On the Compactness of the Composition Operator C_σ

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Abstract

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z)=\sum_{n=0}^{\infty} f^{\wedge}(n)\ z^n$ holomorphic on U such that $\sum_{n=0}^{\infty}\left|f^{\wedge}(n)\right|^2<\infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Let ψ be a holomorphic self-map of U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f\circ\psi \qquad (f\in H^2)$$

In this paper we have studied the compactness of the composition operator induced by the bijective map σ and discussed the adjoint the compactness of the composition operator of the map σ .We give some theorems on compactness of the composition operators. We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the map ψ in U.

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

Introduction

This paper consists of two sections . In section one ,we are going to study the map σ and properties of σ , and also discuss σ as an inner map .

In section two, we are going to study the Composition Operator C_σ induced by the map σ and properties of C_σ, and also discuss the adjoint of Composition Operator C_σ induced by the map σ and also discuss C_σ as a compact operator .

Section One

Definition(1.1) : [4]

Let $U=\{z\in C:|z|<1\}$ is a unit ball in complex plane C and $\partial U=\{z\in C:|z|=1\}$ is a boundary of U

Definition (1.2):

For $\lambda\in U$, define $\sigma(z)=\dfrac{2\bar{\lambda}z+2}{2z+2\lambda}$ $(z\in U)$. Since the denominator equal zero only at $z=-\lambda$,then the map σ is holomorphic on the ball $\{|z|<|\lambda|\}$. Since $\lambda\in U$, then this ball contain U . Hence $\sigma: U\to U$ be holomorphic map on U .

Proposition (1.3) :

For $\lambda \in U$, then

$$|\sigma(z)|^2 - 1 = \frac{4\left(1-|z|^2\right)\left(1-|\lambda|^2\right)}{|2z+2\lambda|^2}$$

Proof:

$$|\sigma(z)|^2 - 1 = \left|\frac{2\bar{\lambda}z+2}{2z+2\lambda}\right|^2 - 1 = \frac{|2\bar{\lambda}z+2|^2}{|2z+2\lambda|^2} - 1 = \frac{|2\bar{\lambda}z+2|^2 - |2z+2\lambda|^2}{|2z+2\lambda|^2}$$

$$= \frac{\left(2\bar{\lambda}z+2\right)\left(2\lambda\bar{z}+2\right) - \left(2z+2\lambda\right)\left(2\bar{z}+2\bar{\lambda}\right)}{|2z+2\lambda|^2}$$

$$= \frac{4|\lambda|^2|z|^2 + 4\bar{\lambda}z + 4\lambda\bar{z} + 4 - 4|z|^2 - 4\bar{\lambda}z - 4\lambda\bar{z} - 4|\lambda|^2}{|2z+2\lambda|^2} = \frac{4\left(1-|z|^2\right)\left(1-|\lambda|^2\right)}{|2z+2\lambda|^2}$$

Proposition (1.4) :

If $\lambda \in U$, then σ take ∂U into ∂U .

Proof :

Let $z \in \partial U$, then $|z| = 1$, hence $|z|^2 = 1$. By (1.3) $|\sigma(z)|^2 - 1 = 0$, therefore $|\sigma(z)|^2 = 1$, hence $|\sigma(z)| = 1$, hence $\sigma(z) \in \partial U$, hence σ take ∂U into ∂U .

Definition(1.5): [7]

Let $\psi : U \to U$ be holomorphic map on U, ψ is called an inner map if $|\psi(z)| = 1$.

Proposition (1.6):

σ as an inner map .

Proof :

From (1.4) σ take ∂U into ∂U ., hence $|\sigma(z)| = 1$. By (1.5) σ as an inner map .

Remark(1.10) :

For $\lambda \in U$, we have $\sigma'(0) = \frac{|\lambda|^2 - 1}{\lambda^2}$, $\sigma'(\lambda) = \frac{|\lambda|^2 - 1}{4\lambda^2}$.

Section Two

Definition(2.1): [4]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ where $f^{\wedge}(n)$ denotes the Taylor coefficient of f .

Remark (2.2) :[1]

We can define an inner product of the Hardy space functions as follows:

f(z) = $\sum_{n=0}^{\infty} f^{\wedge}(n)\, z^n$ and $g(z) = \sum_{n=0}^{\infty} g^{\wedge}(n)\, z^n$, then the inner product of f and g is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f^{\wedge}(n)\, \overline{g^{\wedge}(z)}$$

Definition (2.3) :[11]

Let $\alpha \in U$ and define $K_\alpha(z) = \dfrac{1}{1-\overline{\alpha}z}$ $(z \in U)$. Since $\alpha \in U$ then $|\alpha| < 1$, hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $K_\alpha \in H^2$ and $K_\alpha(z) = \sum_{n=0}^{\infty} (\overline{\alpha})^n z^n$.

Definition(2.4) : [4]

Let $\psi : U \to U$ be holomorphic map on U, the composition operator C_ψ induced by ψ is defined on H^2 as follows $C_\psi f = f \circ \psi$ $(f \in H^2)$

Definition(2.5) : [2]

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Theorem (2.6) : [12]

If $\psi : U \to U$ is holomorphic map on U , then $f \circ \psi \in H^2$ and

$\|f \circ \psi\| \leq \sqrt{\dfrac{1+|\psi(0)|}{1-|\psi(0)|}}\ \|f\|$ for every $f \in H^2$.

The goal of this theorem $C_\psi : H^2 \to H^2$.

Definition(2.7) :

The composition operator C_σ induced by σ is defined on H^2 as follows $C_\sigma f = f \circ \sigma$

Proposition(2.8) :

Let $\lambda \in U$, for each $f \in H^2$ then $f \circ \sigma \in H^2$ and $\|f \circ \sigma\| \leq \sqrt{\dfrac{1+|\sigma(0)|}{1-|\sigma(0)|}}\ \|f\|$

Proof :

Since $\sigma : U \to U$ is holomorphic map on U, then by (2.6)

$f \circ \sigma \in H^2$ and $\|f \circ \sigma\| \leq \sqrt{\dfrac{1+|\sigma(0)|}{1-|\sigma(0)|}}\ \|f\|$, hence $C_\sigma : H^2 \to H^2$

Remark (2.9) : [4]

1) One can easily show that $C_\kappa C_\psi = C_{\psi\circ\kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$
$= C_{\psi\circ\psi\circ\cdots\circ\psi} = C_{\psi_n}$

2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.

3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition(2.12): [3]

Let T be an operator on a Hilbert space H , The operator T^* is the adjoint of T if $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x, y \in H$.

Theorem (2.13) : [5]

$\{K_\alpha\}_{\alpha\in U}$ forms a dense subset of H^2.

Theorem (2.14) : [11]

If $\psi : U \to U$ is holomorphic map on U, then for all $\alpha \in U$

$C^*_\psi K_\alpha = K_{\psi(\alpha)}$

Definition(2.15): [12]

Let H^∞ be the set of all bounded holomorphic maps on U .

Definition(2.16): [6]

Let $g \in H^\infty$, the Toeplits operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z)\ f(z) \quad (f \in H^2,\ z \in U)$$

Theorem (2.17) : [6]

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi T_g = T_{g\circ\psi} C_\psi \quad (g \in H^\infty)$

Remark (2.18) : [8]

For each $f \in H^2$, it is well- know that $T^*_h\ f = T_{\bar h}\ f$, such that $h \in H^\infty$.

Proposition(2.19) :

If $\lambda \in U$, then $C^*_\sigma = T_g\ C_\gamma\ T_h$, where $h(z) = (z + \lambda)$, $g(z) = \dfrac{1}{\bar\lambda - z}$, $\gamma(z) = \dfrac{\lambda z - 1}{\bar\lambda - z}$

Proof :

By (2-16), $T^*_h\ f = T_{\bar h}\ f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\left\langle T^*_h\ f, K_\alpha\right\rangle = \left\langle T_{\bar h} f, K_\alpha\right\rangle = \left\langle f, T^*_{\bar h}\ K_\alpha\right\rangle \cdots\cdots (2-1)$$

On the other hand ,

$$\left\langle T^*_h\ f, K_\alpha\right\rangle = \left\langle f, T_h K_\alpha\right\rangle = \left\langle f, h(\alpha) K_\alpha\right\rangle \cdots\cdots (2-2)$$

From (2-1)and (2-2) one can see that $T^*_{\bar h}\ K_\alpha = h(\alpha)\ K_\alpha$. Hence $T^*_h\ K_\alpha = \overline{h(\alpha)}\ K_\alpha$.

Calculation give :

$$C^*_\sigma K_\alpha(z) = K_{\sigma(\alpha)}(z) = \frac{1}{1 - \overline{\sigma(\alpha)}\ z} = \frac{1}{1 - \dfrac{(2\lambda\bar\alpha + 2)\ z}{2\bar\alpha + 2\bar\lambda}}$$

$$= \frac{1}{\dfrac{2\bar\alpha + 2\bar\lambda - 2\lambda\bar\alpha z - 2z}{2\bar\alpha + 2\bar\lambda}} = \frac{2\bar\alpha + 2\bar\lambda}{(2\bar\lambda - 2z) - \bar\alpha(2\lambda z - 2)}$$

$$= \overline{(\alpha+\lambda)} \cdot \frac{1}{\bar{\lambda}-z} \cdot \frac{1}{1-\bar{\alpha}\left(\frac{\lambda z-1}{\bar{\lambda}-z}\right)}$$

$$= \overline{h(\alpha)} \cdot T_g\, k_\alpha(\lambda(z)) = T_g\, \overline{h(\alpha)}\, k_\alpha(\gamma(z))$$

$$= T_g\, \overline{h(\alpha)}\, C_\gamma\, k_\alpha(z) = T_g\, C_\gamma\, \overline{h(\alpha)}\, k_\alpha(z)$$

$$= T_g\, C_\gamma\, T_h^*\, k_\alpha(z) \text{ , therefore}$$

$$C_\sigma^* k_\alpha(z) = T_g\, C_\gamma\, T_h^*\, k_\alpha(z) \quad (z \in U)\,.$$

But $\overline{\{K_\alpha\}_{\alpha\in U}} = H^2$, then $C_\sigma^* = T_g\, C_\gamma\, T_h^*$

<u>Definition (2.20)</u> : [13]

Let T be an operator on a Hilbert space H , T is called compact if every sequence x_n in H is weakly converges to x in H , then Tx_n is strongly converges to Tx .Moreover ($x_n \xrightarrow{W} x$ if $\langle x_n,u\rangle \to \langle x,u\rangle, \forall u\in H$ and $x_n \xrightarrow{S} x$ if $\|x_n - x\| \to 0$.)

<u>Theorem (2.21)</u> : [11]

If $\psi : U \to U$ is holomorphic map on U, then C_ψ is not compact if and only if ψ take ∂U into ∂U

<u>Proposition(2.22)</u> :

If $\lambda \in U$, then C_σ is not compact composition operator

<u>Proof :</u>

From (1.4) σ take ∂U into ∂U . By (2.21) C_σ is not compact composition operator .

<u>Theorem (2.23)</u> :

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi C_\sigma^*$ is compact if and only if $C_\psi C_\gamma$ is compact , where $C_\sigma^* = T_g\, C_\gamma\, T_h^*,\ \gamma(z) = \dfrac{\lambda z-1}{\bar{\lambda}-z}$

<u>Proof:</u>

Suppose that $C_\psi C_\gamma$ is compact . Note that

$$C_\psi C_\sigma^* = C_\psi\, T_g\, C_\gamma\, T_h^* \quad (\text{since } C_\sigma^* = T_g\, C_\gamma\, T_h^* \text{ by (2.19)})$$

$$= T_{g\circ\psi}\, C_\psi\, C_\gamma\, T_h^* \quad (\text{since } C_\psi\, T_g = T_{g\circ\psi} C_\psi \text{ by (2.17)}).$$

Since $C_\psi C_\gamma$ is compact operator , $T_{g\circ\psi}$ and T_h^* are bounded operators then $C_\psi C_\sigma^*$ is compact

Conversely , suppose that $C_\psi C_\sigma^*$ is compact . Note that

$$C_\psi C_\gamma = C_\psi \left(C_\gamma^*\right)^* = C_\psi \left(T_g\, C_\sigma\, T_h^*\right)^* = C_\psi\, T_h\, C_\sigma^*\, T_g^*$$

$$= T_{h\circ\psi}\, C_\psi\, C_\sigma^*\, T_g^* \quad (\text{since } C_\psi\, T_h = T_{h\circ\psi} C_\psi \text{ by (2.17)}).$$

Since $C_\psi C_\sigma^*$ is compact operator , $T_{h\circ\psi}$ and T_g^* are bounded operators then $C_\psi C_\gamma$ is compact .

Corollary (2.24) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\psi C_\sigma^*$ is not compact if and only if there exist points $z_1, z_2 \in \partial U$ such that $(\gamma \circ \psi)(z_1) = z_2$.

Proof:

By (2.23) $C_\psi C_\phi^*$ is not compact if and only if $C_\psi C_\gamma = C_{\gamma \circ \psi}$ is not compact . Since $\gamma : U \to U$ and $\psi : U \to U$ are holomorphics on U , then also $\gamma \circ \psi$. Thus by (2.21) $C_{\gamma \circ \psi}$ is not compact if and only if $\gamma \circ \psi$ take ∂U into ∂U . So, there exist points $z_1, z_2 \in \partial U$ such that $(\gamma \circ \psi)(z_1) = z_2$.

Theorem (2.25) :

If $\psi : U \to U$ is holomorphic on U, then $C_\sigma^* C_\psi$ is compact if and only if $C_\gamma C_\psi$ is compact , where $C_\sigma^* = T_g\ C_\gamma\ T_h^*$, $\gamma(z) = \dfrac{\lambda z - 1}{\bar{\lambda} - z}$

Proof:

Suppose that $C_\gamma C_\psi$ is compact . Note that

$$C_\sigma^* C_\psi = T_g\ C_\gamma\ T_h^*\ C_\psi \quad (\text{since } C_\sigma^* = T_g\ C_\gamma\ T_h^* \text{ by (2.19)})$$
$$= T_g\ C_\gamma\ T_{\bar{h}}\ C_\psi \quad (\text{by (2.18)})$$
$$= T_g\ T_{\bar{h} \circ \gamma}\ C_\lambda\ C_\psi \quad (\text{since } C_\gamma\ T_{\bar{h}} = T_{\bar{h} \circ \lambda} C_\gamma \text{ by (2.17)}).$$

Since $C_\gamma C_\psi$ is compact operator , T_g and $T_{\bar{h} \circ \gamma}$ are bounded operators then $C_\sigma^* C_\psi$ is compact

Conversely ,Suppose that $C_\sigma^* C_\psi$ is compact . Note that

$$C_\gamma C_\psi = (C_\gamma^*)^* C_\psi$$
$$= (T_g\ C_\sigma\ T_h^*)^* C_\psi \quad (\text{since } C_\gamma^* = T_g\ C_\sigma\ T_h^*)$$
$$= T_h\ C_\sigma^*\ T_g^*\ C_\psi$$

Note that , by (2.13) it is enough to prove the compactness on the family $\{K_\alpha\}_{\alpha \in U}$. Hence for each $z \in U$ we have

$$C_\gamma C_\psi K_\alpha(z) = T_h\ C_\sigma^*\ T_g^*\ C_\psi K_\alpha(z)$$
$$= T_h\ C_\sigma^*\ T_g^*\ K_\alpha(\psi(z))$$
$$= T_h\ C_\sigma^*\ \overline{g(\alpha)}\ K_\alpha(\psi(z)) \qquad (\text{since } T_g^*\ K_\alpha = \overline{g(\alpha)}\ K_\alpha)$$
$$= \overline{g(\alpha)}\ T_h\ C_\sigma^*\ K_\alpha(\psi(z))$$
$$= \overline{g(\alpha)}\ T_h\ C_\sigma^*\ C_\psi\ K_\alpha(z)$$

Since $C_\sigma^* C_\psi$ is compact , T_g is bounded and $g \in H^\infty$, then $C_\gamma C_\psi$ is compact .

Corollary (2.26) :

If $\psi : U \to U$ is holomorphic map on U, then $C_\sigma^* C_\psi$ is not compact if and only if there exist points $z_1, z_2 \in \partial U$ such that $(\psi \circ \gamma)(z_1) = z_2$.

Proof:

By (2.25) $C_\sigma^* C_\psi$ is not compact if and only if $C_\gamma C_\psi = C_{\psi \circ \gamma}$ is not compact . Since $\gamma : U \to U$ and $\psi : U \to U$ are holomorphics on U , then also $\psi \circ \gamma$. Thus by (2.21) $C_{\psi \circ \gamma}$ is not compact if and only if $\psi \circ \gamma$ take ∂U into ∂U . So, there exist points $z_1, z_2 \in \partial U$ such that $(\psi \circ \gamma)(z_1) = z_2$.

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حول تراص المؤثر التركيبي C_σ

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المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال التحليلية على U بحيث أن $f(z)=\sum_{n=0}^{\infty} f^\wedge(n)\, z^n$ و $\sum_{n=0}^{\infty} |f^\wedge(n)|^2 < \infty$ و $f^\wedge(n)$ يرمز إلى معاملات تيلر للدالة f

لتكن $\psi : U \to U$ دالة تحليلية على U ، المؤثر التركيبي المحتث من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_\psi f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث تراص المؤثر التركيبي المحتث من الدالة σ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي الغير مرصوص المحتث من الدالة σ . وأعطينا بعض المبرهنات على تراص المؤثرات التركيبية . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ.

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج .