Received: 18/1/2022 **Accepted: 15/2/2022** **Published: 2022**

Weakly Quasi-prime Modules and Coprime Modules

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Abstract:
Let R be a commutative ring with unity and let M be a unitary R-module. The goal of this work is to investigate the relationships between weakly Quasi-prime modules and coprime modules(dual notion of prime modules) .Also we give some basic properties of the two concepts.

Keywords:prime module,coprime module ,weakly Quasi-prime module.

1 .introduction

Throughout this paper all rings are commutative with identity and all modules are unital.A module M is said to be prime if for each submodule W of M, $ann_R M=ann_R W$, where $ann_R M=\{r \in R:rx=0 \text{ for each } x\in M\}$,[5]. W is said to be a weakly Quasi-prime module(briefly w.q.p)module if $ann_R W=ann_R rW$ for every $r \notin nn_R W$,[6].M is a coprime R-module (dual notion of prime R-module) if and only if $ann_R M=ann_R M\backslash N$ for each submodule $N \neq 0$ of M [10].

Note that $ann_R M\backslash W=[W:M]$,equivalently M is coprime R-module if and only if M is a second module,[10],where M is a second module if for every $r \in R$, the homothety r^* on M is either zero or surjective where a homothety r on M means $r^* \in End (M)$ and $r^*(x)=rx$ for each x in M [10].Ali,I.M. in [1,corollary 9] prove that M is a coprime R-module if and only if for every r in R,$r\neq0$,either rM=0 or rM=M(that is, M is a second module).

In this work we study the relations between weakly Quasi Prime and coprime modules and give the necessary and (or) sufficient condition under which the two concepts are equivalent .

Remarks and Examples (2-1)

1-Z as Z-module is not coprime,see[1],but it is a W.q.p module, [6].
2-For all $n,m \in Z$;$n\neq m$,the Z-module $M=Z_n \oplus Z_m$ is not coprime,[1].
3-The Z-module $M=Z \oplus Z_p$;p is prime number,is a W.q.p module,[6].
4-Every simple R-module is a W.q.p module and a coprime R-module but the convers is not true for example:Q as a Z-module is coprime[1] and w.q.p ,but not simple.

5-M as R-module is coprime ifand only if $ann_R M$ is a prime ideal,[9].

Anderson in [3],defined a divisible R-module M as :M is called divisible if for all $r\neq 0$ in R ;rM=M, [3].

A module M is called multiplication if and only if there exist an ideal B of R for any submodule W of M, such that BM=W[8].

6-If M is a multiplication coprime ,then M is a w.q.p module.

proof:since M is coprime multiplication module ,then M is simple and hence M is prime[10], this implies M is quasi-prime[5],so M is a w.q.p module[6].

3.The Main Results

Theorem(3-1)

Every divisible coprime module is a w.q.p module.

Proof: let W be a divisible coprime module, then rW=W,so ann_R rW= $ann_R W$, Suppose $a\in ann_R W$ so aW=0,but aW =W implies W=0 which is a controduction ,so $a\notin ann_R w$ which means W is a w.q.p module.

Theorem (3-2)

If W is a divisible R-module, then the following statements are equivalent.

1-W is coprime.

2-W is a w.q.p module.

Proof:1→ 2 the proof follows by theorem (3-1). 2 → 1 by [1] we must prove rW=W or rW=0. For each $r\in R$,suppose that rW $\neq$W and rW$\neq 0$, so let rW = B, for a submodule B of W, then $r\in$ [B:W],which mean rW $\subseteq$ B, so $ann_R B \subseteq ann_R rW$,but W is a w.q.p module,so $ann_R rW=ann_R W$ for each $r\notin ann_R W$,implies $ann_R B\subseteq ann_R W$ which is acontroduction.

The divisibility condition is necessary in theorem(3-2) for example:]. Z as a Z-module is a (w.q.p) module,[6],but not coprime,see[10 where Z as a Z-module is not divisible.So rW=0 or rW=W for all $r\in R$

In [8], an R-module M is called finitely generated (briefly f.g)

If there exist r_1 in R such that $(1-rr_1)M=0$.`

Theorem (3-3)

If K is a f.g .w.q.p R- module, then K is a coprime module.

Proof

Since,K is a w.q.p R- module so $ann_R K=ann_R rK$ $\forall$ $r\notin ann_R K$,so rK$\neq 0$.But K is f.g ,so there exist $r_1\in$ R such that $(1-rr_1)K=0$,this implies K=rK which means K is coprime.

Theorem(3-4)

For a finitely generated R-module W, the following statement are equivalent:

1-W is a w.q.p R-module.
2-W is coprime.
3-W is a prime module.

Proof:

1→ 2 by theorem(3-3) 2 → 1,since W is f.g so there exist $r_1 \in R$ such that $(1-rr_1)W=0$,so rW=W,since W is coprime f.g so W is prime module .By [2],which means $ann_R W=ann_R N$ for each submodule N of W.

To prove $ann_R W=ann_R rW$ for every $r\notin ann_R W$,since $rW\subseteq W$,

So $ann_R W \subseteq ann_R rW$,to prove $ann_R rW \subseteq ann_R W$.Let $x \in ann_R rW$so xrW=0, $xr \in ann_R M$,since W is prime module so by[5],$ann_R W$ is a prime ideal so either $x\in ann_R W$ or $r\in ann_R W$.If $r\in ann_R W$ implies rW=0,but rW=W so W=0,which is controduction.

Therefore $x\in ann_R W$,so $ann_R rW\subseteq ann_R W$ which mean $ann_R W=ann_R Rw$, For each $r\notin ann_R W$.

2 → 3,the proof follows by [2].

3 → 1 ,since W is f.g so there exist $r_1 \in R$;$(1-rr_1)W=0$, [7].

So rW=W,we must show that $ann_R W=ann_R rW$ for each $r\notin annW$

Since $rW\subseteq W$ so $ann_R W\subseteq ann_R rW$,to prove $ann_R rW\subseteq ann_R W$.

Let $x\in ann_R rW$ so xrW=0,which mean $xr\in ann_R W$,by[5]implies $ann_R W$ Is a prime ideal so either $x\in ann_R W$ or $r\in ann_R W$.If $r\in ann_R W$,so rW=0 which is a controduction,since W is f.g ,so $x\in ann_R W$,therefore $ann_R rW\subseteq ann_R W$,so $ann_R rW=ann_R W$ for each $r\notin ann_R W$,so W is a w.q.p module.

theorem (3-4)

If K is a cyclic R-module ,then the following statements are equivalent :

1-K is a coprime R- module.
2-K is a prime R-module.
3-K is a w.q.p R-module.

Proof:

1→2,the proof follows by[1].

2→ 3,the proof follows by [6].

3 → 1,since K is a w.q.p R-module and cyclic ,then K is prime by [6].

We must prove $ann_R N\subseteq ann_R K\backslash N$ for each submodule N of K.

Let $x\in ann_R N=ann_R K$,but K is a w.q.p R-module,so $ann_R rK=ann_R K$ for each $r\notin ann_R K$,so $x\in ann_R rK$ implies xrK=0,so that $xr\in ann_R K$ but $ann_R K$ Is prime ideal(since K is prime module by [6]),so either $x\in ann_R K$ or $r\in ann_R K$,but $r\notin ann_R K$,implies $x\in ann_R K/N$(since $K/N \subseteq M$ so $ann_R K\subseteq ann_R K/N$).

In[2] A.M.Inaam submitted a non-torsion module,where an B- module W over an integral domain B is a non –torsion if there exist w ∈ W such that $ann_B(w)=0$.

Proposition (3-5)

Every non-torsion coprime R-module over an integral domain R is a w.q.p module.

Proof

Let M be a non-torsion coprime R-module.Let x∈ $ann_R rM$, so xrM=0,but M is coprime so rM=0 or rM=M.If rM=0 implies x.0=0,so rM=M which mean xM=0 so x∈$ann_R M$ implies $ann_R M=ann_R rM$ .suppose r∈$ann_R M$,but M is non- torsion so r=0 which is controduction.So $r\notin annM$.

Recall that a submodule B of an R-module M is called a direct summand of M if and only if there exists a submodule C of M such that $M=B\oplus C$, [4]

Anderson submitted in [3] that $ann_R(U\oplus V)=ann_R U\cap ann_R V$,where U,V are R-modules.

Theorem (3-6)

Let M_1 and M_2 be two coprime R-modules.Then $M=M_1\oplus M2$ Is a w.q.p module.

Proof:

To prove $ann_R rM \subseteq ann_R M$.Let x∈ $ann_R rM$ so xrM=0 implies $xr(M_1\oplus M_2)=(0,0)$.So $xrM_1=0$ and $xrM_2=0$,so xr ∈ $ann_R M_1$ and xr∈ $ann_R M_2$ .But M_1 and M_2 are coprime so by [9] there exists a proper submodule N_1 of M_1 and a proper submodule N_2 of M_2 such that xr∈ $[N_1:M_1]$ and xr ∈ $[N_2:M_2]$ implies $xrM_1 \subseteq N_1$ and $xrM_2 \subseteq N_2$,so x∈ $[N_1:M_1]$ and x∈ $[N_2:M_2]$ which mean x∈ $ann_R M_1$ and x∈ $ann_R M_2$ so x∈ $ann_R M_1\cap ann_R M_2$ so x∈ $ann_R(M_1\oplus M_2)$,which mean x $ann_R M$,so $ann_R rM \subseteq ann_R M$,which implies $ann_R rM=ann_R M$,we must prove r∉ $ann_R M$ .Suppose r∈ $ann_R M=ann(M_1\oplus M_2)$,so r $ann_R M_1\cap ann_R M_2$.But $M_1$and $M_2$are coprime so by [9], r∈ $[N_1:M_1]$ and r∈ $[N_2:M_2]$,implies $rM_1 \subseteq N_1$ and $rM_2\subseteq N_2$,but $M_1$and M_2 re coprime so $rM_1=M_1$ and $rM_2=M_2$implies $M_1\subseteq N_1$ and $M_2\subseteq N_2$ which is controduction.So r∉ $ann_R M$,which mean M is a w.q.p module.

Next ,we describe the relation between weakly quasi prime module and coprime module of fractional R_S-module M_S,where a subset S of a ring R is called multiplicatively closed if 1∈ S,0∉ S and ab∈S for each a,b inS.Let M be an R- module and S be a multiplicatively closed on R ;S≠ ∅ ,0∉ S .Let R_S be the set of all fractionals r/s where r ∈ R,s ∈ S,$M_S$be the set of all fractional x/s where x ∈ M ,s ∈ S ;$x_1/s_1=x_2/s_2$ ifand only if there exist t ∈ S such that $t(s_1x_2-s_2x_1)=0$.So we can make M_S into R_S-module by setting x/s+y/t=(tx+sy)/st, r/t.x/s=rx/ts for every x,y ∈M and r∈ R ,s,t ∈ S.So we can define the two maps ψ :R→ R_S;ψ (r)=r/1

:M→M ; $\varphi(m) = \frac{m}{1}$ $for\ each\ m\ \in\ M.$ φ For each r ∈ R,

If N is a submodule of an R-module M and S is a multiplicatively closed in R, then N_S ={n/s;n in N ,s in S}is a submodule of the R_S-module M_S,[8].

Theorem (3-7)

Let M_S be a coprime R_S-module ,then M_S is a w.q.p R_S-module for every multiplicatively closed set S of R.

Proof :

Since ann $_{RS}M_S$⊆ann_{RS}r/sM_S .Let x/t ∈ann_{RS}r/sM_S ,then x/t. r/t_1M_S=0 ,implies xr/tt_1M_S=0,but $M_S$is a coprime R_{S-} module ,which mean r/t_1M_S=M_S,i.e r/$t_1M_S \neq 0$,so r/t_1 ∉annM_S, x/t ∈ $ann_{RS}M_S$. So M_S is w.q.p R_S-module.

Conclusion:

From this research we conclude that the condition make two module coprime and w.q.p are equivalent,which is divisible,finitely generated,cyclic,if M a non-torsion coprime R-module over integral domain R,then M is w.q.p module. we conclude that if M_1 , M_2 are two coprime , then M=$M_1 \oplus M_2$ is w.q.p module..

Acknowledgement

The researchers would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq)Baghdad,Iraq.for its support in the present Research.

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الموديولات الشبه اوليه الضعيفة والمودبولات العكس اولية

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الخلاصة:
لتكن R حلقة ابدالية تحوي المحايد وليكن M موديول حيادي .الهدف من هذا العمل هو ايجاد بعض العلاقات بين الموديولات الشبه اولية الضعيفة والموديولات العكس اولية واعطينا بعض الخواص المهمة للمفهومين .

كلمات مفتاحية :
الموديولات الاولية،الموديولات العكس اولية،المودبولات الشبه اولية الضعيفة.

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