Properties Chaotic of Rabinovich-Fabrvikant Equations

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Abstract:

We give a new map named (Rabinovich-Fabrvikant equations) and find five fixed points we study only one fixed point $x_0(0,0,0)$, and all general properties of them We prove that the contracting and expanding area of this point , thought the study of the chaotic of the point by use the Wiggins defined and we proof that the lyapunov exponent of the point xo (0,0,0) is positive .We use matlab program to show sensitive dependence on the initial conditions and transitivity of (R-F).

Keywords: The Rabinovich-Fabrvikant equations, fixed point, Jacobin of Rabinovich-Fabrvikant equations, sensitive dependends on intial condition ,transitivity, Lyapunov exponents of the Rabinovich-Fabrvikant equations Lyapunov dimension , topological entropy of Rabinovich-Fabrvikant equations

ألخلاصة

درسنا معادله رابينوفيتش ودرسنا الخواص العامه لها ووجدنا مناطق التقلص والتوسيع وكذلك وجدنا خواصها الفوضوية حيث برهنا أنها تحتوي على تبولوجي انتروبي موجبا وتمتلك حساسية عند الشروط الابتدائية وإنها متعديه باستخدام برنامج ماتلاب وبرهنا توسيع ليبانوف الموجب وأخيرا درسنا بعد ليبانوف لهذه ألداله

الكلمات المفتاحيه: معادله رابينوفيتش ونقطه ثابته، جاكوبين، الحساسيه معتمده على الشروط الابتدائيه والتعدي لمعادله رابينوفيتش، توسيع ليبونوف وبعد ليبانوف واخيرا تبولوجي انتروبي .

1.Introduction

Rabinovich -Fabrikant[1979] Recently, we look more closely into the Rabinovich-Fabrikant system, after a decade of the study in [Danca & Chen,2004], discovering some new characteristics such as cycling chaos, Tollmien-Schlichting waves in hydrodynamic ows, wind waves onwater, concentration waves during chemical reactions in a medium where diusion occur, Langmuir waves in plasma, etc[Michal *et al.*, 2015]. the Rabinovich – Fabrikant equaction is three dimensions and two parameters$t$ and α and the system is chaotic :-

$$\begin{pmatrix} zy - y + yx^2 + tx \\ 3xz + x - x^3 + ty \\ -2z\alpha - 2zxy \end{pmatrix}$$

Definition and Notations:-

Let $F:R^3 \to R^3$ such that $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix}$ be a map. We called fixed point of three dimensions if only

Pair $\begin{pmatrix} p \\ g \\ h \end{pmatrix}$ such that $f\begin{pmatrix} p \\ g \\ h \end{pmatrix}=p$,$g\begin{pmatrix} p \\ g \\ h \end{pmatrix}=g$ and $h\begin{pmatrix} p \\ g \\ h \end{pmatrix}=h$ .then fixed point is attracting if and only if$F^n\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} p \\ g \\ h \end{pmatrix}$ as $n \to \infty$ for every $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the disk centered of $\begin{pmatrix} p \\ g \\ h \end{pmatrix}$

F is called diffeomorphism provided if F is one –to –one ,onto , inverse and c^∞ .The mixed k^{th} partial derivatives exist and one continuous is called c^∞ .

Assume that the first partials of the coordinate maps f_1, f_2 and f_3 of F exists at v_0. We find Jacobin of F at v_0 by the determined of DF (vo) and its denoted by $J = |\det(DF(V_0))| F$ is called area contracting at vo if $|\det(DF(V_0))| < 1$, and F is area expanding at vo if $|\det(DF(V_0))| > 1$.

Definition (2.1):- (Kolyada and Snoha, 1997):-

The $f: X \to X$ is said to be sensitive dependence on initial conditions if there exists $\in > 0$ such that for any $x_0 \in X$ and any open set U⊂ X containing x_0 there exist $y_0 \in U$ and $n \in Z^+ such\ that\ d(f^n(x_0), f^n(y_0)) > \in\ that\ is\ \exists \in > 0, \forall x, \forall \delta > 0, \exists y \in B\ \delta\ (x)\ , \exists: d(f^n(x_0), f^n(y_0)) \geq \in$

Definition (2.2)(Fotion, 2005):-

Let $f: X \to X$ be a continuous map and X be a metric space. Then the map f is said to be chaotic according to Wiggins or W-chaotic if :

1- f is topologically transitive.
2- f is exhibits sensitive dependent on initial condition.

Definition (2.3)(Sturman, 2006):-

The map $f: R^n \to R^n$ will have n Lyapunov exponent, say $L_1 (x, v)$, $L_2 (x, v)$, ……, $L_n (x, v)$ for a system of n variable . then the Lyapunov exponent is the maximum n Lyapunov exponent that is $L_1 (x, v) = max\{ L_1 (x, v) , L_2(x, v), \ldots \ldots, L_n (x, v)\} . Where\ v = (v_1, v_2, \ldots., v_n)$.

Proposition(2.4):-

If $t \neq 1$ then the Rabinovich - Fabrikant equation has five fixed points $P_0(0,0,0)$

$$p_{1,2} = (\pm\sqrt{\frac{bR_1+2b}{4b-3a}}, \pm\sqrt{b\frac{4b-3a}{R_1+2}}, \frac{aR_1+R_2}{(4b-3a)R_1+8b-6a}),$$

$$p_{3,4} = (\pm\sqrt{\frac{bR_1-2b}{3a-4b}}, \pm\sqrt{b\frac{4b-3a}{2-R_1}}, \frac{aR_1-R_2}{(4b-3a)R_1-8b+6a}$$

and we study only one fixed points $P_0(0,0,0)$.

Proof:-

$zy - y + yx^2 + tx = x$ …….(1)

$3xz + x - x^3 + ty = y$………….(2)

$-2z\alpha - 2zxy = z$ ……..(3)

Since

$-2z\alpha - 2zxy - z = 0$, $z(-2\alpha - 2xy - 1) = 0$ therefore $z = 0$ …..(4)

We put (4) in (2) we get

$x - x^3 + ty - y = 0\ hence\ y = \frac{x-x^3}{t-1}$ …………(5)

We put (5)and (4) in (1)

Then $-(\frac{x-x^3}{t-1}) + \left(\frac{x-x^3}{t-1}\right)x^2 + tx - x = 0$, $\frac{-x+x^3-x^5+x^3+t^2x-tx-tx+x}{t-1} = 0andx(2x^2 - x^4 + t^2 - 2t) = 0$

$If\ x = 0\ then\ y = 0\ and\ z = 0$ then $p_0(0,0,0)$

Or

The other four points by[5]

$$x_{1,2} = (\pm\sqrt{\frac{bR_1 + 2b}{4b - 3a}}, \pm\sqrt{b\frac{4b - 3a}{R_1 + 2}}, \frac{aR_1 + R_2}{(4b - 3a)R_1 + 8b - 6a}$$

$$x_{3,4} = (\pm\sqrt{\frac{bR_1 - 2b}{3a - 4b}}, \pm\sqrt{b\frac{4b - 3a}{2 - R_1}}, \frac{aR_1 - R_2}{(4b - 3a)R_1 - 8b + 6a}$$

Where $R_1 = \sqrt{3a^2 - 4ab + 4}$ and $R_2\ 4ab^2 - 7a^2b + 3a^3 + 2a$

Proposition (2.5):-

If t≠1 and α≠0 the Jacobain of the Rabinovich –Fabrikant equation is $2\alpha(-1 - t^2)$

Proof:-

The differential matrix of Rabinovich –Fabrikant equation of

$$DR_{\alpha,t}\ (vo) = \begin{vmatrix} \frac{\partial f(vo)}{\partial x} & \frac{\partial fz(v0)}{\partial y} & \frac{\partial f(v0)}{\partial z} \\ \frac{\partial f2(vo)}{\partial x} & \frac{\partial f2(x)}{\partial y} & \frac{\partial f(vo)}{\partial z} \\ \frac{\partial f2(vo)}{\partial x} & \frac{\partial f2(vo)}{\partial y} & \frac{\partial f(u)}{\partial z} \end{vmatrix} \text{ then , } DR_{\alpha,t}\ (Po) = \begin{vmatrix} t & -1 & 0 \\ 1 & t & 0 \\ 0 & 0 & -2\alpha \end{vmatrix}$$

Therefore J=det $DR_{\alpha,t}\ (Po)$ =-2α det $\begin{vmatrix} t & -1 \\ 1 & t \end{vmatrix} = -2\alpha(t^2+1) = 2\alpha(-1 - t^2)$■

Proposition(2.6):-

1. If $|t| < 1\ and\ |\alpha| < \frac{1}{(2(-1-t^2)}$ since t ≠1 then Rabinovich –Fabrikant equation is area contracting at p_0 .
2. If$|t| > 1\ and\ |\alpha| > \frac{1}{(2(-1-t^2)}$ then since t≠1 Rabinovich –Fabrikant equation is area expanding at P_0 .

Proof :-

1. If $|\alpha| < \frac{1}{(2(-1-t^2)}$ such that $t < 1$ then the absolute value of Jacobain of Rabinovich –Fabrikant equation is least than 1 that is $R_{\alpha,t}$ is area contracting map.
2. If $t > 1$ since $|J| = \left|\det\left(DR_{\alpha,t}\ (P_0)\right)\right| = |2\alpha(-1 - t^2)| > 1$
   By hypotheses $|\alpha| > \frac{1}{2(-1-t^2)}$ then $R_{\alpha,t}$ is area expanding ■

Remark (2.7):-

The $R_{\alpha,t}$ is on to and is not one to one then $R_{\alpha,t}$ is not diffemorephism.

Proposition (2.8):-

Then $R_{\alpha,t}\ is\ c^{\infty}$

Proof :-

$\frac{\partial R_1}{\partial X} = 2xy + t$, $\frac{\partial R_2}{\partial X} = 3z + 1 - 3x^2$, $\frac{\partial R_3}{\partial X} = -2zy$

$\frac{\partial R_1^2}{\partial X^2} = 2Y$, $\frac{\partial R_2^2}{\partial X^2} = -6X$, $\frac{\partial R_3^2}{\partial X^2} = 0$

$\frac{\partial R_1^3}{\partial X^3} = 0$, $\frac{\partial R_2^3}{\partial X^3} = -6$, $\frac{\partial R_3^3}{\partial X^3} = 0$............ $\frac{\partial R_1^n}{\partial X^n} = 0$, $\forall n \in N\ and\ \frac{\partial R_2^n}{\partial X^n} = 0\ \forall n \in N$ and these partial derivatives are exist and continous then $R_{\alpha,t}\ is\ c^{\infty}$■

Proposition (2.9):-

$\lambda_{1,2} = t \pm i \quad , \lambda_3 = -2\alpha$.

Proof :-

Det ($DR_{\alpha,t}$ (v)- λI)= det $\begin{vmatrix} t-\lambda & -1 & 0 \\ 1 & t-\lambda & 0 \\ 0 & 0 & -2\alpha-\lambda \end{vmatrix} = 0$ *then*

(t- $\lambda)^2$ (-2α- λ) + 1 = 0 hance (t- $\lambda)^2$ (-2α- λ) = −1 then if (-2α- λ) = 0 $therefore - 2\alpha = \lambda$

Or (t- $\lambda)^2$= -1 ,Then $\lambda = t \pm i$ $therefore$ $\lambda_{1,2} = t \pm i$ and λ_3=-2α <0 $\forall t \in R$ $is\ the\ eigenvalues\ of$ $DR\alpha$, t (P_0)■

Proposition (2.10):-

1.If α<0 and t>0 then P_0 is repelling fixed point .

2. If t<0 and α>0 then P_0 is saddle fixed point .

Proof:-

1. Since$\lambda_{1,2} = t \pm i$, $\lambda_3 = -2\alpha$ and $\alpha < 0$, $t > 0$ $then\ the\ real\ part\ of$ $\lambda_{1,2}$ is positive they by [1] thenP_0 IS repelling fixed point.
2. Since $\lambda_{1,2} = t \pm i$, $\lambda_3 = -2\alpha$ and $t < 0$, $\alpha > 0$ the real part of $\lambda_{1,2}$ is negative so by [1] P_0 is saddle fixed point ■

Proposition(2.11):-

The set of fixed points of Rabinovich –Fabrikant is closed.

Proof:-

Let A be the set of fixed points of Rabinovich –Fabrikant then

A=$\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : R_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\ and\ R_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\}$, A⊏ R^3 ,To show that A is closed set

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in A^c$ then $R_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and since A⊏ R^3 we have three distinct

elements in R^3then there exist two disjoint open set M,N ⊏ R^3 such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$

M and $R_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in N$, hence $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M\ \cap\ (R_{\propto,t}^{-1})$(N) since N open subset in R^3 and

$R_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is continuous map we have $(R_{\propto,t}^{-1})$(N) is open subset in R^3

Let $M\ \cap\ (R_{\propto,t}^{-1})$(N =U , we chain that U⊏ A^C to show this , let $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in U$ then

$\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in M$

and $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in (R_{\propto,t}^{-1}(N))$ so $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in M, R_{\alpha,t}\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in MR_{\alpha,t}\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in N$ but since M∩ $N =$

Φ then $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \neq R_{\alpha,t}\begin{pmatrix} u \\ r \\ s \end{pmatrix}$

Hence $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in A^c$, that is our claim is true , Hance for each $\begin{pmatrix} u \\ r \\ s \end{pmatrix} \in A^c$ we could find the open set U such that U⊏ A^c so A^c is open then A is closed■

3. Properties chaotic of the Rabinovich –Fabrikant:-

Proposition (3.1):-

If $|t|>1$ then Rabinovich –Fabrikant is sensitive dependent on initial condition .

Proof:-

If $|\alpha| > 1 , |t| > 1$ *then*

Let $x = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ be a point in R^3 , $R_{\alpha,t} < \begin{pmatrix} zy - y + yx^2 + tx \\ 3xz + x - x^3 + ty \\ -2z\alpha - 2zxy \end{pmatrix}$ *then*

Case(1):-

If $|x| \leq 1$ by hypothesis and by definition

$R_{\alpha,t}(x) < \begin{pmatrix} -2z\alpha ty - ty \\ t^2y \\ 4z\alpha^2 \end{pmatrix}$, that is $R_{\alpha,t}^n(x) < \begin{pmatrix} -2z\alpha^{2n}t^{3n}y - t^{3n}y \\ t^{2n}y \\ (2n)z\alpha^{2n} \end{pmatrix}$

Thus if$|\alpha|>1$ and $|t| > 1$ then $n \longrightarrow \infty$

let $x_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in R^3 , x_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \ni R_{\alpha,t}(x_1) < \begin{pmatrix} -2z\alpha ty - ty \\ t^2y \\ 4z\alpha^2 \end{pmatrix}$ and $R_{\alpha,t}(x_2) < \begin{pmatrix} -2z\alpha ty - ty \\ t^2y \\ 4z\alpha^2 \end{pmatrix}$

that $dim(R_{\alpha,t}(x_1), R_{\alpha,t}(x_2)) = \sqrt{(-2z\alpha ty - ty)^2 + (t^2y)^2 + (4z\alpha^2)^2}$

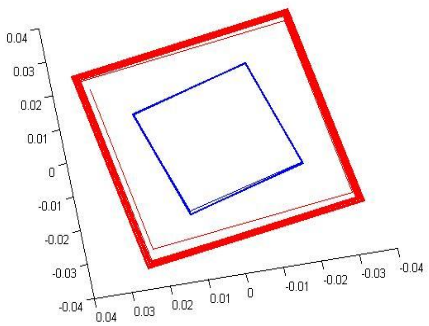
$= \sqrt{(-2z\alpha ty - ty)^n + (t^2y)^n + (4z\alpha^2)^n}$

If $|\alpha|>1, |t| > 1$ and $n \longrightarrow \infty$, $d(R_{\alpha,t}^n(x), R_{\alpha,t}^n(y)) \rightarrow \infty$

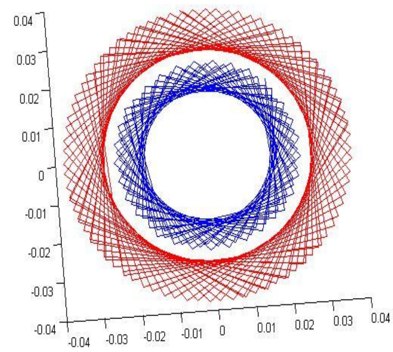
Hence $R_{\alpha,t}$ *has* sensitive dependent on initial condition

If $|x| > 1$ form of the Rabinovich –Fabrikant equation . they are diverge on the iterates of this map . thus it has sensitive dependent on initial condition

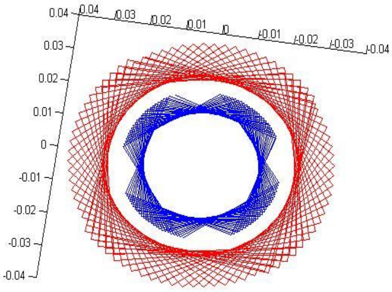
We get pictures satisfy the sensitive dependence on initial conditions to the Rabinovich –Fabrikant in different parameters by use the Matlab program .



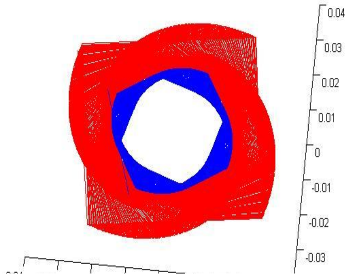
m_1= -0.3, m_2= -0.4 $t_{1,2}$=0 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)



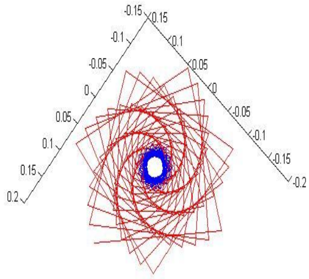
m_1= -0.1, m_2= -0.2 t_1=0.05 ,t_2=0.04 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)



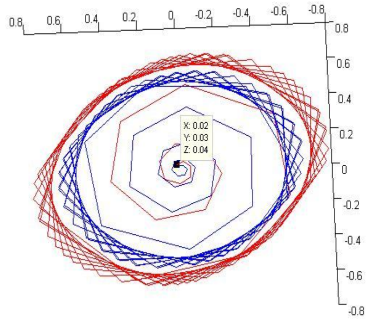
m_1= -0.1, m_2= -0.2 t_1=0.01,t_2=0.02 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)



m_1= -0.3, m_2= -0.4 $t_{1,2}$=0 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)

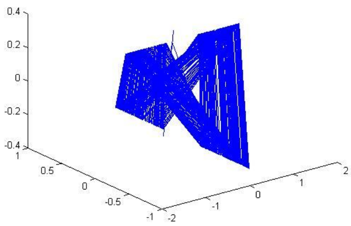


m_1= 0.1, m_2= 0.2 t_1=-0.1, t_2=-0.2 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)

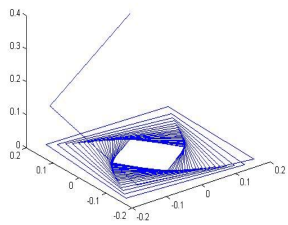


m_1= -0.1, m_2= -0.2, t_1=0.6 ,t_2=0.7 with initial points (0.01,0.02,0.03) and (0.02,0.03,0.04)

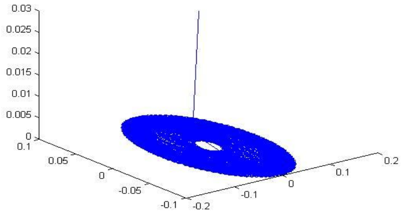
We get pictures satisfy the transitive to the Rabinovich –Fabrikant in different parameters by use the Matlab program:-



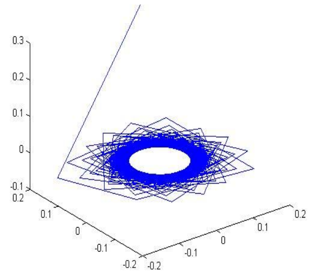
m =0.5 ,t = 0 with initial points (0.1,0.2,0.3)



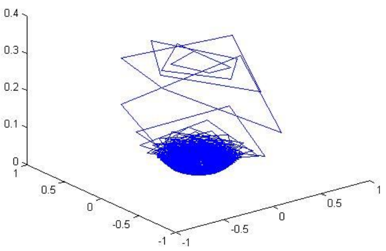
m =0.02 , t = 0 with initial points (0.1,0.2,0.3)



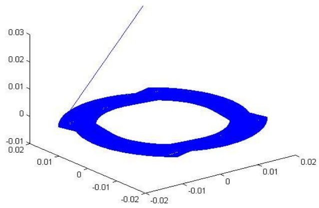
a =-0.05 , b = -0.1 with initial points (0.01,0.02,0.03)



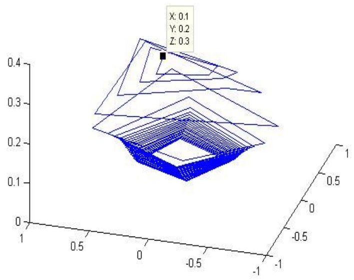
m=0.1, t =0.1 with initial points (0.1,0.2,0.3)



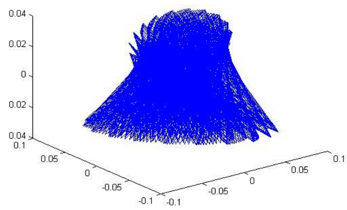
m=-0.5 t=-0.1 with initial points (0.1,0.2,0.3)



t_1=-0.002, m_1=0 with initial points (0.01,0.02,0.03)



t =-0,4 , m = -0.0001 with initial points (0.1,0.2,0.3)



m=0.5 , t =-0.1 with initial points (0.01,0.02,0.03)

Definition (3.2):-

The Rabinovich –Fabrikant is strange attractor if Lyapunovdimension.

Definition(3.3)[Gulick ,1992]:-

Let V be a subset of R^2, and suppose that F: V→R^2 has continuous partial derivatives. Assume also that v_0 is in V, with orbit $\{v_n\}_{n=0}^{\infty}$.For each n=1,2,…… we define $D_n F(v_0)$ by the formula $D_n\, F(v_0) = [DF(v_{n-1})][DF(v_{n-2})] \ldots\ldots [DF(v_0)]$ where $DF(v_k)$ denotes the 2×2 matrix identified with the differential of F at v_k . Then

$D_n F(v_0)$ is 2×2 matrix (depending on n). If $D_n F(v_0)$ has nonzero real eigenvalues, we denote their absolute values of the eigenvalues by $d_{n1}(v_0)$ and $d_{n2}(v_0)$. For convenience we will assume that $d_{n1}(v_0) \geq d_{n2}(v_0)$. The Lyapunov numbers $r_1(v_0)$ and $r_2(v_0)$ of V at v_0:-$r_1(v_0) = \lim_{n\to\infty}[d_{n1}(v_0)]^{\frac{1}{n}}$, $r_2(v_0) = \lim_{n\to\infty}[d_{n2}(v_0)]^{\frac{1}{n}}$ provided that the limits exist

proposition (3.4):-

let $R_{\alpha,t} : R^3 \to R^3$ be the R-F equation the lyapunov exponents of $R_{\alpha,t}$ is positive .

proof:-

By properties (2-9)

If t<1 and $\alpha < 1$ by proposition $|\lambda_{1,2}| = |t + i|$, If t<1 since $x_{1,2} = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_{1,2}\right) = \lim_{n\to\infty}\frac{1}{n}\left|DR_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_{1,2}\right| < 0$, But if $\alpha < 1$ then $x_3 = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_3\right) = \lim_{n\to\infty}\frac{1}{n}\left|DR_{\alpha,t}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, v_3\right| > 0$ so $x_v = \max\{x_1^{\pm}(x, v_1), x_2^{\pm}(x, v_2), x_3^{\pm}(x, v_3),\}$ then $x_v > 0$ so in the same way we can prove if t>1 and $\alpha > 1$ then L(v)>0

Proposition (3.5):-

If $|\alpha| < 0$ and t<0 then $R_{\alpha,t}$ has a stranger attractor then either

Dim A_F=1-$\frac{ln|-2\alpha|}{in\ |t+i|}$ or Dim A_F=1-$\frac{ln|t+i|}{in\ |-2\alpha|}$

Proof:-

Case1:- Since $|\alpha| < 0$ and t<0 then $|t + i| \leq -2\alpha$ by definition (3.3)

Dn_1= max eigenvalues of $D_n R_{\alpha,t}$.

Dn_2=min eigenvalues of $D_n R_{\alpha,t}$..

Then

$r_1 = \lim_{n\to\infty}(dn_1) = \lim_{n\to\infty}(-2\alpha)$

$r_2 = \lim_{n\to\infty}(dn_2) = \lim_{n\to\infty}(t + i)$

then dim A_F =1 -$\frac{\ln|-2\alpha|}{\ln|t+i|}$

case (2):-If $|\alpha| > 0$ and $t > 0$, then we can prove by the same way : dim A_F =1 -$\frac{\ln|t+i|}{\ln|-2\alpha|}$

we recall the theorem (3.35) in [3]by

proposition(3.6):-

The upper estimate of topological entropy of

$$h_{top}(R_{\alpha,t}) \leq \frac{\log\alpha}{2\log t + \log\alpha}$$

Proof:-

By theorem(3.35) on [3] we get

$$\begin{aligned} h_{top}(R_{\alpha,t}) &\leq lo\,g\,max_{x\in R^n} \max L \in T_{x\in R^n} \left|\det(DR_{\alpha,t}(x)|L)\right| \\ &\leq lo\,g\,max_{x\in R^n} \max L \in T_{x\in R^n} |2\alpha(-(1 + t^2)| \\ &\leq \frac{2\log\alpha}{4\log t + 2\log\alpha} \leq \frac{\log\alpha}{2\log t + \log\alpha} \end{aligned}$$

We find estimate of topological entropy of Rabinovich –Fabrikant equation

We recall the theorem (3.25) on [4] by

Proposition (3.7):-

1. If $\alpha < 0\ then\ |-2\alpha| > |t+i|$ therefore
$$h_{top}\left(R_{\alpha,t}\right) \geq \log|-2\alpha|$$
2. If $t > 0,\ \alpha > 0\ then\ |t+i| > |-2\alpha|$ then
$$h_{top}\left(R_{\alpha,t}\right) \geq \log|t+i|$$

proof:-

case (1) :-

by proposition (2.9) and by hypothesis then
$$h_{top}\left(R_{\alpha,t}\right) \geq \log|-2\alpha|$$
By using the same way, we can prove this case

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