The Complete Solution For Special Kinds of Nonlinear Second Order Partial Differential Equations With Three Independent Variables

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Abstract

The main aim of this search is to find the complete solution for special kinds of nonlinear second order partial differential equations with three independent variables , which have the general form

$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{xt} + D\,Z_{yt} + E\,Z_{yy} + F\,Z_{tt} + G\,Z_x + H\,Z_y + I\,Z_t + J\,Z = 0,$

where $A , B , C , D , E , F , G , H , I$ and J are functions of $x , y , t , Z , Z_x , Z_y , Z_t ,$ $Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt} , Z_{yy}$ and Z_{tt} . And $Z , Z_x , Z_y , Z_t , Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt} ,$ Z_{yy} and Z_{tt} in these functions are of first degree and not multiplied with together.

1. Introduction

The differential equations play important role in plenty of the fields of the sciences as Physics, Chemistry and other sciences, and therefore the plenty of the scientists were studying this subject and they are trying to find modern methods for getting rid up the difficulties that facing them in the solving of some of these equations.

The researcher Kudaer [4],2006 studied the linear second order ordinary differential equations ,which have the form $y'' + P(x)\,y' + Q(x)\,y = 0$, and used the assumption $y(x) = e^{\int Z(x)\,dx}$ to find the general solution of it , and the solution depends on the forms of *P(x)* and *Q(x)* .

The researcher Abd Al-Sada [2], 2006 studied the linear second order partial differential equations with constant coefficients and which have the form

$A\,Z_{xx} + B\,Z_{yy} + C\,Z_{xy} + D\,Z_x + E\,Z_y + F\,Z = 0$,

where A , B , C , D , E and F are arbitrary constants , and used the assumption $Z(x, y) = e^{\int u(x)\,dx + \int v(y)\,dy}$ to find the complete solution of it.

The researcher Hani [3],2008, studied the linear second order partial differential equations, which have three independent variables , and its general form

$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{xt} + D\,Z_{yt} + E\,Z_{yy} + F\,Z_{tt} + G\,Z_x + H\,Z_y + I\,Z_t + J\,Z = 0,$

where *A,B,C,…,I* and *J* are arbitrary constants , and used the assumption $Z(x, y, t) = e^{\int u(x)dx + \int v(y)dy + \int w(t)dt}$ to find the complete solution of it .

The researcher Hanoon [5], 2009, studied the linear second order partial differential equations, with variable coefficients which have the form

$$A(x,y)\,Z_{xx} + B(x,y)\,Z_{yy} + C(x,y)\,Z_{xy} + D(x,y)\,Z_x + E(x,y)\,Z_y + F(x,y)\,Z = 0,$$

where A , B , C , D , E and F are functions of x or y or both x and y . To solve this kind of equations , she used the assumptions

$$Z(x,y) = e^{\int \frac{u(x)}{x}\,dx + \int v(y)\,dy},\ Z(x,y) = e^{\int u(x)\,dx + \int \frac{v(y)}{y}\,dy} \text{ and } Z(x,y) = e^{\int \frac{u(x)}{x}\,dx + \int \frac{v(y)}{y}\,dy},$$

these assumptions represent the complete solution of the above last equation .

The researcher Mohsin [1], 2010, studied the nonlinear second order partial differential equations, of homogeneous degree which have the general form
$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{yy} + D\,Z_x + E\,Z_y + F\,Z = 0$,
where A, B, C, D, E and F are linear functions of dependent variable Z and its partial derivatives with respect to the independent variables x and y, she using the following assumptions

$$Z(x,y) = e^{\int u(x)dx + \int v(y)dy},\ Z(x,y) = e^{\int \frac{u(x)}{x}dx + \int v(y)dy},\ Z(x,y) = e^{\int u(x)dx + \int \frac{v(y)}{y}dy},$$

and $Z(x,y) = e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy}$ to find the complete solutions of the above last kind equation.

In our work, we will solve the nonlinear second order partial differential equations , which have the general form
$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{xt} + D\,Z_{yt} + E\,Z_{yy} + F\,Z_{tt} + G\,Z_x + H\,Z_y + I\,Z_t + J\,Z = 0$,
by using the assumptions

$$Z(x,y,t) = e^{\int u(x)dx + \int v(y)dy + \int w(t)dt},\ Z(x,y,t) = e^{\int u(x)dx + \int v(y)dy + \int \frac{w(t)}{t}dt},$$

$$Z(x,y,t) = e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy + \int w(t)dt} \text{ and } Z(x,y,t) = e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy + \int \frac{w(t)}{t}dt},$$

where A , B , C , D , E , F , G , H , I and J are functions of x , y , Z , Z_x , Z_y , Z_t, Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt} , Z_{yy} and Z_{tt} . And Z , Z_x , Z_y , Z_t , Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt}, Z_{yy} and Z_{tt} in these functions are of first degree and not multiplied with together. And by the above last assumptions we will get the complete solutions of the above last equation.

2. The Suggested method for getting the Complete Solution of some Nonlinear second order partial differential equations with Three Independent Variables

Let us consider the general form of nonlinear second order partial differential equation with three independent variables:-
$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{xt} + D\,Z_{yt} + E\,Z_{yy} + F\,Z_{tt} + G\,Z_x + H\,Z_y + I\,Z_t + J\,Z = 0$,
where A , B , C , D , E , F , G , H , I and J are functions of x , y , Z , Z_x , Z_y , Z_t,

Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt} , Z_{yy} and Z_{tt} . And Z , Z_x , Z_y , Z_t , Z_{xx} , Z_{xy} , Z_{xt} , Z_{yt} , Z_{yy} and Z_{tt} in these functions are of first degree and not multiplied with together.

So, for this purpose we will search about functions $\mathrm{u}(x)$, $\mathrm{v}(y)$ and $\mathrm{w}(t)$ such that the assumptions

$Z(x,y,t)=e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$, $Z(x,y,t)=e^{\int u(x)dx+\int v(y)dy+\int \frac{w(t)}{t}dt}$,

$Z(x,y,t)=e^{\int \frac{u(x)}{x}dx+\int \frac{v(y)}{y}dy+\int w(t)dt}$ and $Z(x,y,t)=e^{\int \frac{u(x)}{x}dx+\int \frac{v(y)}{y}dy+\int \frac{w(t)}{t}dt}$

give the complete solution to the above equation and to do this we will classify the above equation to the following cases (the cases have so much big numbers):-

<u>Case(1) </u>:

$$\mathrm{A}_1 Z_t Z_{xx} + \mathrm{A}_2 Z_y Z_{xy} + \mathrm{A}_3 Z Z_{tt} + \mathrm{A}_4 Z_x Z_{yy} = 0 .$$

<u>Case(2)</u> :

$$\mathrm{A}_1 t Z_y Z_{xt} + \mathrm{A}_2 Z_y Z_x + \mathrm{A}_3 t Z_t Z_{yy} + \mathrm{A}_4 t^2 Z_{xy} Z_{tt} = 0 .$$

<u>Case(3)</u> :

$$\mathrm{A}_1 Z Z_{tt} + \mathrm{A}_2 y^2 Z Z_{yy} + \mathrm{A}_3 x^2 Z_x^2 + \mathrm{A}_4 y^2 Z_y^2 = 0 .$$

<u>Case(4)</u> :

$$\mathrm{A}_1 x^2 \mathrm{y}^2 Z_{xx} Z_{yy} + \mathrm{A}_2 t^2 y Z_y Z_{tt} + \mathrm{A}_3 x^2 t^2 Z_{xt}^2 + \mathrm{A}_4 x y Z Z_{xy} = 0 .$$

Where A_1 , A_2 , A_3 and A_4 are real constants.

Now we will find the complete solution of these cases as follows :

<u>Case(1) </u>: By using the assumption

$Z(x,y,t)=e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$, we get

$$Z_x = \mathrm{u}(x)\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$\Rightarrow Z_{xx} = (\mathrm{u}'(x)+\mathrm{u}^2(x))\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$Z_{xy} = \mathrm{u}(x)\, \mathrm{v}(y) e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$Z_y = \mathrm{v}(y)\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$\Rightarrow Z_{yy} = (\mathrm{v}'(y)+\mathrm{v}^2(x))\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$Z_t = \mathrm{w}(t)\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

$$\Rightarrow Z_{tt} = (\mathrm{w}'(t)+\mathrm{w}^2(t))\, e^{\int u(x)dx+\int v(y)dy+\int w(t)dt}$$

So, the equation

$$A_1 Z_t Z_{xx} + A_2 Z_y Z_{xy} + A_3 Z Z_{tt} + A_4 Z_x Z_{yy} = 0.$$

Transforms to the form

$$\begin{pmatrix} A_1 (\mathrm{w}(t)(\mathrm{u}'(x)+\mathrm{u}^2(x))) + A_2(\mathrm{v}^2(y)\,\mathrm{u}(x)) + \\ A_3(\mathrm{w}'(t)+\mathrm{w}^2(t)) + A_4(\mathrm{u}(x)(\mathrm{v}'(y)+\mathrm{v}^2(y)) \end{pmatrix} e^{2[\int u(x)dx+\int v(y)dy+\int w(t)dt]} = 0,$$

Since $e^{2[\int u(x)dx+\int v(y)dy+\int w(t)dt]} \neq 0$

So,

$$A_1(\mathrm{w}(t)(\mathrm{u}'(x)+\mathrm{u}^2(x))) + A_2(\mathrm{v}^2(y)\,\mathrm{u}(x)) + A_3(\mathrm{w}'(t)+\mathrm{w}^2(t)) + A_4(\mathrm{u}(x)(\mathrm{v}'(y)+\mathrm{v}^2(y)) = 0$$

Here we can't separate the variables, so we suppose that $\mathrm{u}(x)=\lambda_1$ where λ_1 is an arbitrary constant, and the last equation becomes :

$$A_1\lambda_1^2\mathrm{w}(t) + A_2\,\lambda_1\,\mathrm{v}^2(y) + A_3(\mathrm{w}'(t)+\mathrm{w}^2(t)) + A_4\lambda_1((\mathrm{v}'(y)+\mathrm{v}^2(y)) = 0 \qquad \ldots(1)$$

This equation is variables separable equation [6], we can solve it as follows :

$$A_3(\mathrm{w}'(t)+\mathrm{w}^2(t)) + A_1\lambda_1^2\mathrm{w}(t) = -[A_4\lambda_1((\mathrm{v}'(y)+\mathrm{v}^2(y)) + A_2\,\lambda_1\,\mathrm{v}^2(y)] = \lambda_2^2$$

So $\mathrm{w}'(t)+\mathrm{w}^2(t)) + \dfrac{A_1\lambda_1^2}{A_3}\mathrm{w}(t) - \dfrac{\lambda_2^2}{A_3} = 0$; λ_2 is constant , $A_3 \neq 0$

Let $B_1 = \dfrac{A_1\lambda_1^2}{A_3}$ and $B_2 = \dfrac{-\lambda_2^2}{A_3}$,

then the last equation becomes

$$\mathrm{w}'(t)+\mathrm{w}^2(t)+B_1\mathrm{w}(t)+B_2 = 0 \qquad \ldots(2)$$

Also $\dfrac{A_4}{A_2+A_4}\mathrm{v}'(y)+\mathrm{v}^2(y)+\dfrac{\lambda_2^2}{(A_2+A_4)\lambda_1} = 0$; $A_2 \neq -A_4$ *and* $\lambda_1 \neq 0$

Let $C_1 = \sqrt{\dfrac{A_4}{A_2+A_4}}$ and $C_2 = \dfrac{\lambda_2}{\sqrt{(A_2+A_4)\lambda_1}}$, then the last equation becomes

$$C_1^2\mathrm{v}'(y)+\mathrm{v}^2(y)) + C_2^2 = 0 \qquad \ldots(3)$$

The equation (2) is variables separable [6], we can solve it as follows:

$$\frac{dw}{\left(w(t)+\frac{B_1}{2}\right)^2 - d_1^2} + dt = 0 \qquad ; \ d_1^2 = \frac{B_1^2}{4} - B_2$$

i) If $\frac{B_1^2}{4} \neq B_2$, then by integrating both sides of last equation we get

$$-\frac{1}{d_1}\tanh^{-1}\left(\frac{w(t)+\frac{B_1}{2}}{d_1}\right) = b_1 - t \qquad ;(-1 < \left(\frac{w(t)+\frac{B_1}{2}}{d_1}\right) < 1)$$

$$\Rightarrow \quad w(t) = d_1 \tanh(d_1 t - d_1 b_1) - \frac{B_1}{2}$$

ii) If $\frac{B_1^2}{4} = B_2$, then

$$\frac{-1}{w(t)+\frac{B_1}{2}} = b_2 - t \qquad \Rightarrow \qquad w(t) = \frac{1}{t-b_2} - \frac{B_1}{2}$$

Also, equation(3) is variables separable [6], we can solve it as follows :

$$\frac{C_1^2\, dv}{v^2(y)+C_2^2} + dy = 0 \quad \Rightarrow \quad \frac{C_1^2}{C_2}\tan^{-1}\left(\frac{v(y)}{C_2}\right) = C - y \ ; C \text{ is constant and } C_2 \neq 0$$

$$\Rightarrow \quad v(y) = C_2 \tan\left(\frac{C_2}{C_1^2}C - \frac{C_2}{C_1^2}y\right)$$

So, the complete solution of equation (1), is given by :

i) If $\frac{B_1^2}{4} \neq B_2$, we get

$$Z(x,y,t) = e^{\int \lambda_1 dx + \int C_2 \tan\left(\frac{C_2}{C_1^2}C - \frac{C_2}{C_1^2}y\right)dy + \int (d_1 \tanh(d_1 t - d_1 b_1) - \frac{B_1}{2})dt}$$

$$= e^{\lambda_1 x + C_1^2 \ln\cos\left(\frac{C_2}{C_1^2}C - \frac{C_2}{C_1^2}y\right) + \ln\cosh(d_1 t - d_1 b_1) - \frac{B_1}{2}t + g}$$

$$= K e^{\lambda_1 x - \frac{B_1}{2}t}\left(\cos\left(\frac{C_2}{C_1^2}C - \frac{C_2}{C_1^2}y\right)\right)^{C_1^2}\cosh(d_1 t - d_1 b_1) \qquad ; \quad K = e^g$$

$$= K e^{\lambda_1 x - \frac{B_1}{2} t} \left(\cos\left(\frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} C - \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} y\right) \right)^{\frac{A_4}{A_2 + A_4}}$$

$$\cosh\left(\sqrt{\frac{A_1^2 \lambda_1^4}{4 A_3^2} + \frac{\lambda_2^2}{A_3}}\ t - b_1 \sqrt{\frac{A_1^2 \lambda_1^4}{4 A_3^2} + \frac{\lambda_2^2}{A_3}}\right) \quad ; A_4, A_3, \lambda_1 \neq 0$$

$$Z(x,y,t) = e^{\lambda_1 x - \frac{B_1}{2} t} \left(K_1 \cos \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} y + K_2 \sin \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} y \right)^{\frac{A_4}{A_2 + A_4}}$$

$$\left(K_3 \cosh \frac{1}{2A_3} \sqrt{A_1^2 \lambda_1^4 + 4\lambda_2^2 A_3}\ t + K_4 \sinh \frac{1}{2A_3} \sqrt{A_1^2 \lambda_1^4 + 4\lambda_2^2 A_3}\ t \right)$$

Where $K_1 = K^{\frac{A_2 + A_4}{A_4}} \cos \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} C$, $K_2 = K^{\frac{A_2 + A_4}{A_4}} \sin \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} C$,

$K_3 = \cosh b_1 \frac{1}{2A_3} \sqrt{A_1^2 \lambda_1^4 + 4\lambda_2^2 A_3}$, $K_4 = \sinh b_1 \frac{1}{2A_3} \sqrt{A_1^2 \lambda_1^4 + 4\lambda_2^2 A_3}$,

λ_1 and λ_2 are arbitrary constants.

ii) If $\frac{B_1^2}{4} = B_2$, we get

$$Z(x,y,t) = e^{\int \lambda_1 dx + \int C_2 \tan\left(\frac{C_2}{C_1^2} C - \frac{C_2}{C_1^2} y\right) dy + \int \left(\frac{1}{t - b_2} - \frac{B_1}{2}\right) dt}$$

$$= e^{\lambda_1 x + C_1^2 \ln \cos\left(\frac{C_2}{C_1^2} C - \frac{C_2}{C_1^2} y\right) + \ln(t - b_2) - \frac{B_1}{2} t + g}$$

$$= K e^{\lambda_1 x - \frac{B_1}{2} t} (t - b_2) \left(\cos\left(\frac{C_2}{C_1^2} C - \frac{C_2}{C_1^2} y\right) \right)^{C_1^2} \quad ; \quad K = e^g$$

$$= e^{\lambda_1 x - \frac{B_1}{2} t} (t - b_2) \left(K_1 \cos \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} y + K_2 \sin \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} y \right)^{\frac{A_4}{A_2 + A_4}} , \ A_4, \lambda_1 \neq 0$$

Where $K_1 = K^{\frac{A_2 + A_4}{A_4}} \cos \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} C$, $K_2 = K^{\frac{A_2 + A_4}{A_4}} \sin \frac{\lambda_2 \sqrt{A_2 + A_4}}{A_4 \sqrt{\lambda_1}} C$,

λ_1 and λ_2 are arbitrary constants.

<u>Case(2) </u>: By using the assumption

$$Z(x,y,t)=e^{\int u(x)dx+\int v(y)dy+\int \frac{w(t)}{t}dt}\quad \text{, we get}$$

$$Z_{xy}=\frac{u(x)\,w(t)}{t}\,e^{\int u(x)dx+\int v(y)dy+\int \frac{w(t)}{t}dt}$$

$$Z_t=\frac{w(t)}{t}\,e^{\int u(x)dx+\int v(y)dy+\int \frac{w(t)}{t}dt}$$

$$\Rightarrow\ Z_{tt}=\left(\frac{t\,w'(t)+w^2(t)-w(t)}{t^2}\right)e^{\int u(x)dx+\int v(y)dy+\int \frac{w(t)}{t}dt}$$

And by using Z_x, Z_{xx}, Z_y, Z_{yy} and Z_{xy} from the case(1), then the equation

$$A_1\,t\,Z_y\,Z_{xt}+A_2\,Z_y\,Z_x+A_3\,t\,Z_t\,Z_{yy}+A_4\,t^2 Z_{xy}Z_{tt}=0.$$

Transforms to the form

$$\left(\begin{array}{l} A_1\,t\,(\frac{v(y)\,u(x)\,w(t)}{t})+A_2 v(y)\,u(x)+ \\ A_3 t\,(\frac{w(t)}{t})(v'(y)+v^2(y)+A_4 t^2 u(x)\,v(y)(\frac{t\,w'(t)+w^2(t)-w(t)}{t^2}) \end{array}\right)e^{2[\int u(x)dx+\int v(y)dy+\int w(t)dt]}=0,$$

Since $e^{2[\int u(x)dx+\int v(y)dy+\int w(t)dt]}\neq 0$
So,

$$\begin{array}{l} A_1\,v(y)\,u(x)\,w(t)+A_2 v(y)\,u(x)+A_3 w(t)(v'(y)+v^2(y)+ \\ A_4 u(x)\,v(y)(t\,w'(t)+w^2(t)-w(t))=0 \end{array}\qquad \ldots(4)$$

Here we can't separate the variables, so we suppose that $w(t)=\lambda_1$ where λ_1 is an arbitrary constant, and the last equation becomes :

$$A_1\,\lambda_1 v(y)\,u(x)+A_2 v(y)\,u(x)+A_3\,\lambda_1(v'(y)+v^2(y))+A_4(\lambda_1^2-\lambda_1)u(x)\,v(y)=0$$

This equation is variables separable [6], we can solve it as follows :

$$[A_1\,\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)]\,u(x)=-\frac{A_3\,\lambda_1(v'(y)+v^2(y))}{v(y)}=\lambda_2^2$$

Therefore $\quad u(x)=\dfrac{\lambda_2^2}{A_1\,\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}\quad ;A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)\neq 0$

Also, $\quad v'(y)+v^2(y)+\dfrac{\lambda_2^2}{A_3\,\lambda_1}v(y)=0\quad ;\lambda_1$ *and* $A_3\neq 0$

This equation is similar to Bernoulli equation [7], then the solution of it is given by :

$$v(y)=\frac{e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}}{\int e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}\,dy}$$

So, the complete solution of equation (4), is given by :

$$Z(x,y,t)=e^{\int\frac{\lambda_2^2}{A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}dx+\int\frac{e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}}{\int e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}dy}dy+\int\frac{\lambda_1}{t}dt}$$

$$=e^{\frac{\lambda_2^2}{A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}x+\ln(\int e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}dy)+\lambda_1\ln t+g}\ ;\ t>0$$

$$=K\,t^{\lambda_1}\,e^{\frac{\lambda_2^2}{A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}x}\left(\int e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}dy\right)\qquad ;\quad K=e^{g}$$

$$=K\,t^{\lambda_1}\,e^{\frac{\lambda_2^2}{A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}x}\left(-\frac{A_3\lambda_1}{\lambda_2^2}e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}+d\right)$$

$$=t^{\lambda_1}\,e^{\frac{\lambda_2^2}{A_1\lambda_1+A_2+A_4(\lambda_1^2-\lambda_1)}x}\left(K_1e^{-\frac{\lambda_2^2}{A_3\lambda_1}y}+K_2\right)\quad ;\ A_3\lambda_1\neq 0$$

Where $K_1=-K\frac{A_3\lambda_1}{\lambda_2^2}$, $K_2=K\,d$, λ_1 and λ_2 are arbitrary constants.

Case(3) : By using the assumption

$Z(x,y,t)=e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int w(t)dt}$, we get

$$Z_x=\frac{u(x)}{x}e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int w(t)dt}$$

$$Z_y=\frac{v(y)}{y}e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int w(t)dt}$$

$$\Rightarrow \quad Z_{yy} = \left(\frac{y\,v'(y) + v^2(y) - v(y)}{y^2}\right) e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy + \int w(t)dt}$$

And using Z_{tt} from the case(1), then the equation

$$A_1 \; Z\, Z_{tt} + A_2 \; y^2 Z Z_{yy} \; + A_3 \, x^2 \, Z_x^2 \; + A_4 \; y^2 Z_y^2 = 0 .$$

Transforms to the form

$$\begin{pmatrix} A_1 \; (w'(t) + w^2(t)) + A_2 (v'(y) + v^2(y) - v(y)) + \\ A_3 \, u^2(x) + A_4 v^2(y) \end{pmatrix} e^{2[\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy + \int w(t)dt]} = 0 ,$$

Since $\quad e^{2[\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy + \int w(t)dt]} \neq 0$

So, $A_1 \; (w'(t) + w^2(t)) + A_2 (y\, v'(y) + v^2(y) - v(y)) + A_3 \, u^2(x) + A_4 v^2(y) = 0 \quad \ldots(5)$

This equation is variables separable [6], we can solve it as follows :

Let $A_1 \; (w'(t) + w^2(t)) = \lambda_1^2 \;\;, \;\; A_2 (y\, v'(y) + v^2(y) - v(y)) + A_4 v^2(y) = \lambda_2^2$,

Therefore, $\quad A_3 \, u^2(x) = -(\lambda_1^2 + \lambda_2^2) \quad \Rightarrow \; u(x) = \mp \sqrt{\frac{\lambda_1^2 + \lambda_2^2}{A_3}}\, i$

Also $\quad w'(t) + w^2(t) - \frac{\lambda_1^2}{A_1} = 0 \quad ; A_1 \neq 0 \qquad \ldots(6)$

And $\quad y\, v'(y) + (\frac{A_4}{A_2} + 1) v^2(y) - v(y) - \frac{\lambda_2^2}{A_2} = 0 \;\; ; A_1 \neq 0$

$$\Rightarrow \; \frac{A_2}{A_2 + A_4} y\, v'(y) + v^2(y) - \frac{A_2}{A_2 + A_4} v(y) - \frac{\lambda_2^2}{A_2 + A_4} = 0 \;\; ; A_2 \neq A_4$$

Let $B_1 = \frac{A_2}{A_2 + A_4}$ and $B_2 = \frac{\lambda_2^2}{A_2 + A_4}$ then the last equation becomes

$$B_1 y\, v'(y) + v^2(y) - B_1 v(y) - B_2 = 0 \qquad \ldots(7)$$

The equation(6) is variables separable [6], we can solve it as follows :

$$\frac{dw}{w^2(t) - \frac{\lambda_1^2}{A_1}} + dt = 0 \qquad \Rightarrow -\frac{\sqrt{A_1}}{\lambda_1} \tanh^{-1}(\frac{\sqrt{A_1}}{\lambda_1} w(t)) = c - t$$

$$\Rightarrow \quad w(t) = \frac{\lambda_1}{\sqrt{A_1}} \tanh(\frac{\lambda_1}{\sqrt{A_1}} t - \frac{\lambda_1}{\sqrt{A_1}} c)$$

Also , equation(7) is variables separable [6], we can solve it as follows :

$$\frac{B_1\, dv}{\left(v(y)-\frac{B_1}{2}\right)^2 - d^2} + \frac{dy}{y} = 0 \qquad ;\ d^2 = B_2 + \frac{B_1^2}{4}$$

i) If $B_2 \neq -\frac{B_1^2}{4}$, we get

$$-\frac{B_1}{d}\tanh^{-1}\left(\frac{v(y)-\frac{B_1}{2}}{d}\right) = -\ln(c\,y) \qquad ; \quad \left|\frac{v(y)-\frac{B_1}{2}}{d}\right| < 1 \quad \textit{and}\ c\,y > 0$$

$$\Rightarrow \qquad v(y) = d\tanh\left(\frac{d}{B_1}\ln c\,y\right) + \frac{B_1}{2}$$

ii) If $B_2 = -\frac{B_1^2}{4}$, we get

$$\frac{-B_1\, dv}{(v(y)-\frac{B_1}{2})^2} + \frac{dy}{y} = 0 \qquad \Rightarrow \qquad \frac{-B_1}{(v(y)-\frac{B_1}{2})} = -\ln c\,y \quad ; c\,y > 0$$

$$\Rightarrow \ v(y) = \frac{B_1}{\ln c\,y} + \frac{B_1}{2}$$

Then the complete solution of the equation(5), is given by :

i) If $B_2 \neq -\frac{B_1^2}{4}$, we get

$$Z(x,y,t) = e^{\int \frac{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\, i}{x} dx + \int \frac{d\tanh(\frac{d}{B_1}\ln c\,y)+\frac{B_1}{2}}{y} dy + \int \frac{\lambda_1}{\sqrt{A_1}} \tanh(\frac{\lambda_1}{\sqrt{A_1}} t - \frac{\lambda_1}{\sqrt{A_1}} c)\, dt}$$

$$= e^{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\, i \ln x + B_1 \ln\cosh(\frac{d}{B_1}\ln c\,y) + \frac{B_1}{2}\ln y + \ln\cosh(\frac{\lambda_1}{\sqrt{A_1}} t - \frac{\lambda_1}{\sqrt{A_1}} c) + g}$$

$$= x^{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\, i}\ y^{\frac{A_2}{2(A_2+A_4)}} \left(\cosh\sqrt{\frac{4(A_1+A_2)\lambda_2^2 + A_2^2}{4A_2^2}}\ \ln cy\right)^{\frac{A_2}{A_2+A_4}}$$

$$\left(F_1 \cosh\frac{\lambda_1}{\sqrt{A_1}} t + F_2 \sinh\frac{\lambda_1}{\sqrt{A_1}} t\right)$$

Where $F_1 = e^g \cosh\frac{\lambda_1}{\sqrt{A_1}} c$, $F_2 = e^g \sinh\frac{\lambda_1}{\sqrt{A_1}} c$, ; A_1, A_2, A_3 *and* $c \neq 0$

λ_1, λ_2 and c are arbitrary constants.

ii) If $B_2 = -\frac{B_1^2}{4}$, we get

$$Z(x,y,t)=e^{\int \frac{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\,i}{x}dx+\int\left(\frac{B_1}{y\ln c\,y}+\frac{B_1}{2y}\right)dy+\int \frac{\lambda_1}{\sqrt{A_1}}\tanh\left(\frac{\lambda_1}{\sqrt{A_1}}t-\frac{\lambda_1}{\sqrt{A_1}}c\right)dt}$$

$$=e^{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\,i\ln x+B_1\ln(\ln c\,y)+\frac{B_1}{2}\ln y+\ln\cosh\left(\frac{\lambda_1}{\sqrt{A_1}}t-\frac{\lambda_1}{\sqrt{A_1}}c\right)+g}$$

$$=F\,x^{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\,i}\;y^{\frac{B_1}{2}}\;(\ln cy)^{B_1}\cosh\left(\frac{\lambda_1}{\sqrt{A_1}}t-\frac{\lambda_1}{\sqrt{A_1}}c\right)\qquad ;\ F=e^{g}$$

$$=x^{\mp\sqrt{\frac{\lambda_1^2+\lambda_2^2}{A_3}}\,i}\;y^{\frac{3A_2}{2(A_2+A_4)}}\;\left(F_1\cosh\frac{\lambda_1}{\sqrt{A_1}}t+F_2\sinh\frac{\lambda_1}{\sqrt{A_1}}t\right)\qquad ;\ A_1, A_2\ and\ A_3\neq 0$$

Where $F_1=F\,c^{\frac{A_2}{A_2+A_4}}\cosh\frac{\lambda_1}{\sqrt{A_1}}c$, $F_2=F\,c^{\frac{A_2}{A_2+A_4}}\sinh\frac{\lambda_1}{\sqrt{A_1}}c$,

λ_1, λ_2 and c are arbitrary constants.

<u>Case(4) :</u> By using the assumption

$Z(x,y,t)=e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt}$, we get

$$Z_{xx}=\left(\frac{x\,u'(x)+u^2(x)-v(x)}{x^2}\right)e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt}$$

$$Z_{xt}=\frac{u(x)\,w(t)}{x\,t}\,e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt}$$

$$Z_{xy}=\frac{u(x)\,v(y)}{x\,y}\,e^{\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt}$$

And by using $Z_y, Z_{yy}\ Z_{tt}$ from the above cases, then the equation

$$A_1\,x^2\,y^2\ Z_{xx}\,Z_{yy}+A_2\,t^2\,y\,Z_y\,Z_{tt}+A_3\,x^2\,t^2Z_{xt}^2+A_4\,x\,y\,Z\,Z_{xy}=0.$$

Transforms to the form

$$\begin{pmatrix}A_1\,(xu'(x)+u^2(x)-u(x))(yv'(y)+v^2(y)-v(y))+\\ A_2\,v(y)(tw'(t)+w^2(t)-w(t))+A_3u^2(x)w^2(t)+A_4u(x)v(y)\end{pmatrix}e^{2[\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt]}=0,$$

Since $e^{2[\int\frac{u(x)}{x}dx+\int\frac{v(y)}{y}dy+\int\frac{w(t)}{t}dt]}\neq 0$

So $$A_1\,(xu'(x)+u^2(x)-u(x))(yv'(y)+v^2(y)-v(y))+ A_2\,v(y)(tw'(t)+w^2(t)-w(t))+A_3u^2(x)w^2(t)+A_4u(x)v(y)=0 \qquad \ldots(8)$$

Here we can't separate the variables, so we suppose that v(y)= λ_1 and w(t)= λ_2 where λ_1 and λ_2 are arbitrary constants, and the last equation becomes :

$$A_1\,(\lambda_1^2-\lambda_1)(xu'(x)+u^2(x)-u(x))+A_2\,\lambda_1(\lambda_2^2-\lambda_2)+$$

$$A_3\,\lambda_2^2u^2(x)+A_4\lambda_1u(x)=0$$

$$\Rightarrow A_1\,(\lambda_1^2-\lambda_1)xu'(x)+(A_1\,(\lambda_1^2-\lambda_1)u^2(x)-u(x))+A_2\,\lambda_1(\lambda_2^2-\lambda_2)+$$

$$A_3\,\lambda_2^2u^2(x)+A_4\lambda_1u(x)=0$$

So $$A_1\,(\lambda_1^2-\lambda_1)\,xu'(x)+(A_1\,(\lambda_1^2-\lambda_1)+A_3\,\lambda_2^2)\,u^2(x)+$$

$$(A_4\lambda_1-A_1\,(\lambda_1^2-\lambda_1))\,u(x)+A_2\,\lambda_1(\lambda_2^2-\lambda_2)=0$$

Let $B_1=\dfrac{A_1\,(\lambda_1^2-\lambda_1)}{A_1\,(\lambda_1^2-\lambda_1)+A_3\,\lambda_2^2}$, $B_2=\dfrac{A_4\,\lambda_1-A_1\,(\lambda_1^2-\lambda_1)}{A_1\,(\lambda_1^2-\lambda_1)+A_3\,\lambda_2^2}$

and $B_3=\dfrac{A_2\,\lambda_1(\lambda_2^2-\lambda_2)}{A_1\,(\lambda_1^2-\lambda_1)+A_3\,\lambda_2^2}$; $A_1\,(\lambda_1^2-\lambda_1)+A_3\,\lambda_2^2\neq 0$,

then the last equation becomes

$$B_1\,xu'(x)+u^2(x)+B_2\,u(x))+B_3=0 \qquad \ldots(9)$$

This equation is variables separable [6], we can solve it as follows :

$$\frac{B_1\,du}{\left(u(x)+\frac{B_2}{2}\right)^2+d^2}+\frac{dx}{x}=0 \qquad ;\ d^2=B_3-\frac{B_2^2}{4}$$

i) If $B_3\neq\dfrac{B_2^2}{4}$, we get

$$\frac{B_1}{d}\tan^{-1}\left(\frac{u(x)+\frac{B_2}{2}}{d}\right)=-\ln(c\,x);\quad c\,x>0$$

$$\Rightarrow \quad u(x)=-d\tan\left(\frac{d}{B_1}\ln c\,x\right)-\frac{B_2}{2}\ ;\ B_1\neq 0$$

ii) If $B_3=\dfrac{B_2^2}{4}$, we get

$$\frac{B_1\, du}{(u(x)+\frac{B_2}{2})^2}+\frac{dx}{x}=0 \qquad \Rightarrow \qquad \frac{-B_1}{(u(x)+\frac{B_2}{2})}=-\ln c\,x\,;\, c\,x>0$$

$$\Rightarrow\ u(x)=\frac{B_1}{\ln c\,x}-\frac{B_2}{2}$$

Then the complete solution of the equation(8), is given by :

i) If $B_3 \neq \frac{B_2^2}{4}$, we get

$$Z(x,y,t)=e^{\int \frac{-d\tan(\frac{d}{B_1}\ln c\,x)-\frac{B_2}{2}}{x}dx+\int\frac{\lambda_1}{y}dy+\int\frac{\lambda_2}{t}dt}$$

$$=e^{B_1\ln\cos(\frac{d}{B_1}\ln c\,x)-\frac{B_2}{2}\ln x+\lambda_1\ln y+\lambda_2\ln t+g} \qquad ;cx,x,y\ and\ t>0$$

$$=F\,x^{-\frac{B_2}{2}}\,y^{\lambda_1}\,t^{\lambda_2}\,(\cos(\frac{d}{B_1}\ln c\,x))^{B_1} \qquad ;\ F=e^{g}\ \text{and}\ B_1\neq 0$$

$$=F\,y^{\lambda_1}t^{\lambda_2}\,x^{\frac{A_1(\lambda_1^2-\lambda_1)-A_4\lambda_1}{2A_1(\lambda_1^2-\lambda_1)+2A_3\lambda_2^2}}$$

$$\left(\cos\left(\frac{1}{2(A_1(\lambda_1^2-\lambda_1))}\sqrt{4(A_1(\lambda_1^2-\lambda_1)+A_3\lambda_2^2)(A_2\lambda_1(\lambda_2^2-\lambda_2)-(A_4\lambda_1-A_1(\lambda_1^2-\lambda_1))^2}\ \ln c\,x\right)\right)^{\frac{A_1(\lambda_1^2-\lambda_1)}{A_1(\lambda_1^2-\lambda_1)+A_3\lambda_2^2}}$$

$;\ A_1, A_3, c\neq 0$

Where F,λ_1,λ_2 and c are arbitrary constants.

ii) If $B_3=\frac{B_2^2}{4}$, we get

$$Z(x,y,t)=e^{\int(\frac{B_1}{x\ln c\,x}-\frac{B_2}{2x})dx+\int\frac{\lambda_1}{y}dy+\int\frac{\lambda_2}{t}dt}$$

$$=e^{B_1\ln\ln c\,x-\frac{B_2}{2}\ln x+\lambda_1\ln y+\lambda_2\ln t+g} \qquad ;cx,x,y\ and\ t>0$$

$$=F\,x^{-\frac{B_2}{2}}\,y^{\lambda_1}\,t^{\lambda_2}\,(\ln c\,x)^{B_1} \qquad ;\ F=e^{g}$$

$$=F\,y^{\lambda_1}\,t^{\lambda_2}\,x^{\frac{A_1(\lambda_1^2-\lambda_1)-A_4\lambda_1}{2A_1(\lambda_1^2-\lambda_1)+2A_3\lambda_2^2}}\,(\ln c\,x)^{\frac{A_1(\lambda_1^2-\lambda_1)}{A_1(\lambda_1^2-\lambda_1)+A_3\lambda_2^2}} \qquad ;\ A_1, A_3\ and\ c\neq 0$$

Where F,λ_1,λ_2 and c are arbitrary constants.

3. **References**

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المستخلص

الهدف الرئيسي من هذا البحث هو إيجاد الحل الكامل لأنواع خاصة من المعادلات التفاضلية الجزئية اللاخطية من الرتبة الثانية ذات ثلاث متغيرات والتي صيغتها العامة

$$A\,Z_{xx} + B\,Z_{xy} + C\,Z_{xt} + D\,Z_{yt} + E\,Z_{yy} + F\,Z_{tt} + G\,Z_{x} + H\,Z_{y} + I\,Z_{t} + J\,Z = 0,$$

حيث

$A, B, C, D, E, F, G, H, I, J$ دوال ل $x, y, t, Z, Z_x, Z_y, Z_t, Z_{xx}, Z_{xy}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{tt}$

وان $Z, Z_x, Z_y, Z_t, Z_{xx}, Z_{xy}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{tt}$

لهذة الدوال هي من الدرجة الأولى وغير مضروبة ببعضها