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Inferential Methods for the Dagum Regression Model

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ABSTRACT

The Dagum Regression Model, introduced to address limitations in traditional econometric models, provides enhanced flexibility for analyzing data characterized by heavy tails and asymmetry, which is common in income and wealth distributions. This paper develops and applies the Dagum model, demonstrating its advantages over other distributions such as the Log-Normal and Gamma distributions. The model's parameters are estimated using Maximum Likelihood Estimation (MLE) and the Method of Moments (MoM). A simulation study evaluates both methods' performance across various sample sizes, showing that MoM tends to offer more robust and precise estimates, particularly in small samples. These findings provide valuable insights into the analysis of income inequality and wealth distribution using the Dagum model.

1. Introduction

The analysis of income and wealth distributions is fundamental in economics and social science research. Disparities in these distributions often exhibit heavy tails and asymmetry, characteristics inadequately captured by traditional models such as the Log-Normal or Gamma distributions. The Dagum distribution, introduced by Camilo Dagum in 1977, addresses these limitations by providing a more flexible model that can handle such data features effectively. Its flexibility makes it particularly suitable for modeling variables with skewness and kurtosis.

When examining global, national, or regional data, we often encounter significant disparities in income and health, where some individuals are poor while others are wealthy. Consequently, modeling income and wealth distribution has been extensively researched across various scientific disciplines. In 1977, Camilo Dagum introduced a new distribution model to address the limitations of traditional distributions like Log-Normal and Gamma, which inadequately represent the tails of actual distributions. Since then, numerous studies have adopted the Dagum model to analyze income and wealth, particularly in economic and financial contexts.
[2] Review and analyze various models of income and wealth distribution, focusing on their differences, similarities, strengths, and weaknesses, while emphasizing the inefficiencies found among them. [3] Examine and contrast different techniques for handling families of distributions with multiple parameters.
Numerous factors contribute to income disparities among individuals, including health

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issues, pandemics, and more [4-5]. Econometric studies frequently use regression techniques to explore the varying relationships between income and health. In this context, methods like quantile regression and generalized additive models have been developed as advanced distributional regression approaches to better understand these dynamics [6-9].

Numerous studies analyze how changes in explanatory variables influence the average of dependent variables using linear and generalized linear regression models [10]. Camilo Dagum and his colleagues [11-12] conducted extensive research on income distributions across various countries, demonstrating that the Dagum distribution more accurately represents disparities and inequality compared to traditional distributions like log-normal and Pareto. The Dagum model has been widely applied to economic data, including income, expenditure, and wealth, as well as to social issues such as health outcomes and educational attainment. Research [13] has highlighted the model's versatility in handling different types of data. Its capability to manage heavy tails and skewness has proven especially beneficial in financial econometrics. For instance, researchers have utilized the Dagum regression model to analyze asset returns and other financial variables that exhibit heavy-tailed behavior [14].

Researchers have extended the Dagum distribution to handle data with multiple dependent variables that have heavy-tailed distributions [15]. Bayesian techniques have been used to estimate parameters in Dagum regression models, allowing for the integration of prior knowledge and management of parameter uncertainty [16]. Enhanced flexibility and applicability of the Dagum distribution have been achieved through generalized versions, which create new distribution families by transforming existing ones [17].

This paper focuses on applying the Dagum regression model to analyze income and wealth distributions, estimating its parameters using MLE and MoM. Both methods are compared through a simulation study, assessing their performance under different sample sizes and parameter configurations. The remainder of the paper is structured as follows: Section 2 presents the theoretical framework of the Dagum distribution and the parameter estimation methods. Section 3 details the simulation study, and Section 4 discusses the results. The paper concludes with a summary of findings and suggestions for future research.

2. Methodology

2.1 The Dagum Regression Model

The Dagum regression model is used to analyze the distribution of a dependent variable Y based on explanatory variables X. The Dagum distribution is characterized by its flexibility in modeling data with varying degrees of skewness and kurtosis. In this study, we estimate the parameters of the Dagum distribution using the maximum likelihood estimation method [18-19].

The probability density function (PDF) of the Dagum distribution for a continuous variable y is defined as:

$$f(y) = \frac{\alpha\sigma\, y^{\alpha\sigma-1}}{\beta^{\alpha\sigma}(1+\left(\frac{y}{\beta}\right)^{\alpha})^{\sigma+1}}, \qquad (1)$$

where:

$\alpha > 0$ is the shape parameter,

$\beta > 0$, is the scale parameter,

$\sigma > 0$, is the second shape parameter.

These parameters control the skewness and tail behavior of the Dagum distribution. For each observation i, the PDF is given by:

$$f(y_i) = \frac{\alpha_i\sigma_i\, {y_i}^{\alpha_i\sigma_i-1}}{{\beta_i}^{\alpha_i\sigma_i}(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i})^{\sigma_i+1}}, \qquad (2)$$

In the regression setting, the parameters α_i, β_i, σ_i for each observation are modeled as functions of explanatory variables x_i:

where

$$\alpha_i = b_{10} + b_{11}x_i \qquad (3)$$

$$\beta_i = b_{20} + b_{21}x_i \qquad (4)$$

$$\sigma_i = b_{30} + b_{31}x_i, \qquad (5)$$

where $b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31}$ are the coefficients to be estimated.

2.2 Maximum Likelihood Estimation

The likelihood function $L(\varphi|y_i)$ for n observations is the product of the individual density functions:

$$L(\varphi|y_i) = \prod_{i=1}^{n} \frac{\alpha_i\sigma_i\, y_i^{\alpha_i\sigma_i-1}}{\beta_i^{\alpha_i\sigma_i}(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i})^{\sigma_i+1}} \quad (6)$$

To simplify the maximization process, we take the natural logarithm of the likelihood function, resulting in the log-likelihood function $\ell(\varphi|y_i)$

$$\ell(\varphi|y_i) = \log \prod_{i=1}^{n} \frac{\alpha_i\sigma_i\, y_i^{\alpha_i\sigma_i-1}}{\beta_i^{\alpha_i\sigma_i}(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i})^{\sigma_i+1}} \quad (7)$$

Using the logarithm properties, the log-likelihood function can be expressed as:

$$\ell(\varphi|y_i) = \sum_{i=1}^{n} \log\left[\frac{\alpha_i\sigma_i\, y_i^{\alpha_i\sigma_i-1}}{\beta_i^{\alpha_i\sigma_i}(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i})^{\sigma_i+1}}\right] \quad (8)$$

Expanding the log-likelihood function further gives:

$$\ell(\varphi|y_i) = \sum_{i=1}^{n}\Big[\log\alpha_i + \log\sigma_i + (\alpha_i\sigma_i - 1)\log y_i - \alpha_i\sigma_i\log\beta_i - (\sigma_i+1)\log(1 + \left(\frac{y_i}{\beta_i}\right)^{\alpha_i})\Big] \quad (9)$$

where $\varphi = (b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31})$ are estimated by maximizing the log-likelihood function using numerical optimization techniques. The optimization aims to find the values of φ that best fit the data, minimizing the difference between the observed and predicted values of y_i.

3. Implementation

The estimation procedure is implemented in a statistical software environment, utilizing optimization algorithms such as the Newton-Raphson method or the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. These algorithms iteratively update the parameter estimates to maximize the log-likelihood function.

3.1 Partial Derivatives

Derivatives with respect to b_{10} and b_{11}:

These derivatives involve the parameter α_i, which is influenced by b_{10} and b_{11}:

$$\frac{\partial\ell}{\partial b_{10}} = \sum_{i=1}^{n}\left[\frac{1}{\alpha_i} + \sigma_i\log y_i - \sigma_i\log\beta_i - \frac{(\sigma_i+1)}{\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)}\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\log\frac{y_i}{\beta_i}\right] \quad (10)$$

$$\frac{\partial\ell}{\partial b_{11}} = \sum_{i=1}^{n} x_i\left[\frac{1}{\alpha_i} + \sigma_i\log y_i - \sigma_i\log\beta_i - \frac{(\sigma_i+1)}{\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)}\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\log\frac{y_i}{\beta_i}\right] \quad (11)$$

Derivatives with respect to b_{20} and b_{21}:

These derivatives involve the parameter β_i, which is influenced by b_{20} and b_{21}:

$$\frac{\partial\ell}{\partial b_{20}} = -\sum_{i=1}^{n}\left[\frac{\alpha_i\sigma_i}{\beta_i} + \frac{(\sigma_i+1)\alpha_i}{\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)}\left(\frac{y_i}{\beta_i}\right)^{\alpha_{i-1}}\frac{-y_i}{\beta_i^{\,2}}\right] \quad (12)$$

$$\frac{\partial\ell}{\partial b_{21}} = -\sum_{i=1}^{n} x_i\left[\frac{\alpha_i\sigma_i}{\beta_i} + \frac{(\sigma_i+1)\alpha_i}{\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)}\left(\frac{y_i}{\beta_i}\right)^{\alpha_{i-1}}\frac{-y_i}{\beta_i^{\,2}}\right] \quad (13)$$

Derivatives with respect to b_{30} and b_{31}:

These derivatives involve the parameter σ_i, which is influenced by b_{30} and b_{31}:

$$\frac{\partial\ell}{\partial b_{30}} = \sum_{i=1}^{n}\left[\frac{1}{\sigma_i} + \alpha_i\log y_i - \alpha_i\log\beta_i - \log\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)\right] \quad (14)$$

$$\frac{\partial\ell}{\partial b_{31}} = \sum_{i=1}^{n} x_i\left[\frac{1}{\sigma_i} + \alpha_i\log y_i - \alpha_i\log\beta_i - \log\left(1+\left(\frac{y_i}{\beta_i}\right)^{\alpha_i}\right)\right] \quad (15)$$

3.1.1 Newton-Raphson Iterative Process

The Newton-Raphson method updates the parameter estimates by iteratively solving the following equation:

$$\varphi^{(k+1)} = \varphi^{(k)} - H^{-1}\left(\varphi^{(k)}\right) g\left(\varphi^{(k)}\right) \quad (16)$$

where:

$\varphi^{(k)}$ is the vector of parameter estimates at iteration k,

$H\left(\varphi^{(k)}\right)$ is the Hessian matrix of the second derivatives of the log-likelihood function,

$g\left(\varphi^{(k)}\right)$ is the gradient vector of the first derivatives of the log-likelihood function.

The process is repeated until the change in the log-likelihood function or the parameter estimates is sufficiently small, indicating convergence to the maximum likelihood estimates.

Hessian Matrix

The Hessian matrix $H\left(\varphi^{(k)}\right)$ contains the second-order partial derivatives of the log-likelihood function with respect to the parameters. For the Dagum regression model, the Hessian matrix is derived from the second derivatives of the log-likelihood function with respect to $b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31}$.

3.1.2 Implementation in Statistical Software

The Newton-Raphson algorithm is implemented in statistical software, which requires specifying the initial parameter values, computing the gradient and Hessian at each iteration, and updating the parameters until convergence. Common software tools, such as R, Python (with libraries like SciPy), and MATLAB, provide built-in functions to perform this optimization efficiently.

Convergence and Model Selection

Upon convergence, the final parameter estimates $\varphi = (b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31})$ are used to assess the model's goodness-of-fit.

By leveraging the Newton-Raphson method, we ensure precise and robust estimation of the Dagum regression model parameters, leading to reliable insights into the relationships between the dependent variable and explanatory variables.

3.2 Method of Moments

The Method of Moments (MoM) is an alternative approach to parameter estimation that relies on equating sample moments (such as the mean, variance, and skewness) to theoretical moments derived from the probability distribution. For the Dagum distribution, the moments can be used to estimate the parameters α, β, and σ by matching the sample moments with the corresponding population moments [20-21].

Theoretical Moments of the Dagum Distribution

The Dagum distribution has known expressions for its theoretical moments. The k^{th} moment of the Dagum distribution is given by:

$$E(Y^k) = \beta^k \frac{\Gamma(1-\frac{k}{a})\Gamma(\sigma+\frac{k}{a})}{\Gamma(\sigma)} \quad (17)$$

where:

- $E(Y^k)$ is the k^{th} moment of Y,
- $\Gamma(.)$ is the Gamma function,
- a, β, and σ are the parameters of the Dagum distribution.

For the first three moments, we have:

- Mean $(\mu) = E(Y)$

$$= \beta \frac{\Gamma(1-\frac{1}{a})\Gamma(\sigma+\frac{1}{a})}{\Gamma(\sigma)} \quad (18)$$

- $\text{Var(Y)} = E(Y^2) - \left(E(Y)\right)^2 = \left[\beta^2 \frac{\Gamma\left(1-\frac{2}{a}\right)\Gamma\left(\sigma+\frac{2}{a}\right)}{\Gamma(\sigma)} - (\beta \frac{\Gamma(1-\frac{1}{a})\Gamma(\sigma+\frac{1}{a})}{\Gamma(\sigma)})^2\right]$ (19)
- $\text{Skewness(Y)} = \frac{\beta^3 \frac{\Gamma(1-\frac{3}{a})\Gamma(\sigma+\frac{3}{a})}{\Gamma(\sigma)} - 3\beta\mu\, Var(Y) - \mu^3}{Var(Y)^{\frac{3}{2}}}$ (20)

Here, we consider

$$\alpha = b_{10} + b_{11}\bar{X} \quad (21)$$

$$\beta = b_{20} + b_{21}\bar{X} \quad (22)$$

$$\sigma = b_{30} + b_{31}\bar{X} \quad (23)$$

Estimating Parameters Using the Method of Moments

To estimate the parameters α, β, and σ using the Method of Moments (MoM), we begin by computing the sample moments from the observed data $\text{Y} = \{\text{y}_1, \text{y}_2, \text{y}_3, \dots, \text{y}_n\}$:

- Sample Mean $(\bar{\text{Y}})$

$$= \frac{1}{n}\sum_{i=1}^{n} \text{y}_i \quad (24)$$

- Sample Variance (S_Y^2)

$$= \frac{1}{n-1}\sum_{i=1}^{n} (\text{y}_i - \bar{\text{Y}})^2 \quad (25)$$

- SkewnessY

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} (\text{y}_i - \bar{\text{Y}})^3}{S_Y^3} \quad (26)$$

Using the above equations, we solve for the parameters $b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31}$ by equating the equation of the theoretical moment to the sample moment, given the data (X, Y). This often involves iterative numerical methods or optimization techniques because of the non-linear nature of the Gamma functions involved.

4. Simulation

We conducted a simulation study to evaluate the performance of Maximum Likelihood Estimation (MLE) and Method of Moments (MoM) for parameter estimation in the Dagum regression model under various scenarios. The simulation was designed as follows:

- We generated data sets under three sample sizes n=(60, 100, 150) ,where x_i was drawn from $N(0,1)$.
- Three sets of parameter vectors were considered: vector $\varphi_1 = (0.2, 1, 0.4, 2, 0.8, 3)$, $\varphi_2 = (0.5, 1.5, 0.8, 2.5, 1, 3)$, $\varphi_3 = (0.8, 2, 1.2, 3, 1.8, 3.8)$.
- Using equations (3)-(5), we computed $\alpha_i, \beta_i, \sigma_i$.
- y_i generated from Dagum distribution according to calculated $\alpha_i, \beta_i, \sigma_i$.
- The final data set (x_i, y_i) is considered to evaluate the estimation methods. Both MLE and MoM were applied using Multivariate Newton-Raphson to estimate $b_{10}, b_{11}, b_{20}, b_{21}, b_{30}, b_{31}$.

Assessment the accuracy

The simulation process was repeated for R=500 replications.

Average Mean square error is calculated for each replication as:

$$MSE = \frac{\sum_{r=1}^{R}(\varphi_r - \hat{\varphi}_r)^2}{R}$$

5. Results

We use the above formula of MSE to compute the accuracy of estimating according to the different scenarios.

Table 1: MSE values of 500 replications for the first set of parameters across three sample sizes for both MLE and MoM.

Sample size	Method	$b_{10} = 0.2$	$b_{11} = 1$	$b_{20} = 0.4$	$b_{21} = 2$	$b_{30} = 0.8$	$b_{31} = 3$
n=60	MLE	0.3784	0.5978	0.2256	0.6387	0.4890	0.8936
	MOM	0.2232	0.4823	0.2698	0.6121	0.4418	0.8846
n=100	MLE	0.3133	0.4665	0.2173	0.5721	0.4176	0.8174
	MOM	0.2054	0.3749	0.2419	0.5531	0.3933	0.7917
n=150	MLE	0.2790	0.3487	0.2020	0.5198	0.3754	0.7649
	MOM	0.2512	0.3102	0.2213	0.4829	0.3278	0.7166

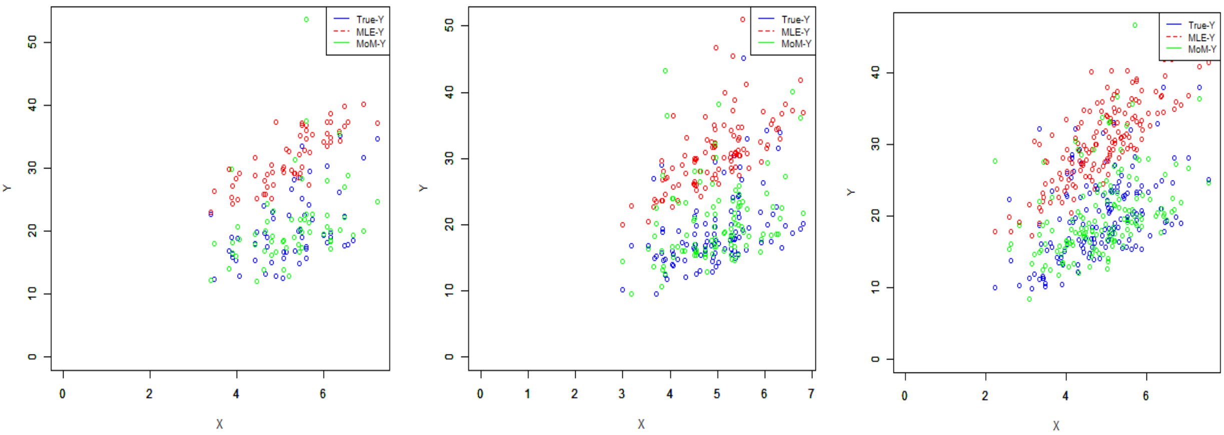

Figure 1: The plots represent the scatter of generated X and real Y compared to estimated ones. The graphs from left to right represent sample sizes 50, 100 and 150, respectively, according to the first scenario of parameters.

Table 2: MSE values of 500 replications for the second set of parameters across three sample sizes for both MLE and MoM.

Sample size	Method	$b_{10} = 0.5$	$b_{11} = 1.5$	$b_{20} = 0.8$	$b_{21} = 2.5$	$b_{30} = 1$	$b_{31} = 3$
n=60	MLE	0.4129	0.8723	0.4567	0.9111	0.6689	1.2345
	MOM	0.4028	0.7324	0.4311	0.8565	0.5199	0.9946
n=100	MLE	0.3908	0.7566	0.3765	0.7734	0.5572	0.9852
	MOM	0.3821	0.7255	0.3911	0.7126	0.4876	0.8789
n=150	MLE	03276	0.5255	0.3833	0.6212	0.5113	0.7255
	MOM	0.2943	0.3229	0.3737	0.5933	0.3992	0.5633

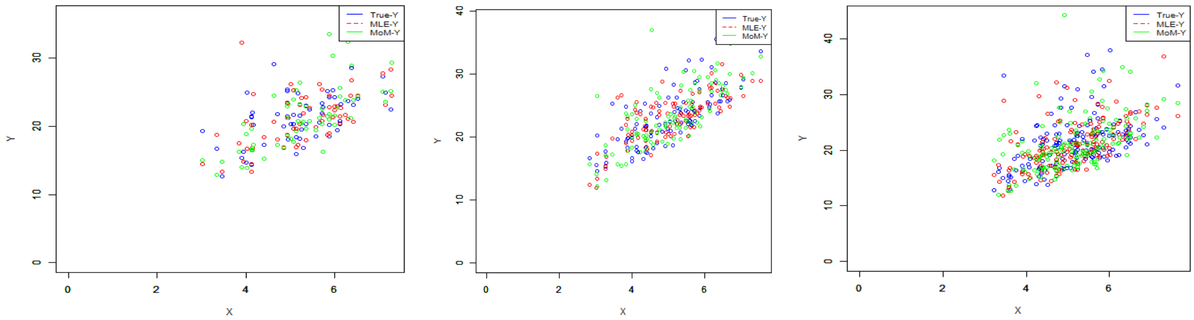

Figure 2: The plots represent the scatter of generated X and real Y compared to estimated ones. The graphs from left to right represent sample sizes 50, 100 and 150, respectively, according to the second scenario of parameters.

Table 3: MSE values of 500 replications for the third set of parameters across three sample sizes for both MLE and MoM.

Sample size	Method	$b_{10} = 0.8$	$b_{11} = 2$	$b_{20} = 1.2$	$b_{21} = 3$	$b_{30} = 1.8$	$b_{31} = 3.8$
n=60	MLE	0.6673	0.8847	0.7633	1.5467	0.9455	1.3767
	MOM	0.6121	0.8767	0.7298	1.477	0.7435	1.2445
n=100	MLE	0.5925	0.8245	0.6144	1.1876	0.5299	0.8956
	MOM	0.5839	0.7933	0.6098	0.9924	0.5711	0.8877
n=150	MLE	0.3965	0.5833	0.5340	0.8744	0.3777	0.4345
	MOM	0.3944	0.4564	0.1733	0.5446	0.3044	0.3288

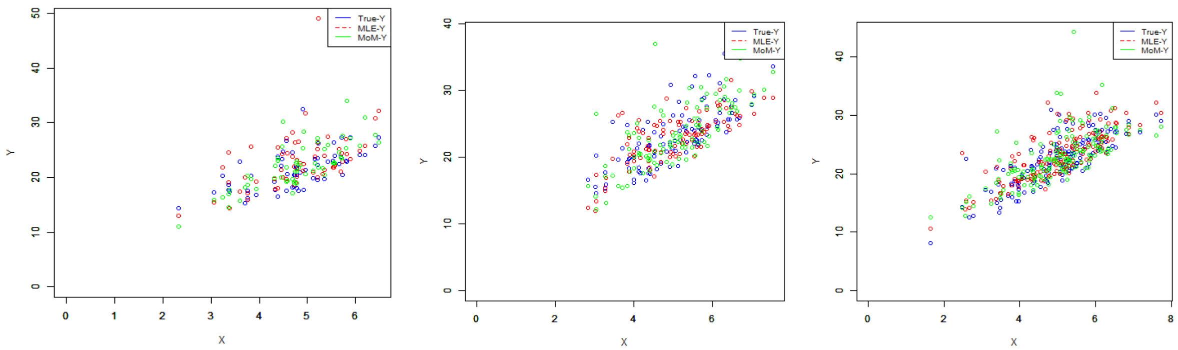

Figure 3: The plots represent the scatter of generated X and real Y compared to estimated ones. The graphs from left to right represent sample sizes 50, 100 and 150, respectively, according to the third scenario of parameters.

6. Conclusion

This study highlights the effectiveness of the Dagum regression model in analyzing skewed and heavy-tailed distributions, commonly observed in income and wealth data. A comprehensive simulation study was conducted to assess the accuracy of parameter estimation using Maximum Likelihood Estimation (MLE) and the Method of Moments (MoM) under various sample sizes and parameter settings, as illustrated in Tables 1-3. Overall, both estimation methods produced reasonable results, though MoM consistently outperformed MLE by yielding lower Mean Square Error (MSE) values, especially with smaller sample sizes. These results indicate that

MoM may offer a more robust alternative for parameter estimation when the model's distributional assumptions hold.

Figures 1-3 further demonstrate the performance of the two methods by comparing scatter plots of actual versus estimated Y values. The green points (MoM estimates) are tightly clustered around the blue points (actual values), while the red points (MLE estimates) are more widely dispersed, indicating that MLE produces less accurate estimates. This supports the conclusion that MoM offers more precise estimates that closely align with the true values.

Future research could investigate alternative estimation techniques or apply the Dagum model to more complex datasets. Additionally, expanding the model to accommodate multiple dependent variables or exploring Bayesian approaches presents promising avenues for further exploration.

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