Semi-Totally Semi-Continuous Functions in Topological Spaces

Hula .M Salih

AL-Mustansiriya University

Collage of Education

Department of Mathematics

Abstract

In this paper ,we introduce and study a new generalization of continuity called semi-totally semi-continuity ,which is stronger than semi-continuity and weaker than semi-totally continuity is introduced and studied "Further ,some properties of these functions are investigated .Also semi-totally semi-open functions in topological spaces are introduced and studied.

Key words:

Semi-open ,clopen , semi-clopen, totally continuity , semi-total continuity semi-continuity ,semi-total semi-continuity ,semi-totally semi-open function .

1-Introduction

N.Levine[4] introduced the concept of semi-continuous function in 1963.In 1980 , Jain[5] introduced totally continuous functions . In 1995 , T.M.Nour [6] introduced the concept of totally semi-continuous functions as a generalization of totally continuous functions . In 2011,S.S. Benchalli and Umadevi I Neeli [7] introduced the concept of semi- totally continuous function and several properties of semi-totally continuous functions were obtained . In this work, a new generalization of continuity called semi-totally semi-continuity ,which is stronger than semi-continuity and weaker than semi-totally continuity is introduced and studied ,Further ,some properties of these functions are investigated .Also semi-totally semi-open functions in topological spaces are introduced and studied.

2-Preliminaries

A subset A of X is said to be semi-open [4] if $A \subset Cl(Int(A))$.The complement of a semi-open set is called semi-closed set .The family of semi-open sets of space X is denoted by SO(X) .The set which is both open and closed is called clopen set .

Definition 2- 1

A function $f : X \to Y$ is said to be :

1- Semi-continuous [4] if the inverse image of each open subset of Y is semi-open in X .

2- Totally-continuous [5] if the inverse image of every open subset of Y is clopen subset of X .

3- Strongly-continuous [1] if the inverse image of every subset of Y is clopen subset of X .

4- Totally semi-continuous [8] if the inverse image of every open subset of Y is semi- clopen in X .

5- Strongly semi-continuous [8] if the inverse image of every subset of Y is semi-clopen in X .

6- Semi-Totally continuous [7] if the inverse image of every semi- open subset of Y is clopen in X .

7- Irresolute continuous [6] if the inverse image of every semi- open set in Y is semi-open in X .

Definition 2-2

A function $f : X \to Y$ is said to be :

1) semi-open [2] if f(U) is semi- open in Y for each open set U in X .

2) semi-closed [3] if f(F) is semi-closed in Y for each closed set F in X .

3-Main Results

In this section ,We introduce the concept of semi-totally semi-continuous functions as a generalization of the concept of totally continuous functions and study the relationships between semi-totally semi-continuous functions and other simile functions also several properties of semi-totally semi-continuous functions are obtained .

Definition 3-1

A function $f: X \to Y$ is said to be semi-totally semi-continuous function if the inverse image of every semi-open subset of Y is semi- clopen in X.

Example 3-2

Let X={a,b,c} and Y={1,2,3}, τ={X,Ø,{a},{b,c}} and σ={Y,Ø,{1}} Then SO(Y)={Y,Ø,{1},{1,2},{1,3}} Define f(b) = f(c) =1 and f(a) =3.Clearly the inverse image of each semi-open is semi-clopen in X. Therefore f is a semi-totally semi-continuous function .

Theorem 3-3

A function $f: X \to Y$ is semi-totally semi-continuous function if and only if the inverse image of every semi-closed subset of Y is semi- clopen in X.

Proof: Let F be any semi-closed set in Y. Then Y-F is semi-open set in Y, by definition $\mathbf{f^{-1}(Y-F)}$ is semi-clopen in X. That is $\mathbf{X-f^{-1}(F)}$ is semi-clopen in X,this implies $\mathbf{f^{-1}(F)}$ is semi-clopen in X.

On the other hand, if V is semi-open in Y, then Y-V is semi-closed in Y, by hypothesis,$\mathbf{f^{-1}(Y-V)=X-f^{-1}(V)}$ is semi-clopen in X,which implies $\mathbf{f^{-1}(V)}$ is semi-clopen in X. thus, inverse image of every semi-open set in Y is semi-clopen in X. Therefore f is semi-totally semi-continuous function .

Theorem 3-4

Every totally continuous function is a semi-totally semi-continuous function.

Proof: Suppose $f: X \to Y$ is totally continuous and U is any open subset of Y. since every open set is semi-open, U is semi-open in Y and $f: X \to Y$ is totally continuous it follows $\mathbf{f^{-1}(U)}$ is clopen in X , hence $\mathbf{f^{-1}(U)}$ is semi-clopen in X .Thus inverse image of every semi-open set in Y is semi-clopen in X. Therefore the function f is semi-totally semi-continuous.

The converse of the above theorem need not be true ,as shown by the following :

Example 3-5

Let X={a,b,c} and τ={X,Ø,{a},{b},{a,b}} be a topology on X , Let Y={1,2} and σ={Y,Ø,{1}} be a topology on Y, Define a function $f:(X,\tau) \to (Y,\sigma)$ such that f(a)=1 , f(b)=f(c)=2 .Then clearly f is semi-totally semi-continuous function , but not totally continuous function .

Theorem 3-6

Every strongly continuous function is semi-totally semi-continuous function.

Proof: Suppose $f: X \to Y$ is strongly continuous function and A be any semi-open set inY ,By definition $\mathbf{f^{-1}(A)}$ is semi-clopen in X. Thus the inverse image of each semi-open set in Y is semi-clopen in X .Therefore f is semi-totally semi-continuous function .

The converse of the above theorem need not be true ,as shown by the following :

Example 3-7

Let X={a,b,c} and τ={X,Ø,{a},{b},{a,b},{a,c}} be a topology on X , Let Y={1,2,3} and σ={Y,Ø,{1},{2,3}} be a topology on Y, Define a function $f:(X,\tau) \to (Y,\sigma)$ such that f(a) = 2 , f(b) = 1, f(c) = 3 .Then clearly f is semi-totally semi-continuous function, since the inverse image of every semi-open set in Y is semi-clopen in X, but not strongly continuous ,because for the set {2}, $\mathbf{f^{-1}\{2\} = \{a\}}$ is not clopen in X .

Theorem 3-8

Every semi-totally continuous function is semi-totally semi-continuous function.

Proof: Suppose $f: X \to Y$ is semi-totally continuous function and A be any open set inY , since every open set is semi-open and $f: X \to Y$ is semi-totally continuous ,it follows that $\mathbf{f^{-1}(A)}$ is clopen and hence semi-clopen in X. Thus the inverse image of each semi-open set in Y is semi-clopen in X .Therefore f is semi-totally semi-continuous function .

The converse of the above theorem need not be true ,as shown by the following :

Example 3-9

Let X={a,b,c} and τ={X,Ø,{a},{b},{a,b}} be a topology on X , Let Y={1,2,3} and σ={Y,Ø,{1}} be a topology on Y , SO(X)={X,Ø,{a},{b,c},{a,b},{a,c}}and SO(Y)={Y,Ø,{1},{1,2},{1,3}} Define a function $f:(X,\tau)\rightarrow(Y,\sigma)$ such that f(a)=1 , f(b)= f(c)=2 .Then clearly f is semi-totally semi-continuous function, since the inverse image of every semi-open set in Y is semi-clopen in X, but f is not semi-totally continuous ,because for the set {1}, $f^{-1}\{1\}=\{a\}$ is not clopen in X .

By the same way of theorem(3-8) we can prove that Every totally semi- continuous function is semi-totally semi-continuous function.

Theorem 3-10

Every semi-totally semi-continuous function is semi-continuous function.

Proof: Suppose $f:X\rightarrow Y$ is a semi-totally semi-continuous function and A be any open set inY ,By definition $\mathbf{f^{-1}(A)}$ is semi-clopen in X. this implies $\mathbf{f^{-1}(A)}$ is semi-open in X . Thus the inverse image of each open set in Y is semi-open in X .Therefore f is semi-continuous function .

Example 3-11

Let X={a,b,c} and τ={X,Ø,{a}} be a topology on X , Let Y={1,2,3} and σ={Y,Ø,{1},{1,2}} be a topology on Y , SO(X) ={X,Ø,{a},{a,b},{a,c}}and SO(Y)={Y,Ø,{1},{1,2},{1,3}} Define a function $f:(X,\tau)\rightarrow(Y,\sigma)$ such that f(a)=1 , f(b)= 2, f(c)=3 .Then clearly f is semi-continuous function, , but f is not semi-totally semi- continuous ,because for the set {1}, $f^{-1}\{1\}=\{a\}$ is not semi- clopen in X .

Thus we have the following relationship :Strong continuous →semi-totally continuous →totally continuous →totally semi-continuous →semi-totally semi-continuous →semi-continuous ,the converses are not true in general .

Theorem 3-12

A function $f:X\rightarrow Y$ is semi-totally semi-continuous if and only if for each x∈ X and each semi-open set V in Y with f(x) ∈ V, there is a semi- clopen set U in X such that x ∈U and f(U)⊂V.

Proof: Suppose $f: X \rightarrow Y$ is a semi-totally semi-continuous function and V be any semi-open set in Y containing f(x) so that $\mathbf{x \in f^{-1}(V)}$,since f is semi-totally semi-continuous ,$\mathbf{f^{-1}(V)}$ is semi-clopen in X. Let $\mathbf{U = f^{-1}(V)}$,Then U is semi-clopen set in X and x $\in$ U .Also $\mathbf{f(U) = f(\,f^{-1}(V)) \subset V}$, this implies f (U)$\subset$V. On the other hand Let V be semi-open in Y, Let $\mathbf{x \in f^{-1}(V)}$ be any arbitrary point ,this implies f(x) $\in$ V , therefore by above there is a semi-clopen set f(G_x)$\subset$X containing x such that f(G_x)$\subset$V, which implies $G_x \subset \mathbf{f^{-1}(V)}$ .We have x $\in G_x \subset \mathbf{f^{-1}(V)}$,this implies $\mathbf{f^{-1}(V)}$ is semi-clopen neighbourhood of x ,since x is arbitrary ,it implies $\mathbf{f^{-1}(V)}$ is semi-clopen neighbourhood of each of its points ,hence it is semi-clopen set in X. therefore f is semi-totally semi-continuous .

Remark 3-13

The semi-totally semi-continuous function not necessary to be strongly semi-continuous function . To illustrate that, consider the following example

Example 3-14

Let X={a,b,c} and τ={X,Ø,{a},{b,c}} be a topology on X , Let Y={1,2,3} and σ={Y,Ø,{1}} be a topology on Y , then the identity function $f:(X,\tau) \rightarrow (Y,\sigma)$ is semi-totally semi-continuous function, but not strongly semi- continuous.

Now, the following theorem provides a condition in order to make remark (3-13) is true.

Theorem 3-15

Every semi-totally semi-continuous function into a T_1 space is strongly semi-continuous function.

Proof: Suppose $f: X \rightarrow Y$ is a semi-totally semi-continuous function in a T_1space ,singletons are closed sets .Hence f^{-1}(B) is semi-clopen in X for every subset B of Y, therefore f is strongly semi-continuous function.

Now we study some properties on semi-totally semi-continuous

Theorem 3-16

A function $f : X \to Y$ is semi-totally semi-continuous and A is semi-clopen subset of X, then the restriction $f \setminus_A : A \to Y$ is semi-totally semi-continuous .

Proof: Consider the function $f \setminus_A : A \to Y$ and V be any semi-open set in Y, since f is semi-totally semi-continuous, $f^{-1}(V)$ is semi-clopen subset of X. since A is semi-clopen subset of X and $(f \setminus_A)^{-1}(V) = A \cap f^{-1}(V)$ is semi-clopen in A , it follows $(f \setminus_A)^{-1}(V)$ is semi- clopen in A, hence $f \setminus_A$ is semi-totally semi-continuous.

Theorem 3-17

The composition of two semi-totally semi-continuous function is semi-totally semi-continuous.

Proof: Let $f : X \to Y$ and $g: Y \to Z$ be two semi-totally semi-continuous functions .Let V be a semi-open set in Z. since g semi-totally semi-continuous functions $g^{-1}(V)$ is semi-clopen and hence semi-open in Y . Further ,since f is semi-totally semi-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi-clopen in X .Hence $g \circ f : X \to Z$ is semi-totally semi-continuous .

Theorem 3-18

If $f : X \to Y$ is semi-totally semi-continuous and $g: Y \to Z$ is irresolute, then $g \circ f : X \to Z$ is semi-totally semi-continuous.

Proof: Let $f : X \to Y$ be semi-totally semi-continuous and $g: Y \to Z$ be irresolute , Let V be semi-open in Z ,since g is irresolute ,$g^{-1}(V)$ is semi-open in Y. Now since f is semi-totally semi-continuous , $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi-clopen in X .Hence $g \circ f : X \to Z$ is semi-totally semi-continuous.

Theorem 3-19

If $f : X \to Y$ is semi-totally semi-continuous and $g: Y \to Z$ is semi-continuous then $g \circ f : X \to Z$ is totally semi-continuous.

Proof: Let V be open in Z ,since g is semi-continuous ,$g^{-1}(V)$ is semi-open in Y. Now since f is semi-totally semi-continuous , $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi-clopen in X .hence $g \circ f : X \to Z$ is totally semi-continuous.

4-Semi-totally Semi-open function

In this section , we introduce a new class of function called semi-totally semi-open function and also study some of their basic properties.

Definition 4-1

A function $f : X \to Y$ is said to be semi-totally semi-open function if the image of every semi-open set of X is semi- clopen in Y.

Theorem 4-2

If a bijective function $f : X \to Y$ is semi-totally semi-open then the image of each semi-closed set of X is semi- clopen in Y.

Proof: Let F be any semi-closed set in X. Then Y-F is semi-open set in Y, Since f is semi-totally semi-open , $\mathbf{f(X-F) = Y - f(F)}$ is semi-clopen in Y ,this implies $\mathbf{f(F)}$ is semi-clopen in Y.

Theorem 4-3

A subjective function $f : X \to Y$ is semi-totally semi-open if and only if for each subset B of Y and for each semi-closed set U containing $\mathbf{f^{-1}(B)}$, there is a semi-clopen set V of Y such that B $\subset$V and $\mathbf{f^{-1}(V) \subset U}$.

Proof: Suppose $f : X \to Y$ is a surjective semi-totally semi-open function and B is subset of Y. Let U be semi-closed set of X such that $\mathbf{f^{-1}(B) \subset U}$ .Then V=Y- f (X-U) is semi-clopen subset of Y containing B such that $\mathbf{f^{-1}(V) \subset U}$.

On the other hand ,suppose F is a semi-closed set of X. Then $\mathbf{f^{-1}(Y - f(F)) \subset X - F}$ and X-F is semi-open ,by hypothesis, there exists a semi-clopen set Vof Y such that Y-f (F) $\subset$V ,which implies $\mathbf{f^{-1}(V) \subset X - F}$

Therefore $\mathbf{F \subset X - f^{-1}(V)}$.Hence$\mathbf{Y - V \subset f(F) \subset f\left(X - f^{-1}(V)\right) \subset Y - V}$ This implies f (F) =Y-V ,Which is semi-clopen in Y . thus, the image of a semi-open set in X is semi-clopen in Y, Therefore f is a semi-totally semi-open function .

Theorem 4-4

For any bijective function $: X \rightarrow Y$, the following statements are equivalent (i) Inverse of f is semi-totally semi-continuous (ii) f is semi-totally semi-open .

Proof: (i)→(ii) Let U be a semi-open set of X. by assumption $(\mathbf{f}^{-1})^{-1}(\mathbf{U}) = \mathbf{f}(\mathbf{U})$ is semi- clopen in Y . So f is semi-totally semi-open.

(ii)→(i) Let F be semi-open in X, Then f(V) is semi-clopen in Y. That is $(\mathbf{f}^{-1})^{-1}(\mathbf{V})$ is semi-clopen in Y. Therefore $\mathbf{f}^{-1}$ is semi-totally semi-continuous.

Theorem 4-5

A function $\mathbf{f}: X \rightarrow Y$ is semi-totally semi-open and A is semi-clopen subset of X, then the restriction $f \backslash_A$:A→Y is semi-totally semi-open.

Proof: Consider the function $f \backslash_A$:A→Y and B be any semi-open set in A . Since A is semi-clopen in X, Then B is semi-open in X . Since f is semi-totally semi-open ,hence f(B) is semi-clopen in Y ,But f(B) = f_A(B). Then f_A(B) is semi-clopen in Y, Hence $f \backslash_A$:A→Y is semi-totally semi-open.

Theorem 4-6

The composition of two semi-totally semi-open function is semi-totally semi-open.

Proof: Let f : X→Y and g: Y→Z be two semi-totally semi-open functions ,Then their composition g∘f :X→Z .Let V be a semi-open set in X. since f semi-totally semi-open functions f(V) is semi-clopen in Y and hence it is semi-open in Y, which implies f(V) is semi-open in Y ,since g is semi-totally semi-open g(f(V)) = (g∘f)(V) is semi-clopen in Z .Hence g∘f :X→Z is semi-totally semi-open .

References:

[1] M. Stone, (1937), Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41, 374.

[2] N. Biswas, (1969), Some mappings in topological spaces, Bull. Cal. Math. Soc., 61,127-135.

[3] N. Biswas, (1970), Characterization of semi-continuous mappings , Atti. Accad. Naz. Lience. Rend.CL. Sci. Fis. Math. Nat., 8,48,399-402.

[4] N.Levine, (1963), Semi-open sets and semi-continuity in topological space , Amer.Math.Monthly.,70, 39-41.

[5] R.C. Jain, (1980) ,The role of regularly open sets in general topological ,Ph.D.thesis, Meerut University, Institute of advanced studies, Meerut-India.

[6] S.G. Crossley and S.K. Hildebr, (1972), Semi Topological properties, Fund. Math. ,74, 233-254.

[7] S.S. Benchalli and U. I Neeli , (2011), Semi-totally Continuous function in topological spaces, Inter. Math. Forum , 6 ,10,479-492.

[8] T. M. Nour, (1995), Totally semi-continuous function ,Indian J. Pure Appl.Math.,7, 26, 675-678.

الدوال شبه كلية شبه مستمرة في الفضاءات التبولوجية

م .حلا محمد صالح

الجامعة المستنصرية | كلية التربية | قسم الرياضيات

المستخلص

في هذا البحث ، قدمنا ودرسنا تعميم جديد للاستمرارية يسمى شبه كلية شبه مستمرة ، الذي هو اقوى من شبه المستمر واضعف من شبه كلية مستمر ، بالاضافة الى بعض الخصائص لهذا الدوال تحققت ، كذلك قدمنا ودرسنا الدوال شبه كلية شبه مفتوحة في الفضاءات التبولوجية .

الكلمات المفتاحية

شبه مفتوح ،مجموعة مغلقة مفتوحة ،مجموعة شبه مغلقة شبه مفتوحة ،مستمر كلياً، شبه مستمر كلياً، شبه مستمر،شبه كلية شبه مستمر، دالة شبه كلية شبه مفتوحة