On δ-open sets in bitopological space

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Abstract: In this paper we define the concept of δ-open sets in bitopological space and we use them in order to study some types of continuous functions between bitopological spaces

1. Introduction:

The concept of bitopological space was introduced by Keliy [1]. The definition of δ-open and weakly δ-open sets in (X,τ) were introduced by Velicko in 1, Mayada Ghassab 2003 [2] respectively.

In this paper we define δ-open and weakly δ-open sets in bitopological space .If (X,τ,τ') is a bitopological space and $A \subseteq X$, then A is δ-open iff for any $a \in A$ there is a regular open set C such that $a \in G \subseteq A$. The complement of δ-open set is called δ-closed set. In 2003, Mayada G. Mohammed introduced and studied a new of class open sets, called weakly δ-open sets. This class of open sets is defined in (X,τ). In this paper we study δ-open and weakly δ-open sets in (X,τ,τ') and we introduce new types of continuous and functions between bitopological space namely δ-continuous function, δ-open function ,w-δ-open. Almost w-δ-open, always open function, we characterize these classes of function, study some of their basic properties and investigate their relationships to continuous functions and to other types of functions between bitopological spaces.

Definition 2): Let (X,τ,τ') be bitopological space and $A \subseteq X$, then A is called regular open set if $A = int\tau^c|\tau' A$.

A set F is called regular closed set if $F = cl\tau int\tau F$. the family of all regular open (resp. regular closed) sets of a space X is denoted by RO (X) (resp. RF (X)).

Definition (1.2): Let (X,τ,τ') be a topological space and $w \subseteq X$, then W is called δ-open set iff for any $x \in W$ there exists a regular open G such that $x \in G \subseteq w$. or equivalently: W is δ-open set iff W is union of regular open sets .the complement of δ-open set is called δ-closed set.

Definition (1.3): - A set W of a space X is called open set (resp. F closed set) in (X,τ,τ') if intτintτ´ W=W (resp. CL τclτ´ F=F). i.e W is open in (X,τ,τ') iff W is open in (X, τ) and open in (X, τ´) (resp. F is closed set in (X,τ,τ') if F is closed set in (X,τ) and closed set in (X, τ´).

Remarks (1.4):

1) Every regular open set is δ-open set.

2) Every regular closed set is δ-closed set

The converse of (l) and (2) is not true in general: the following example explains that:

Example (1.5): - let X ={1, 2, 3}

τ={ φ ,X,{1},{2},{1,2}}

τ´={ φ ,X,{1},{2,3}}

Let A={l,2}

We note that A is δ-open set but it is not regular open set. Also, we note that {1,3} is closed set but it is not δ-closed set.

Remarks (1.6); -

1) The concept of open and δ-open set are independent

2) The concept of open and regular open are independent.

The following example shows that: -

Example: -letX={a,b,c}; τ ={ φ , X, {a}, {b}, {a,b}}, τ´={ φ ,X,{a}}

It is clear that {a} is open set but it is not regular and it is not δ-open set.

In example (1.5), we note that {1,2} is δ-open set but it is not open set.

3) The union of two regular open sets need not be regular open set .in example (1.5), {1}, {2} are regular open sets but {١,٢}is not regular open set.

Theorem (1.7): - the union of δ-open sets is δ-open set.

Proof: -let {Ai} is a family of δ-open sets

Let a $\in$ u Ai then there is Aj such that a $\in$ A j, but Aj isδ-open set

then there is a regular open set G such that a $\in$ G $\subset$ Aj, thus

a $\in$ G $\subset$ Aj; therefore, $\cup$ Ai is δ -open set.

Lemma (l,8):

The intersection of a finite number of regular open sets is regular open set.

Theorem (1.9):

The intersection of a finite number of δ -open sets is δ -open set.

Proof: -let {Ai}Vi be a family of δ -open sets

Let a $\in \cap_i A_i$ then a $\in$ A for any I=l,2...n

Since Ai is δ -open set then there is a regular open set G_i .such that a $\in$ Gi $\subseteq$ Aj, thus a $\in \cap$ Gi $\subseteq \cap_i A_i$ but $\cap_i A_i$ is δ -open set

Definition (1.11): -

a point A is a space X is called δ -interior(resp. δ -boundary, δ -limit) of A c X iff there is a regular open neighbourhood L ofx in X such that x $\in$ G $\subset$ A (resp.iff for any δ -open neighbourhood U of x in X then U $\cap$ A≠Ø and U $\cap$ (X-A) ≠Ø،iff for any δ -open neighbourhood)U of x in X then U-{x} $\cap$ A ≠Ø The set of all δ -interior point of A is said to be δ -interior set of A and it is denoted by int s A (resp. the set of all δ -boundary points of A is said to be δ -boundary set and it is denoted by δ - ∂ A , The set of all δ -limit points of A is said to be δ -limit A and it is denoted by δ -A

Definition (1.12) :
The intersection of all δ-closed sets which are containing W is called δ-closure of A and it is denoted by clδ A،
Proposition (1.13):
Let A ,B be two subsets of X and A $\subseteq$B ,then
1. intδA is δ-open set.
2. intδA$\subseteq$A
3. intδA$\subseteq$intδB.
4. δ-A'$\subseteq$ δ-B
5. clδA$\subseteq$ clδB
6. .intδA=Extδ (X-A)
7. δ-∂A= δ-∂ (X-A)
Theorem (1.14):
Let (X, T,T) be a bi-topological space , let A,B be two subsets ofX, then :
1. intδ A$\cup$ intδ B$\subseteq$ intδ (A$\cup$B).
2. intδ (A $\cap$B)= intδA$\cap$ intδB
3. δ-(A $\cup$B)' = δ- A $\cup$ δ-B'
4. δ-(A$\cap$B)' $\subseteq$ δ-A$\cap$ δB

2) Some types of continuous and open functions between bi-topological spaces:

Definition (2.1) : A function f: (X, τ, τ ')$\rightarrow$(Y,σ ,σ ') is said to be continuous function (resp.δ-continuous)if f $^{-1}$(U) is open set (resp. δ-open), for every open set U (resp. δ-open) set in Y .
Remark (2.2):,
The concept of continuity and δ- continuity are independent
.The following two examples explain that:.
Example: Let X=Y={1,2,3}
τ= {ϕ, X, {3}, {2, 3}}
σ= {ϕ, Y, {l},{2,3}}
σ'={ϕ,Y,{2,3}}
Let f: (X,τ,τ′)→ (Y, σ, σ')be the identity function ,we note that f is continuous function but it is not δ-continuous function
Example:
LetX={l,2,3}=Y
τ ={ϕ , X ,{1} ,{ 1,2} }; τ = {ϕ,X ,{1},{ 2} ,{1,2}}
σ={ϕ,Y,{l},{2,3}}; σ'={ϕ,Y,{2,3}}
let f: (X, τ, τ ') → (Y, σ,σ') be the identity function , f is δ-continuous function
Theorem (2.3): let f: (X, τ, τ) → (Y, σ,σ') be the function then the following statements function
.١f is δ-continuous function
.٢for any δ-closed set F in Y then F'(f) is δ-closed set in X
.٣for any A$\subseteq$X then f(clsδ A) $\subseteq$clδ (f[A))
.٤for any B$\subseteq$Y then clδ (f^{-1} (B)) $\subseteq$f^{-1} (clδ B).
proof: (1) → (2)

Let f: (X, τ τ) → (Y,(σ ,(σ ') be a δ -continuous function
Let f be δ -closed set F in Y then X-F is δ -open set in Y
But f is δ -continuous function so f^{-1} (X-F) is δ -open set in X Thus X- f' (X-F) is δ -closed set in X.
Since X- f' (X-F)= f1 (X); therefore, f1 (F) is δ -closed set in X.
Proof: let A $\subseteq$ X
Since f(A) $\subseteq$ cl δ (f(A)) then f^{-1} (f(A))) $\subseteq$ f^{-1} (cl δ (f(A)))
Since A $\subseteq$ f^{-1}(f(A))so cl δ A $\subseteq$ f-1 (cl δ (f(A)) then f(cls δ A) $\subseteq$ f(f^{-1} (cl δ (f(A)) ,but f(f^{-1} (cl δ (f(A)) cls δ (f(A)) therefore،
f(cls δ A) $\subseteq$ cls δ (f(A)).
(٣) →(٤)
Proof: let B $\subseteq$ Y
By (3) we get f(f^{-1} (cl δ (f(B)) $\subseteq$ cl δ (f(A))
Since f (f^{-1} (B)) $\subseteq$ B then by proposition (1.13), we get cls δ (f)f^{-1}(B)) $\subseteq$ cls δ (B)
Since f (cls δ f-1 (B) $\subseteq$ cls δ (f(f-' (B)) then f' (f(cl δ f' (B))
$\subseteq$ (cls (f(f-1 (B))) thus cls δ (f-1 (B)) $\subseteq$ f^{-1}(cl δ (B.))
(٤) →(١)
Proof: let U be δ -open set in Y then X-U is δ -closed set so X-U =cl δ (X-U)
To prove f (U) is δ -open set in X, we most prove f (U)=cls(X-f^{-1}(U))
Since X-U = cls δ (X-U) then f^{-1} (X-U)=f^{-1} (cl δ (X-U)) thus X-f(X-U)=X-f^{-1}(cl δ (X-U)).
By (4) we know that: cls δ (X-^ (U)) $\subseteq$ f^{-1}cls δ (X-U)) and f^{-1}(cl δ (X-U)) $\subseteq$ f(X-U) $\subseteq$ cls δ f(X-U)), thus we get cl δ (X-f^{-1} (U)) $\subseteq$ X- f^{-1} (U); therefore f^{-1} (U) is δ -open set in X.
Definition (2.4); A point x in X is called δ -adherent point of a set A iff every regular open set containing x has nonempty intersection with A .the set of- all δ -adherent by cls δ A is called the δ -closure of A and it is denoted by cl δ A .and a subset A of a space X is called weakly δ -open set if there is an open set 0such that cl δ A= cl δ O 0
.Definition (2.5):.A function f : (X, τ, τ ') → (Y, σ , σ ') is said to be weakly- δ -continuous function (resp. (super-continuous, w-super continuous)function iff for any weakly δ -open (resp. δ -open ,open set)V in Y then f'(V) is weakly δ -open set (resp. δ -open ,weaKly δ -open set) in X -
Definition (2.6):.A function f : (X^ τ, τ) → (Y, σ , σ) is said to be δ -open (resp. δ -closed ,g δ -closed ,w" δ -open ,almost w- δ -open ,always δ -open) if f(U) is δ -open(δ -closed , g δ -closed,w- δ -open,open) set in Y ; for every (resp. g δ -closed ,w- δ -open, open ,w- δ -open)set U in X .
Theorem (2.7): A function f: (X τ, τ) → (Y, σ , σ ') is said to be δ -open function iff f (intA) $\subseteq$ int δ (f(A)); for any A $\subseteq$ X.
Proof: to prove f(int δ A) $\subseteq$ int δ (f(A)); for any A $\subseteq$ X.
Since int δ A $\subseteq$ A then f(int δ A) $\subseteq$ f(A) so int δ (f(int δ (A)) $\subseteq$ int δ (f(A)) since int δ (A) is δ -open set and f is δ -open function then f(int δ A) is δ -open in Y ,thus (int δ A) $\subseteq$ f(int δ A);therefore , f(int δ A) $\subseteq$ int δ (f(int δ (A)) $\subseteq$ int δ (f(A)).

Now; to prove f is δ -open function.
Let U be δ -open set in X. then intδ U=U; so f (intδ U) =f(U);but intδ (f(U)) $\subseteq$ f(U);therefore f(U) $\subseteq$ f (intδ U) c intδ (f(U)),thus
f(U) $\subseteq$ intδ (f(U)) then f(U) is δ -open set ,so f is δ -open function
Remark (2.8):
.١ if A is open set then A is weakly δ -open set
.٢ A is δ -open set then A is weakly δ -open
.٣the concept of super continuous and continuous are independent
The following two examples clear that- :
Example (1) let X={a, b, c}, τ={ϕ, X, {b}} τ ={^,X, {b}, {b،c}},Y= {l,2,3};σ = {ϕ Y, {2}}; σ = {ϕ Y, {2}, {2,3}}
Let f: (X, τ, τ) → (Y, σ ,σ ') such that f(a)=l, f(b)=2, f(c)=3, f is continuous function but it is not super function
Example (2) let X ={1,2,3} ; τ = {ϕ X ,{1} ,{2},{ 1,2} } ; τ′ ={ϕ,X,{1} ,{2,3}}
Let Y ={a, b, c};σ = {ϕ, Y, {a}, {b},{a,b} }; σ '= { ϕ ,Y}،a},{b,c}}
Let f: (X, τ, τ) → (Y, σ ,σ ') such that: f(l)=b, f(2)=a, f(3)=c
It is clear that f is super continuous function but it is continuous function.
(٤ If f is continuous function then f is W-super continuous function.
(٥If f is super continuous then is w-super continuous function.
(٦If f is W- δ -continuous then f is W-super-continuous function
Thee converse of (4), (5), (6) is not true in general. Example (1) and (2) explain that.
(٧If f is W- δ -open function then f is almost W- If f is W-δ-open function
(٨If f is always open function then f is almost W-If f is W- δ -open function
(٩If f is always open function then fis open function
(١٠if f is open function then f is almost W- δ -open function
the converse of (7),(8) ,(9),(10) is not true in general . the following examples explain that
Example: let X={1,2,3}; τ = {ϕ , X} ; τ '- {ϕ,X}
Let Y ={a,b,c} , };σ = {ϕ Y ,{a} ,{b},{a,b} }; σ = {ϕ Y,{a},{b},{a,b}}
(١١Let f: (X, τ, τ) → (Y, σ ,σ ') such that f(a)=1 ,f(b)=2 , f(c)=30
f is almost W- δ -open but it is not W-δ -open function and f
is not almost open function
In example (2); we note that f is W- δ -open but it is open
function
The following tow diagram are declare the relation among these functions

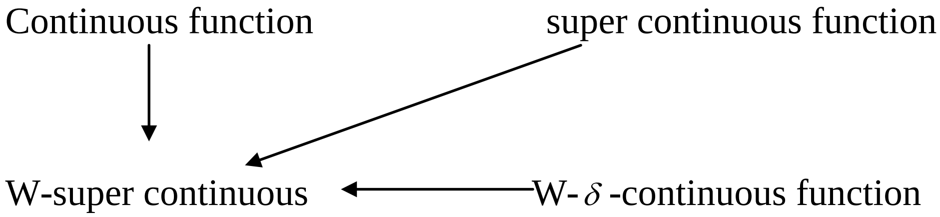

Diagram (1)

$$
\begin{tikzcd}
\text{Open function} \arrow[d] & \text{W-}\delta\text{- open function} \arrow[dl] \\
\text{Almost W-}\delta\text{-open function} & \text{almost open function} \arrow[l] \arrow[ul]
\end{tikzcd}
$$

Diagram (2)

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