f(A)(y)\} \end{aligned}$$

Similerty $f(A)(x \cdot y) \geq \min\{Y(y), f(A)(x)\}$

Hence $f(A)(x \cdot y) \geq \max\{f(A)(x), f(A)(y)\}$. **Therefore f (A) is a fuzzy ideal of Y.**

(2) Let $a, b \in R$ such that $f^{-1}(x) = a$, $f^{-1}(y) = b$, where $x, y \in R'$

$$\begin{aligned} f^{-1}(B)(a-b) &= \sup\{\min\{B(x), B(y)\} \mid x = f(a), y = f(b)\} \\ &\geq \sup\{\min\{Y(x), B(y)\} \mid x = f(a), y = f(b)\} \\ &= \sup\{\min\{X(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \\ &\geq \sup\{\min\{f^{-1}(B)(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \ (9) \\ &\geq \min\{f^{-1}(B)(a), f^{-1}(B)(b)\} \end{aligned}$$

$$\begin{aligned} f^{-1}(B)(a \cdot b) &= \sup\{\min\{B(x), B(y)\} \mid x = f(a), y = f(b)\} \\ &\geq \sup\{\min\{Y(x), B(y)\} \mid x = f(a), y = f(b)\} \\ &= \sup\{\min\{X(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\} \geq \min\{X(a), f^{-1}(B)(b)\} \end{aligned}$$

Similerty $f^{-1}(B)(a \cdot b) \geq \min\{X(b), f^{-1}(B)(a)\}$

Hence $f^{-1}(B)(a \cdot b) \geq \max\{f^{-1}(B)(a), f^{-1}(B)(b)\}$. Therefore $f^{-1}(B)$ is a fuzzy ideal of X.

1.3 QUOTIENT FUZZY RING

In this section, we shall give definition and some properties of quotient fuzzy ring.

DEFINITION 1.3.1 [6] :

Let X be a fuzzy ring of R and A be a fuzzy ideal in X. Define $X / A : R / A_* \to [0,1]$ such that

$$X/A(a+A_*)=\begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a+b)\} & \text{if } a \notin A_*, b \in A_* \end{cases}$$

For all $a + A_* \in R / A_*$, X / A is called a quotient fuzzy ring of X by A.
Note that X /A is a fuzzy ring of R/A_* by remark (3. 3 .1. 2) in[6] .

DEFINITION 1.3.2 [6] :

1. X is a fuzzy ring of R and A is a fuzzy ideal in X, then
$X / A (0 + A_*) = 1$, $[0 + A_* = A_*]$.

2 . If X is a fuzzy ring of R and A and B are fuzzy ideals in X such that $A \subseteq B$. Then
$B /A : R /A_* \to [0,1]$ such that

$$B/A(a+A_*)=\begin{cases} 1 & \text{if } a \in A_* \\ \sup\{B(a+b)\} & \text{if } a \notin A_*, b \in A_* \end{cases}$$

is a fuzzy ideal in X /A.

3. Let X be a fuzzy ring of R such that X(0) = 1 and A be a fuzzy ideal in X. Define
$K : R /A_* \to [0,1]$ such that $K(a + A_*) = \sup\{ X(a + b) \mid b \in A_*\}$. Then X /A = K.

PROPOSITION 1.3.3 :

Let I be an ideal of a ring R, A be a fuzzy ideal of R. Then the fuzzy subset B of R /A define by
$B(a + I) = \sup\{ A(a + x) \mid x \in I\}$ is a fuzzy ideal of the quotient ring in R /I.

PROOF: **Define $B : R / I \to [0,1]$ such that $B(a + I) = \sup\{ A(a + x) \mid x \in I\}$**
since $B(0 + I) = A(0)$, B is non empty fuzzy subset of R / I
Let $a + I, b + I \in R / I$, $a + I = b + I$ then $b = a + c$ for some $c \in I$
$B(a + I)=\sup\{ A(b + x) \mid x \in I\}=\sup\{ A(a + c + x) \mid x \in I\}=\sup\{ A(a + y) \mid y = x + c \in I\}=B(b + I)$
Hence B is a well – defined.

Now, we must prove B is a fuzzy ideal of R / I

1. $B((a + I) - (b + I)) = B((a - b)+ I) = \sup\{ A((a - b) + x) \mid x \in I\}$
$= \sup\{ A((a - b) + (v - w)) \mid x = v - w \in I\}$
$= \sup\{ A((a + v) - (b + w)) \mid w, v \in I\}$
$\geq \sup\{ \min\{ A(a + v), A(b + w)\}\}$
$\geq \min\{ \sup\{ A(a + v) \mid v \in I\}, \sup\{A(b + w)\ w \in I \}\}$
$= \min\{ B(a + I), B(b + I)\}$.

2. $B((a + I)\cdot(b + I)) = B((a\cdot b) + I) = \sup\{A((a\cdot b) + x) \mid x \in I\}$
$\geq \sup\{A(a\cdot b + a\cdot x)) \mid x \in I\},\ a\cdot x \in I$
$= \sup\{A(a(b + x)) \mid x \in I\}$
$\geq \sup\{A(b + x) \mid x \in I\} = B(b + I)$

Similerty, $B((a + I)\cdot(b + I)) \geq = B(b + I)$

Implies that $B((a + I)\cdot(b + I)) \geq = \max\{B(a + I), B(b + I)\}$**. Hence B is a fuzzy ideal of R / I.**

PROPOSITION 1.3.4:

Let I be an ideal of a ring R. Then there is a one – to – one correspondence between the fuzzy

ideals A of R such that $A(x) = A(0)$ **for all** $x \in I$ **and the set of all fuzzy ideals B of R / I.**

PROOF: **Since A is a fuzzy ideal of R, by proposition (1.3.3), we found B such that**

$B(a + I) = \sup\{A(a + x) \mid x \in I\}$ **is a fuzzy ideal of R / I.**

but $A(x) = A(0)$ **for all** $x \in I$, $A(a + x) \geq \min\{A(a), A(x)\} = \min\{A(a), A(0)\} = A(a)$ **and**

$A(a) = A(a + x - x)$ **where** $x \in I$

$\geq \min\{A(a + x), A(+x)\} = A(a + x)$

Hence $A(a + x) = A(a)$ **for each** $x \in I$**. Thus** $B(a + I) = A(a)$**.**

Hence the correspondence $A \to B$ **is injective. Let B be any fuzzy ideal of R / I. Define A on R by** $A(a) = B(a + I)$ **implies that A is a fuzzy ideal of R and for any** $x \in I$, $A(x) = B(x + I) = B(I)$ **then** $A(x) = A(0)$ **if** $x \in I$**.**

PROPOSITION 1.3.5 :

Let $X : R \to [0,1]$ **be a fuzzy ring and A be a fuzzy ideal in X. Then** $f : R \to R / A_*$ **define by**

$f(a) = a + A_*$ **is homomorphism between X and X / A.**

PROOF: $f : R \to R / A_*$ **such that** $f(a) = a + A_*$**. Let** $a, b \in R$

$f(a + b) = (a + b) + A_* = (a + A_*) \oplus (b + A_*) = f(a) \oplus f(b)$

$f(a\cdot b) = (a\cdot b) + A_* = (a + A_*) \otimes (b + A_*) = f(a) \otimes f(b)$

Hence f is a homomorphism between X and X / A.

Note that : since for any $a + A_* \in R / A_*$, $a + A_* = f(a)$**, then f is onto.**

PROPOSITION 1.3.6:

Let X be a fuzzy ring of R and A be a fuzzy ideal of X. Let I be an ideal of R maximal among

those contained in $A_* \cap X_*$**. Then there exists a bijection between the fuzzy ideals of X /A and the fuzzy ideals A of X such that** $I \subseteq A_*$**.**

PROOF: **Let** $B = \{A' \mid A'$ **fuzzy ideal of X,** $I \subseteq A_*\}$, $C = \{A'' \mid A''$ **fuzzy ideal of X /A}**

$f : R \to R / I$ **such that** $f(a) = a + I$, $a \in R$**. Note that f is a homomorphism by proposition (1.3.3). Let** $h : B \to C$ **such that** $h(A') = f(A')$, $h(A')(a) = a + I$

$k : C \to B$ **such that** $k(A'') = f^{-1}(A'')$, $k(A'')(a + I) = a$

Now, to prove that $k \circ h = i_B$, $h \circ k = i_C$

$(k \circ h)(A')(x) = k(h(A')(x)) \qquad x \in R$

$= f^{-1}(f(A')(x)) = \sup\{A'(y) \mid y \in f^{-1}(f(x))\} = \sup\{A'(x+z)) \mid z \in I\} = A'(x)$

Therefore $k \circ h = i_B$

$(h \circ k)(A'')(y) = h(k(A'')(y)) \qquad y \in R / I$

$= f(f^{-1}(A'')(y)) = \sup\{A''(x+z)) \mid z \in I, x \in f(f^{-1}(y)\} = \sup\{A''(y) \mid y \in f(f^{-1}(x))\} = A''(y)$

Hence $h \circ k = i_C$

Now, we must prove that k, h are bijection

1 – 1 : $h(x) = h(y)$, then $x + I = y + I$ implies that $x - y \in I$ and $x = y$ h is 1 – 1

$k(x + I) = k(y + I)$, then $x = y$ k is 1 – 1

onto : for all $x \in R$, there exists $y \in R$ such that $x = y + z$, $z \in I$, then $x \in y + I$. Hence k is onto

for all $x \in R$, there exists $y \in R$ such that $y \in x + I$ implies that $z \in I$, $y = x + z$. Hence h is onto

SECTION TWO

FUZY KERNEL

Now, we shall introduce the concept of a fuzzy kernel of a function and some properties of it.

DEFINITION 2.1.1 ([8],[9]) :

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy rings $f : R \to R'$ be homomorphism between them.

We define the fuzzy kernel of f, $\ker f_{zz}f : R \to [0,1]$ by

$$\ker f_{zz}\, f(x) = \begin{cases} X(0) & x \in \ker f \\ 0 & x \notin \ker f \end{cases}$$

PROPOSITION 2.1.2 :

$\ker f_{zz}f : R \to [0,1]$ is a fuzzy ideal of X.

PROOF: Since $\ker f_{zz}f(0) = X(0)$, then $\ker f_{zz}f \equiv \phi$, let $x, y \in R$, then

1. If $x - y \in \ker f$, $x \cdot y \in \ker f$, hence
   $\ker f_{zz}f(x - y) = X(0) \geq \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$
   $\ker f_{zz}f(x \cdot y) = X(0) \geq \max\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$
2. If $x - y \notin \ker f$, $x \cdot y \notin \ker f$, then $x \notin \ker f$, $y \notin \ker f$
   $\ker f_{zz}f(x - y) = 0 = \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$
   $\ker f_{zz}f(x \cdot y) = 0 = \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$
3. If $x \notin \ker f$, then $\ker f_{zz}f(x) = 0 \leq X(x)$ and
   If $x \in \ker f$, then $\ker f_{zz}f(x) = X(0) = X(x) = Y(0) = Y(x)$.
   $x - y \notin \ker f$, hence $\ker f_{zz}f(x - y) = 0 = \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$
   $x \cdot y \in \ker f$, then $\ker f_{zz}f(x \cdot y) = X(0) \geq \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$

Therefore $\ker f_{zz}f(x - y) \geq \min\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$

$\ker f_{zz}f(x \cdot y) \geq \max\{\ker f_{zz}f(x), \ker f_{zz}f(y)\}$

Hence $\ker f_{zz}f$ is a fuzzy ideal of X.

DEFINITION 2.1.3 ([8],[9]) :

A fuzzy ideal A of a fuzzy ring X of R is called normal if $A(x^{-1} \cdot y \cdot x) \geq \min\{X(x), A(y)\}$ for all $x, y \in R$.

PROPOSITION 2.1.4 [8] :

Let A be a fuzzy ideal of a fuzzy ring X of R. If A is a normal fuzzy ideal, $A(0) \geq t$, then A_t is a normal ideal of X_t.

PROPOSITION 2.1.5 :

$\ker f_{zz}f : R \to [0,1]$ is a normal fuzzy ideal of X.

PROOF: By proposition (2.1.2) $\ker f_{zz}f$ is fuzzy ideal of X. We must prove that $\ker f_{zz}f$ is normal (i.e let $a, b \in R$, $\ker f_{zz}f(a^{-1} \cdot b \cdot a) \geq \min\{\ker f_{zz}f(b), X(a)\}$).

1. If $a \in \ker f$, then $a^{-1} \cdot b \cdot a \in \ker f$ and $\ker f_{zz}f(a^{-1} \cdot b \cdot a) = X(0) \geq \min\{\ker f_{zz}f(b), X(a)\}$

2. If $a \notin \ker f$, then either $a^{-1} \cdot b \cdot a \in \ker f$ or $a^{-1} \cdot b \cdot a \notin \ker f$

$a^{-1} \cdot b \cdot a \in \ker f$ implies that $\ker f_{zz}f(a^{-1} \cdot b \cdot a) = X(0) \geq \min\{\ker f_{zz}f(b), X(a)\}$

$a^{-1} \cdot b \cdot a \notin \ker f$ implies that $\ker f_{zz}f(a^{-1} \cdot b \cdot a) = 0 = \min\{\ker f_{zz}f(b), X(a)\}$

Hence $\ker f_{zz}f$ is a normal fuzzy ideal of X.

PROPOSITION 2.1.5:

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy rings $f : R \to R'$ be homomorphism between them and $A : R \to [0,1]$ is a fuzzy ideal of X and $B : R' \to [0,1]$ is a fuzzy ideal of Y, then

1. f (A) is normal fuzzy ideal of Y, if A is normal and f is surjective.

2. $f^{-1}(B)$ is normal fuzzy ideal of X, if B is normal.

PROOF: (1) f (A) is fuzzy ideal of Y by proposition (1.2.4). Now, we must prove that f (A) is normal. Let $x, y \in R'$ such that $f(a) = x$, $f(b) = y$, where $a, b \in R$

$f(A)(x^{-1} \cdot y \cdot x) = f(A)(f(a^{-1}) \cdot f(b) \cdot f(a)) = f[A(a^{-1} \cdot b \cdot a)] \geq f[\min\{A(b), X(a)\}]$

$= \sup\{\min\{\min\{A(b), X(a)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}\}$

$\geq \min\{\sup\{\min\{A(b), X(a)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}\} \geq \min\{f(A)(y), Y(x)\}$

Hence f (A) is a normal fuzzy ideal of Y.

(2) $f^{-1}(B)$ is fuzzy ideal of Y by proposition (1.2.4)

To prove $f^{-1}(B)$ is normal. Let $a, b \in R$ such that $f(a) = x$, $f(b) = y$, where $x, y \in R'$

$f^{-1}(B)(a^{-1} \cdot b \cdot a) = f^{-1}(B)(f^{-1}(x^{-1}) \cdot f^{-1}(y) \cdot f^{-1}(x)) = f^{-1}[B(x^{-1} \cdot y \cdot x)] \geq f[\min\{B(y), Y(x)\}]$

$= \sup\{\min\{\min\{B(y), Y(x)\} \mid x = f(a), y = f(b)\}\}$

$\geq \min\{\sup\{\min\{B(y), Y(x)\} \mid f(a) = x, f(b) = y\}\} \geq \min\{f^{-1}(B)(b), X(a)\}$

Hence $f^{-1}(B)$ is a normal fuzzy ideal of X.

FUNDAMENTAL THEOREMS

THEOREM 2.2.1 (First Fuzzy Isomorphism Theorem For Fuzzy Rings)

Let X and Y are fuzzy rings and f be onto homomorphism between them. Then X / ker $f_{zz}f \approx f(X)$.

PROOF:Define g : R / ker f → f (R) such that g (*a* + ker f) = f (*a*) for each *a* + ker f ∈ R / ker f By
definition, g is a non empty function of R / ker f since g (0 + ker f) = f (0)
Let *a* + ker f, *b* + ker f ∈R/ker f, *a* + ker f = *b* + ker f implies that *a* – *b*∈ ker f, therefore f (*a* – *b*)=0′ and f is homomorphism, then f (*a*) – f(*b*) = 0′ implies f (*a*) = f(*b*). Thus g (*a* + ker f) = g (*b* + ker f). Hence g is well – define. Now, we must prove g is an isomorphism
First, if g (*a* + ker f) = g (*b* + ker f), then f (*a*) = f(*b*) and f (*a*) – f(*b*) = 0′ implies that
f (*a* – *b*) = 0′. Thus *a* – *b* ∈ ker f therefore *a* + ker f = *b* + ker f g is one – to – one
Second, for any *b* ∈ f (R) there exists *a* ∈ R such that f (*a*) = *b* since f is onto then f (*a*) = g (*a* + ker f) = *b* g is onto
Finally, Let *a* + ker f, *b* + ker f ∈ R / ker f
g [(*a* + ker f) ⊕ (*b* + ker f)]=g [(*a* + *b*) + ker f]=f (*a* +*b*) = f (*a*) +′f (*b*)=g (*a* + ker f) +′ g (*b* + ker f)
g [(*a* + ker f) ⊗ (*b* + ker f)] = g [(*a* · *b*) + ker f] = f (*a* ·*b*) = f (*a*) ·′f (*b*)= g (*a* + ker f) ·′ g (*b* + ker f)
Moreover, X / ker $f_{zz}f$ (*a* + ker f)=sup{X / ker $f_{zz}f$ (*a* + ker f) | *a* + ker f ∈ ker $f_{zz}f$ }=X(*a*)=f(X)(f(*a*)) where f (X) (*a* + ker f) = sup{X(*b*) | *b* ∈ f^{-1}(f(*a*))} = Y(f^{-1}(*a*)) =X(*a*). HenceX/ ker $f_{zz}f \approx f(X)$.

THEOREM 2.2.1(Second Fuzzy Isomorphism Theorem For Fuzzy Rings)

Let A and B be fuzzy ideals of a ring X with A ⊆ B such that $A_* \subseteq B_*$ (i.e B(x) = B(0) whenever A(x) = A(0)). Then (X / A) / (B / A) ≈ (X / B).

PROOF: B / A is a normal fuzzy ideal of X / A by proposition (1.3.6), by proposition (2.1.5). Let B / A(*x* + I_1) = B(*x*) where I_1 is the normal ideal of X maximal among those contained in $A_* \cap X_*$, because B / A (*x* + I_1) = max {B(*y*) | *y* + I_1 = *x* + I_1} and since
$y^{-1} \cdot x \in A_*$, we have $y^{-1} \cdot x \in B_*$ and therefore B(*y*) = B(*x*).
$(X / A)_* = X / I_1$ and $(B / A)_* = B_* / I_1$
So, the normal ideal of X / I_1 maximal among those contained in $(X / A)_* \cap (B / A)_*$ may be written as I_2 / I_1, where I_2 is the normal ideal of X maximal among those contained in $X_* \cap B_*$.
It is well known that f : $(X / I_1) / (I_2 / I_1) \to X / I_2$ such that
f (*x* + I_1 + (I_2 / I_1)) = *x* + I_2 is a ring isomorphism.
Moreover, (X / A) / (B / A) (*x* + I_1 + (I_2 / I_1)) = X / A (*x* + I_1) = X (*x*) = X / B (*x* + I_2)
Hence (X / A) / (B / A) ≈ (X / B).

REFERENCES

1) **Fuzzy Sets, Zadeh L.A., 1965, Information and Control, vol.8, pp.338 – 353.**
2) **Fuzzy Invariant Subgroups and Fuzzy Ideal, Liu W.J., 1982, Fuzzy Sets and Systems, vol.8, pp.133 – 139.**

3) Factorization of Fuzzy Ideals in Dedekind Domains, Lee K.H. and Mordeson J.N., 1997, the journal of fuzzy mathematics, vol.5, No.3, pp.741 – 745.
4) Fuzzy Principal Ideals and Simple Field Extensions, AL- Khamees Y. and Mordeson J.N., 1998, Fuzzy Sets and Systems, vol.96, pp.247 – 253.
5) On Fuzzy Ideals of a Ring I, Mukherjee T.K. and Sen M.K., 1987, Fuzzy Sets and Systems, vol.21, pp. 99 – 104.
6) On Almost Quasi – Frobenius Fuzzy Rings, Hameed A.T., 2000, M.Sc.Thesis, University of Baghdad, College of Education Ibn-AL-Haitham.
7) A First Course in Rings and Ideals, David M., 1970, Addison – Wesley Dublishing Company, London Don Mils, Ontario, University of New Hampshire.
8) Fuzzy Subgroup of Fuzzy Groups and Fuzzy Ideals of Fuzzy Rings, Martines L., 1995, the journal of fuzzy mathematics, vol.3, No.4, pp.833 – 849.
9) Fuzzy Modules Over Fuzzy Rings In Connection With Fuzzy Ideals Of Fuzzy Rings, Martinez L., 1996, the journal of fuzzy mathematics, vol.4, No.4, pp.843 – 857.