The Pitman Estimator of the Gamma Scale Parameter and Comparing it With Some Estimation Methods

مقدر بيتمان لمعلمة قياس كاما ومقارنتها مع بعض الطرائق الاخرى للتقدير

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Abstract

The research studied the Pitman Estimation Method for estimation the scale parameter θ for the two parameter Gamma Distribution, and compared it with some estimation methods for estimation the scale parameter θ. The methods are:

Maximum likelihood Estimator Method (MLE), Standard Bayesian Method (SB), Furthermore; the researcher suggested two mixture methods for estimation; the first Mixture Method between Maximum Likelihood Estimator Method and Pitman Method (MIX I), and the second Mixture Method between Standard Bayesian Method and Pitman Method (MIX II).

Comparison was conducted between the methods mentioned above to estimate the scale parameter in the experimental aspect to find the best estimation method. Generally, a comparison between the estimation methods of estimating the scale parameter is made using the relative efficiency of the risk efficiency under squared error loss function to find the best method through Monte Carlo simulation. Simulation examples are worked out, where the generation of random data, depending on the different sample sizes and run size (L=1000). The Pitman method is found to be the best method for estimation the scale parameter.

المستخلص

في هذا البحث تم دراسة طريقة تقدير بيتمان لتقدير معلمة القياس θ لتوزيع كاما ذي المعلمتين ، ومقارنتها مع بعض الطرائق الأخرى لتقدير معلمة القياس θ والطرق هي:

طريقة الإحتمال الأعظم (MLE) وطريقة بيز القياسية (SB)، واقترحت الباحثة طريقتين تقديريتين للخلط (mixture) . الطريقة الأولى سُميَت (MIX I) بين طريقة الإحتمال الأعظم وطريقة بيتمان, والطريقة الثانية سُميَت (MIX II) بين طريقة بيز القياسية وطريقة بيتمان. وقد أجريت مقارنة بين الطرق المذكورة أعلاه لتقدير معلمة القياس في الجانب التجريبي لإيجاد أفضل طريقة تقدير من خلال أسلوب المحاكاة باستخدام طريقة مونت كارلو (Monte Carlo) وإجراء عدة تجارب مستخدمين الكفاءة النسبية لكفاءة المخاطرة تحت دالة الخسارة التربيعية، وإجراء عدة تجارب و بأحجام عينات مختلفة وحجم التكرار (L = 1000). وتم التوصل بشكل عام الى أن طريقة بيتمان هي الأفضل من بين هذه الطرائق لتقدير معلمة القياس لإمتلاكها أقل دالة مخاطرة مقارنة بالطرائق الاخرى.

1. Introduction

If variable T follows a two-parameter Gamma distribution with shape parameter α and scale parameter θ $(T \sim \Gamma(\alpha, \theta))$, then the following relations hold

Probability density function:

$$f(t; \alpha, \theta) = \begin{cases} \frac{1}{\Gamma\alpha\ \theta^{\alpha}} t^{\alpha-1} e^{-\frac{t}{\theta}} & ; 0 < t < \infty \\ & \alpha, \theta > 0 \\ 0 & elsewhere \end{cases} \quad \dots (1)$$

Here $\Gamma(\alpha)$ is the gamma function and it is expressed as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx = (\alpha - 1)!$$

It is well known that the PDF of Gamma (α, θ) can take different shapes but it is always unimodal. The hazard function of Gamma (α, θ) can be increasing, decreasing or constant depending on $\alpha > 1$, $\alpha < 1$ or $\alpha = 1$ respectively.

If $\alpha = 1$, gamma distribution is reduced to exponential distribution. With integer α, gamma distribution is often called a special Erlangian distribution. It can be derived as the distribution of the waiting time until the kth emission from a Poisson source with intensity parameter θ. Consequently, the sum of α independent exponential variables with parameter θ has a gamma distribution with parameters α and θ and can be used to model.

The main aims for this research are:

1. To use Pittman estimator to estimate the scale parameter of Gamma distribution.
2. To suggest the estimators by finding a new formula to obtain an estimator which have good characteristics in terms of small mean square error; through mixing the Pittman estimator with Bayesian and maximum likelihood estimators. Also, studying the efficiency of this formula in the estimation process to compare estimation methods.

2. Related Work

In 1969, (Choi & Wette) examined the numerical technique of the maximum likelihood method to estimate the parameters of Gamma distribution. (Choi & Wette, 1969)

In 1978, (Padgett and Robinson) considered empirical Bayes (EB) procedures for estimating the reliability for the lognormal failure model. Monte Carlo simulations are presented to compare the EB estimators and the maximum likelihood (ML) estimators of Reliability. (Padgett & Robinson, 1978)

In 1980, (Miller) presented a Bayesian analysis of shape, scale, and mean of the two-parameter gamma distribution. Attention is given to conjugate and “non-informative” priors, to simplifications of the numerical analysis of posterior distributions. (Miller, 1980)

In 2000, (Coit & Jin) developed Maximum likelihood estimators have been for the gamma distribution when there is missing time-to-failure information. (Coit & Jin, 2000)

In 2007, (Freue) used The Pitman estimator of the Cauchy location parameter when the scale parameter is known. Using the squared error loss function, a closed form of the minimum risk equivariant (MRE) estimator. (Freue, 2007)

In 2007, (Akahira, Ohyauchi and Takeuchi) are used the Pitman estimator to obtain the asymptotic expansion of the Pitman estimator and its asymptotic variance. In a non-regular case when the density has an unbounded support. (Akahira, Ohyauchi, & Takeuchi, 2007)

In 2010, (Jasim) derived Bayes' estimator for the Scale parameter in Gamma distribution when the shape parameter is known, depending on squared error and LINEX loss function, then comparisons of risks for scale parameter under squared and LINEX loss function have been made. (Jasim, 2010)

In 2014, (Kishan) compared between maximum likelihood estimator (MLE) and Bayes estimator of scale parameter of Generalized gamma distribution under Squared error loss function when shape parameters are known. (Kishan, 2014)

Background Information:

This section studied properties of two parameter gamma distributions; Pitman Estimator; several estimation methods to estimate the scale parameter of Gamma distribution. The section introduces two new estimation methods through mixing the pitman estimator with each of Bayes estimator and (MLE).

3: Properties of Gamma Distribution (Agarwal, 2003)

1. Mean of gamma distribution is $\alpha\theta$.
2. Variance of gamma variate is $\alpha\theta^2$.

3. If $\theta > 1$, variance > mean; if $\theta < 1$, variance < mean and if $\theta = 1$, variance = mean.
4. Moment generating function of gamma distribution

$M_x(t) = (1-\theta t)^{-\alpha}$

5. Person's coefficients of skewness are,

$\beta_1 = \frac{4}{\alpha}$ or $\gamma_1 = \sqrt{\beta_1} = \frac{2}{\sqrt{\alpha}}$

6. Person's coefficients of Kurtosis are,

$\beta_2 = 3 + \frac{6}{\alpha}$ or $\gamma_2 = \beta_2 - 3 = \frac{6}{\alpha}$.

Since $\gamma_1 > 0$, it ratifies that the gamma frequency curve is positively skewed. Further $\beta_2 > 3$ or $\gamma_2 > 0$ clearly indicate that gamma distribution is leptokurtic.

7. Characteristic function of gamma distribution is $(1-\theta it)^{-\alpha}$
8. If $X_1, X_2, \ldots, X_n$ and n.i.i.d. $\Gamma(\alpha_i, \theta)$ variates for $i = 1,2,\ldots,n$, the sum of the variates, $X_1 + X_2 + \cdots + X_n$ is also a gamma variate with parameters θ and $\sum \alpha_i$. i. e. $\Gamma(\sum \alpha_i, \theta)$. It is known as additive or reproductive property of gamma variate.
9. As $\alpha \to \infty$. Gamma distribution tends to normal distribution. It is called the limiting form of gamma distribution.

4: Loss Function

A loss function or cost function $c(\hat{\theta} - \theta)$ is a scalar valued function which determines the loss of taking the action a when the true parameter value is θ. The action (or control) is the statistical decision to be made based on the currently available information. (Särkkä, 2013)

The three classical loss functions:

1. The "0/1" binary loss function

$$L(\hat{\theta}, \theta) = \begin{cases} 0 & \text{if } |\hat{\theta} - \theta| < \varepsilon \quad ; \quad \text{where } \varepsilon > 0 \\ 1 & \text{if } |\hat{\theta} - \theta| \geq \varepsilon \quad ; \quad \text{where } \varepsilon > 0 \,. \end{cases}$$

2. The "quadratic error" loss function

$$L(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2 \; ; \text{ where } c > 0$$

3. The "absolute error" loss function

$$L(\hat{\theta}, \theta) = c|\hat{\theta} - \theta| \quad ; \text{ where } c > 0$$

Here, ε is going to be very small; and without any loss of generality, let's set a = c = 1.

The "Quadratic Error" Loss Function

Proposed by Legendre (1805) and Gauss (1810), this loss is undoubtedly the most common evaluation criterion. Founding its validity on the ambiguity of the notion of error in statistical settings (i.e., measurement error versus random variation)

The quadratic error loss function is defined as follows:

$$L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2 \qquad \dots (2)$$

The square loss function does care about how different the estimated $\hat{\theta}$ is from the true θ. It is most appropriate when θ lives in a continuous space

with a well-defined metric, so that it makes sense to look at the square distance two different values of θ. (Robert, 2007)

5: Risk Function

The risk function is the average loss function that will be incurred if the estimator $\hat{\theta}$ is used, and is a given by: (Casella & Berger, 2002)

$$R_s(\hat{\theta},\theta) = E[L(\hat{\theta},\theta)] = \int_0^\infty L[\hat{\theta},\theta]\, h(\theta|t)d\theta \qquad \cdots (3)$$

6. Estimation methods for Estimating the scale parameter for the Gamma Distribution

6.1: Maximum Likelihood Estimation Method

Let $(t_1,\dots,t_n)$ be the set of n random lifetime from Gamma distribution with parameters α and θ.

The probability density function of Gamma distribution is given by:

$$f(t;\alpha,\theta) = \frac{1}{\Gamma\alpha\ \theta^\alpha} t^{\alpha-1} e^{-\frac{t}{\theta}}$$

The likelihood function is given by:

$$L = L(t_1,t_2,\dots,t_n;\alpha,\theta) = \frac{1}{(\Gamma\alpha)^n\ \theta^{n\alpha}} \prod_{i=1}^{n} t_i^{\alpha-1}\, e^{-\frac{\sum_{i=1}^n t_i}{\theta}}$$

Then, taking the natural logarithm of the likelihood function: differentiating with respect to θ results in: And $\frac{\partial \ln L(\theta)}{\partial\theta} = 0$,That is, $\theta = \frac{\sum_{i=1}^n t_i}{n\alpha} = \frac{\bar{t}}{\alpha}$

Hence, the MLE of θ is:

$$\hat{\theta}_{MLE} = \frac{\bar{t}}{\alpha} \qquad \ldots (4)$$

6.2: Standard Bayes Estimator for the Gamma Distribution

Consider the two parameter gamma distribution

$$f(t; \alpha, \theta) = \frac{1}{\Gamma\alpha\ \theta^{\alpha}} t^{\alpha-1} e^{-\frac{t}{\theta}}$$

We find Jeffery prior by taking $P(\theta) \propto \sqrt{I_x(\theta)}$, where

$$I_x(\theta) = -E\left(\frac{\partial^2 \ln L(\theta)}{\partial\theta^2}\right) = \frac{n\alpha}{\theta^2}$$

Taking; $P(\theta) = \frac{k\sqrt{n\alpha}}{\theta}$, with k a constant

The joint probability density function $f(t_1, t_2, \ldots, t_n; \alpha, \theta)$ is given by

$$H(t_1, t_2, \ldots, t_n; \alpha, \theta) = \prod_{i=1}^{n} f(t_i, \theta)\, P(\theta) = L(t_1, t_2, \ldots, t_n | \alpha, \theta) P(\theta)$$

$$H(t_1, t_2, \ldots, t_n; \alpha, \theta) = \frac{1}{(\Gamma\alpha)^n\ \theta^{n\alpha}} \prod_{i=1}^{n} t_i^{\ \alpha-1}\, e^{-\frac{\sum_{i=1}^{n} t_i}{\theta}} * \frac{K\sqrt{n\alpha}}{\theta}$$

$$= \frac{K\sqrt{n\alpha}}{(\Gamma\alpha)^n} \theta^{-(n\alpha+1)} \prod_{i=1}^{n} t_i^{\ \alpha-1}\, e^{-\frac{\sum_{i=1}^{n} t_i}{\theta}} \qquad \ldots (5)$$

The marginal probability density function of $(t_1, t_2, \ldots, t_n; \alpha, \theta)$ is given by:

$$P(t_1, t_2, \ldots, t_n) = \int_0^{\infty} H(t_1, t_2, \ldots, t_n; \alpha, \theta)\, d\theta$$

$$= \frac{K\sqrt{n\alpha}}{(\Gamma\alpha)^n} \prod_{i=1}^{n} t_i^{\ \alpha-1}\ \frac{\Gamma n\alpha}{(\sum_{i=1}^{n} t_i)^{n\alpha}} \qquad \ldots (6)$$

The conditional probability density function of θ given the data $(t_1, t_2, \dots, t_n; \alpha, \theta)$ is called posterior distribution of θ, given by

$$h(\theta|t_1, t_2, \dots, t_n) = \frac{H(t_1, t_2, \dots, t_n; \alpha, \theta)}{P(t_1, t_2, \dots, t_n)} \qquad \cdots (7)$$

By substituting equation (5) and (6) in equation (7) we have:

$$h(\theta|t_1, t_2, \dots, t_n) = \frac{\frac{K\sqrt{n\alpha}}{(\Gamma\alpha)^n} \theta^{-(n\alpha+1)} \prod_{i=1}^{n} t_i^{\alpha-1} e^{-\frac{\sum_{i=1}^{n} t_i}{\theta}}}{\frac{K\sqrt{n\alpha}}{(\Gamma\alpha)^n} \prod_{i=1}^{n} t_i^{\alpha-1} \frac{\Gamma n\alpha}{(\sum_{i=1}^{n} t_i)^{n\alpha}}}$$

$$h(\theta|t_1, t_2, \dots, t_n) = \begin{cases} \frac{(\sum_{i=1}^{n} t_i)^{n\alpha}}{\Gamma n\alpha} \theta^{-(n\alpha+1)} e^{-\frac{\sum_{i=1}^{n} t_i}{\theta}} & t > 0 \\ 0 & o \cdot w \end{cases} \qquad \cdots (8)$$

By using quadratic error loss function $L(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2$, we can obtain the Risk function, such that

$$R_s(\hat{\theta}, \theta) = E[L(\hat{\theta}, \theta)] = E\left[c(\hat{\theta} - \theta)^2\right]$$

$$= \int_0^\infty c(\hat{\theta} - \theta)^2 h(\theta|t_1, t_2, \dots, t_n) d\theta$$

$$= c\hat{\theta}^2 - 2c\hat{\theta}E(\theta|t_1, t_2, \dots, t_n) + E(\theta^2|t_1, t_2, \dots, t_n)$$

$$\frac{\partial R_s(\hat{\theta}, \theta)}{\partial \hat{\theta}} = 2c\hat{\theta} - 2cE(\theta|t_1, t_2, \dots, t_n)$$

Let $\frac{\partial R_s(\hat{\theta}, \theta)}{\partial \hat{\theta}} = 0$, then the Bayes estimator is

$$\hat{\theta}_{SB} = E(\theta|t_1, t_2, \dots, t_n) = \int_0^\infty \theta\, h(\theta|t_1, t_2, \dots, t_n)\, d\theta$$

Substituting Equation (8):

$$\hat{\theta}_{SB} = \frac{\sum_{i=1}^{n} t_i}{n\alpha - 1} = \frac{n}{n\alpha - 1}\bar{t} \quad \cdots (9)$$

6.3: The Method of Pitman Estimation

The Pitman estimator has several properties to make it desirable. For example, (i) It is uniquely determined in case of location parameter distribution of any type, (ii) It is usually admissible, (iii) It is attending optimality properties in the same sense that the maximum likelihood estimator is attend optimality properties in the regular case, but it is often better than the maximum likelihood estimator in the non-regular cases, (iv) In small sample situation it coincides with the uniformly minimum variance unbiased estimator if the latter exists.(Akahira, Ohyauchi & Takeuchi, 2007)

6.3.1: Pitman's Estimator for Location Parameter

Let $X_1, X_2, \dots, X_n$ be a random sample from a density $f(x;\theta)$ where θ is a location parameter of the distribution. Then the pitman estimator $\hat{\theta} = t(x_1, x_2, \dots, x_n)$ is given by the formula, (Agarwal, 2003)

$$\hat{\theta}_p = \frac{\int_0^\infty \theta \prod_{i=1}^{n} f(x_i,\theta)d\theta}{\int_0^\infty \prod_{i=1}^{n} f(x_i,\theta)d\theta}$$

$$= \frac{\int_0^\infty \theta L(x|\theta)d\theta}{\int_0^\infty L(x|\theta)d\theta} \quad \cdots (10)$$

6.3.1.1: Properties of Pitman Estimator for Location Which It Usually Holds (Agarwal, 2003)

1-The Pitman estimator for location has smallest mean squared error.

2-A Pitman estimator is function of sufficient statistics.

3-Pitman estimator is a minimax estimator on the real line.

6.3.2: Pitman's Estimator for Scale Parameter

Let $X_1, X_2, \dots, X_n$ be a random sample from a density $f(x;\theta)$ where θ is a scale parameter such that $\theta > 0$. Also $f(x;\theta)$ exists for $x > 0$ and otherwise zero.

Then the Pitman estimator $\hat{\theta} = t(X_1, X_2, \dots, X_n)$ for scale parameter can be obtained by the expression is:

$$\hat{\theta}_p = \frac{\int_0^\infty \frac{1}{\theta^2} \prod_{i=1}^n f(x_i, \theta) d\theta}{\int_0^\infty \frac{1}{\theta^3} \prod_{i=1}^n f(x_i, \theta) d\theta}$$

$$= \frac{\int_0^\infty \frac{1}{\theta^2} L(\theta) d\theta}{\int_0^\infty \frac{1}{\theta^3} L(\theta) d\theta} \qquad \cdots (11)$$

Pitman estimator for scale is also a function of sufficient statistics. (Agarwal, 2003)

6.3.3: Pitman Estimator for the Scale Parameter of the Gamma Distribution

Let t has Gamma(α, θ) distribution then its probability distribution function equals:

$$f(t; \alpha, \theta) = \frac{1}{\Gamma\alpha\ \theta^\alpha} t^{\alpha-1} e^{-\frac{t}{\theta}}$$

The likelihood function is given by:

$$L = L(t_1, t_2, \dots, t_n; \alpha, \theta) = \frac{1}{(\Gamma\alpha)^n\ \theta^{n\alpha}} \prod_{i=1}^n t^{\alpha-1}\ e^{-\frac{\sum_{i=1}^n t_i}{\theta}}$$

$$L(\theta)\ \alpha\ \frac{1}{\theta^{n\alpha}}\ e^{-\frac{\sum_{i=1}^n t_i}{\theta}}$$

And the pitman estimator $\hat{\theta}_p$ of the scale parameter θ is given by:

$$\hat{\theta}_p = \frac{\int_0^\infty \frac{1}{\theta^2} L(\theta) d\theta}{\int_0^\infty \frac{1}{\theta^3} L(\theta) d\theta}$$

$$= \frac{\int_0^\infty \frac{1}{\theta^2} \frac{1}{\theta^{n\alpha}} e^{-\frac{\sum_{i=1}^n t_i}{\theta}} d\theta}{\int_0^\infty \frac{1}{\theta^3} \frac{1}{\theta^{n\alpha}} e^{-\frac{\sum_{i=1}^n t_i}{\theta}} d\theta}$$

$$= \frac{\int_0^\infty \frac{1}{\theta^{n\alpha+2}} e^{-\frac{\sum_{i=1}^n t_i}{\theta}} d\theta}{\int_0^\infty \frac{1}{\theta^{n\alpha+3}} e^{-\frac{\sum_{i=1}^n t_i}{\theta}} d\theta}$$

$$= \frac{\Gamma(n\alpha+1)/(\sum_{i=1}^n t_i)^{n\alpha+1}}{\Gamma(n\alpha+2)/(\sum_{i=1}^n t_i)^{n\alpha+2}}$$

$$= \frac{\sum_{i=1}^n t_i}{n\alpha+1} = \frac{n}{n\alpha+1}\bar{t}$$

Then, the pitman estimator for the parameter θ is:

$$\hat{\theta}_p = \frac{n}{n\alpha+1}\bar{t} \qquad \cdots (12)$$

6.4: First Mixture Method of MLE and Pitman (Mix I)

This method is a suggesting estimator obtained from mixing two estimators which are maximum likelihood estimator and pitman estimator, the aim of this estimator is to find an estimator that minimizes the MSE.

$$\hat{\theta}_{Mix\,I} = P\hat{\theta}_{MLE} + (1-P)\hat{\theta}_p$$

We have to find the value of P which makes the MSE has minimum value According to the following steps:

Subtracting θ from both sides

$$\hat{\theta}_{Mix\,I} - \theta = [P\hat{\theta}_{MLE} + (1-P)\hat{\theta}_p] - \theta$$

Squaring the both sides and taking the expectation

$$E(\hat{\theta}_{Mix\,I} - \theta)^2 = E\{[P\hat{\theta}_{MLE} + (1-P)\hat{\theta}_p] - \theta\}^2$$

$$= P^2E(\hat{\theta}_{MLE})^2 + 2P(1-P)E(\hat{\theta}_{MLE})E(\hat{\theta}_p) + (1-P)^2E(\hat{\theta}_p)^2$$

$$-2PE(\hat{\theta}_{MLE})E(\theta) - 2(1-P)E(\hat{\theta}_p)E(\theta) + E(\theta)^2$$

To find the minimizing value

$$\frac{dE(\hat{\theta}_{Mix\,I}-\theta)^2}{dP} = 2PE(\hat{\theta}_{MLE})^2 + (2-4P)E(\hat{\theta}_{MLE})E(\hat{\theta}_p)$$
$$-2(1-P)E(\hat{\theta}_p)^2 - 2E(\hat{\theta}_{MLE})E(\theta) + 2E(\hat{\theta}_p)E(\theta)$$

And setting $\frac{dE(\hat{\theta}_{Mix\,I}-\theta)^2}{dP} = 0$

$$PE(\hat{\theta}_{MLE})^2 + (1-2P)E(\hat{\theta}_{MLE})E(\hat{\theta}_p) - (1-P)E(\hat{\theta}_p)^2 - \theta E(\hat{\theta}_{MLE})$$

$$+\theta E(\hat{\theta}_p) = 0$$

$$P = \frac{\theta E(\hat{\theta}_{MLE}) - \theta E(\hat{\theta}_p) - E(\hat{\theta}_{MLE})E(\hat{\theta}_p) + E(\hat{\theta}_p)^2}{E(\hat{\theta}_{MLE})^2 - 2E(\hat{\theta}_{MLE})E(\hat{\theta}_p) + E(\hat{\theta}_p)^2}$$

$$E(\hat{\theta}_{MLE}) = E\left(\frac{\sum_{i=1}^n t_i}{n\alpha}\right) = \theta, \quad E(\hat{\theta}_p) = E\left(\frac{\sum_{i=1}^n t_i}{n\alpha+1}\right) = \frac{n\alpha\theta}{n\alpha+1},$$

$$E(\theta) = \theta, \qquad E(\hat{\theta}_{MLE})^2 = \frac{n\alpha+1}{n\alpha}, \quad E(\hat{\theta}_p)^2 = \frac{n\alpha}{n\alpha+1}$$

$$P = \frac{\theta^2 - \frac{n\alpha}{n\alpha+1}\theta^2 - \frac{n\alpha}{n\alpha+1\theta^2} + \frac{n\alpha}{n\alpha+1}\theta^2}{\frac{n\alpha+1}{n\alpha}\theta^2 - 2\frac{n\alpha}{n\alpha+1}\theta^2 + \frac{n\alpha}{n\alpha+1}\theta^2}$$

By simplifying the value of P: $P = \frac{n\alpha}{2n\alpha+1}$

$$\hat{\theta}_{Mix\,I} = P\hat{\theta}_{MLE} + (1-P)\hat{\theta}_p$$

$$= \left(\frac{n\alpha}{2n\alpha+1}\right)\frac{\sum_{i=1}^n t_i}{n\alpha} + \left(1 - \frac{n\alpha}{2n\alpha+1}\right)\frac{\sum_{i=1}^n t_i}{n\alpha+1} = \frac{2\sum_{i=1}^n t_i}{2n\alpha+1}$$

Then the Mix I estimator for the parameter θ is:

$$\hat{\theta}_{Mix\,I} = \frac{2\sum_{i=1}^n t_i}{2n\alpha+1} \qquad \cdots (13)$$

6.5: Second Mixture Method of SB and Pitman (Mix II)

This method is a suggesting estimator obtained from mixing two estimators which are Standard Bayesian estimator and pitman estimator, the aim of is to find an estimator that minimizes the MSE

$$\hat{\theta}_{Mix\,II} = P\hat{\theta}_{SB} + (1-P)\hat{\theta}_{P}$$

We have to find the value of P which makes the MSE has minimum value According to the following steps:

Subtracting θ from both sides

$$\hat{\theta}_{Mix\,II} - \theta = \left[P\hat{\theta}_{SB} + (1-P)\hat{\theta}_{P}\right] - \theta$$

Squaring the both sides and taking the expectation

$$E\left(\hat{\theta}_{Mix\,II} - \theta\right)^2 = E\{\left[P\hat{\theta}_{SB} + (1-P)\hat{\theta}_{P}\right] - \theta\}^2$$

$$= P^2E\left(\hat{\theta}_{SB}\right)^2 + 2P(1-P)E\left(\hat{\theta}_{SB}\right)E\left(\hat{\theta}_{P}\right) + (1-P)^2E\left(\hat{\theta}_{P}\right)^2 -$$

$$2PE(B)E(\theta) - 2(1-P)E\left(\hat{\theta}_{P}\right)E(\theta) + E(\theta)^2$$

To find the minimizing value

$$\frac{dE\left(\hat{\theta}_{Mix\,II} - \theta\right)^2}{dP} = 2PE\left(\hat{\theta}_{SB}\right)^2 + (2-4P)E\left(\hat{\theta}_{SB}\right)E\left(\hat{\theta}_{P}\right)$$

$$-2(1-P)E\left(\hat{\theta}_{P}\right)^2 - 2E\left(\hat{\theta}_{SB}\right)E(\theta) + 2E\left(\hat{\theta}_{P}\right)E(\theta) = 0$$

And setting $\frac{dE\left(\hat{\theta}_{Mix\,I} - \theta\right)^2}{dP} = 0$

$$PE\left(\hat{\theta}_{SB}\right)^2 + (1-2P)E\left(\hat{\theta}_{B}\right)E\left(\hat{\theta}_{P}\right) - (1-P)E\left(\hat{\theta}_{P}\right)^2 - \theta E\left(\hat{\theta}_{SB}\right) + \theta E\left(\hat{\theta}_{P}\right) = 0$$

$$P = \frac{\theta E\left(\hat{\theta}_{SB}\right) - \theta E\left(\hat{\theta}_{P}\right) - E\left(\hat{\theta}_{SB}\right)E\left(\hat{\theta}_{P}\right) + E\left(\hat{\theta}_{P}\right)^2}{E\left(\hat{\theta}_{SB}\right)^2 - 2E\left(\hat{\theta}_{SB}\right)E\left(\hat{\theta}_{P}\right) + E\left(\hat{\theta}_{P}\right)^2}$$

$$E(\hat{\theta}_{SB}) = E\left(\frac{\sum_{i=1}^{n} t_i}{n\alpha-1}\right) = \theta, \qquad E(\hat{\theta}_P) = E\left(\frac{\sum_{i=1}^{n} t_i}{n\alpha+1}\right) = \frac{n\alpha\theta}{n\alpha+1},$$

$$E(\theta) = \theta, \qquad E(\hat{\theta}_{SB})^2 = \frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}, \quad E(\hat{\theta}_P)^2 = \frac{n\alpha}{n\alpha+1}$$

$$P = \frac{\frac{n\alpha}{n\alpha-1}\theta^2 - \frac{n\alpha}{n\alpha+1}\theta^2 - \frac{(n\alpha)^2}{(n\alpha-1)(n\alpha+1)}\theta^2 + \frac{n\alpha}{n\alpha+1}\theta^2}{\frac{n\alpha(n\alpha+1)}{(n\alpha-1)^2}\theta^2 - 2\frac{(n\alpha)^2}{(n\alpha-1)(n\alpha+1)}\theta^2 + \frac{n\alpha}{n\alpha+1}\theta^2}$$

By simplifying the value of P: $P = \frac{n\alpha-1}{2(n\alpha+1)}$

$$\hat{\theta}_{Mix\,II} = P\hat{\theta}_{SB} + (1-P)\hat{\theta}_P$$

$$= \left(\frac{n\alpha-1}{2(n\alpha+1)}\right)\frac{\sum_{i=1}^{n} t_i}{n\alpha-1} + \left(1 - \frac{n\alpha}{2-1(n\alpha+1)}\right)\frac{\sum_{i=1}^{n} t_i}{n\alpha+1}$$

$$= \frac{(n\alpha+2)\sum_{i=1}^{n} t_i}{(n\alpha+1)^2}$$

Then the Mix II estimator for the parameter θ is:

$$\hat{\theta}_{Mix\,II} = \frac{(n\alpha+2)\sum_{i=1}^{n} t_i}{(n\alpha+1)^2} \qquad \cdots (14)$$

7: Risk Efficiency

The risk efficiency is denoted by $R(\hat{\theta},\theta)$ given by $E[L(\hat{\theta},\theta)]$

$$R(\hat{\theta},\theta) = E[L(\hat{\theta},\theta)] = \begin{cases} \int_{R_x} L(\hat{\theta},\theta) f(x;\theta)\,dx \\ \text{or} \\ \sum_{X\in R_X} L(\hat{\theta},\theta) p(x;\theta) \end{cases}$$

Clearly if $R(\hat{\theta}_1,\theta) \le R(\hat{\theta}_2,\theta)$ for all θ and $R(\hat{\theta}_1,\theta) < R(\hat{\theta}_2,\theta)$ for some θ then the risk efficiency for $\hat{\theta}_1$ is better than the risk efficiency for $\hat{\theta}_2$. We say that $\hat{\theta}_2$ is inadmissible. (Jasim, 2010)

7.1: The Risk Efficiency of $\hat{\theta}_{MLE}$ Under Squared Error Loss Function

The risk functions of the estimator under squared error loss is denoted by $R_s(\hat{\theta}_{MLE},\theta)$, is given by:

$$R_s(\hat{\theta}_{MLE},\theta) = \int_0^\infty (\hat{\theta}_{MLE}-\theta)^2 f(x_1,x_2,\dots,x_n|\theta)dx_1dx_2\dots.dx_n$$

Let $S=\sum X_i$, because $X_i, i=1,2,\dots,n$ are identically distributed and independent from gamma distribution with parameters (α,θ) then $S=\sum X_i \sim Gamma(n\alpha,\theta)$, so that:

$$R_s(\hat{\theta}_{MLE},\theta) = \int_0^\infty \left(\hat{\theta}_{MLE}{}^2 - 2\hat{\theta}_{MLE}\theta+\theta^2\right)\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$=\int_0^\infty \hat{\theta}_{MLE}{}^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds - 2\int_0^\infty \hat{\theta}_{MLE}\theta\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$+\int_0^\infty \theta^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

By substituting equation (4):

$$\hat{\theta}_{MLE}=\frac{\sum X_i}{n\alpha}=\frac{S}{n\alpha}$$

$$R_s(\hat{\theta}_{MLE},\theta) = \int_0^\infty \left(\frac{S}{n\alpha}\right)^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$-2\int_0^\infty \left(\frac{S}{n\alpha}\right)\theta\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$+\int_0^\infty \theta^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$=\frac{1}{(n\alpha)^2\Gamma(n\alpha)\theta^{n\alpha}}\int_0^\infty s^{n\alpha+1}e^{-\frac{s}{\theta}}ds$$

$$-\frac{2}{(n\alpha)\Gamma(n\alpha)\theta^{n\alpha-1}}\int_0^\infty s^{n\alpha}e^{-\frac{s}{\theta}}ds+\theta^2$$

$$= \frac{\Gamma(n\alpha+2)\theta^{n\alpha+2}}{(n\alpha)^2\,\Gamma(n\alpha)\theta^{n\alpha}} - \frac{2\Gamma(n\alpha+1)\theta^{n\alpha+1}}{(n\alpha)\Gamma(n\alpha)\theta^{n\alpha-1}} + \theta^2$$

$$= \frac{(n\alpha+1)}{(n\alpha)}\theta^2 - 2\theta^2 + \theta^2 = \frac{\theta^2}{(n\alpha)}$$

Then, the risk efficiency of $\hat{\theta}_{MLE}$ is:

$$R_s(\hat{\theta}_{MLE},\theta) = \frac{\theta^2}{(n\alpha)} \qquad \cdots (15)$$

7.2: The Risk Efficiency of $\hat{\theta}_{SB}$ Under Squared Error Loss Function

The risk functions of the estimator under squared error loss is denoted by $R_s(\hat{\theta}_{SB},\theta)$, is given by:

$$R_s(\hat{\theta}_{SB},\theta) = \int_0^\infty (\hat{\theta}_{SB}-\theta)^2 f(x_1,x_2,\dots,x_n|\theta)dx_1dx_2\dots.dx_n$$

Let $S=\sum X_i$,then $S=\sum X_i \sim Gamma(n\alpha,\theta)$, so that:

$$R_s(\hat{\theta}_{SB},\theta) = \int_0^\infty \left(\hat{\theta}_{SB}{}^2 - 2\hat{\theta}_{SB}\theta + \theta^2\right)\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$= \int_0^\infty \hat{\theta}_{SB}{}^2 \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$-2\int_0^\infty \hat{\theta}_{SB}\theta\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$+\int_0^\infty \theta^2 \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

By substituting equation (9) and simplifying the results, the risk efficiency of $\hat{\theta}_{SB}$ is:

$$R_s(\hat{\theta}_{SB},\theta) = \frac{\theta^2(n\alpha+1)}{(n\alpha-1)^2} \qquad \cdots (16)$$

7.3: The Risk Efficiency of $\hat{\theta}_p$ Under Squared Error Loss Function

The risk functions of the estimator under squared error loss is denoted by $R_s(\hat{\theta}_p,\theta)$, is given by:

$$R_s(\hat{\theta}_p,\theta) = \int_0^{\infty} (\hat{\theta}_p - \theta)^2 f(x_1, x_2, \dots, x_n|\theta) dx_1 dx_2 \dots . dx_n$$

Let $S = \sum X_i$, because $X_i, i = 1,2, \dots, n$ are identically distributed and independent from gamma distribution with parameters (α, θ) then $S = \sum X_i \sim Gamma(n\alpha, \theta)$, so that:

$$R_s(\hat{\theta}_p,\theta) = \int_0^{\infty} \left(\hat{\theta}_p{}^2 - 2\hat{\theta}_p\theta + \theta^2\right) \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1} e^{-\frac{s}{\theta}} ds$$

$$= \int_0^{\infty} \hat{\theta}_p{}^2 \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1} e^{-\frac{s}{\theta}} ds - 2\int_0^{\infty} \hat{\theta}_p \theta \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1} e^{-\frac{s}{\theta}} ds$$

$$+ \int_0^{\infty} \theta^2 \frac{1}{\Gamma n\alpha\ \theta^{n\alpha}} s^{n\alpha-1} e^{-\frac{s}{\theta}} ds$$

By substituting equation (12) and simplifying the results, the risk efficiency of $\hat{\theta}_p$ is:

$$R_s(\hat{\theta}_p,\theta) = \frac{\theta^2}{n\alpha + 1} \qquad \cdots (17)$$

7.4: The Risk Efficiency of $\hat{\theta}_{Mix\,I}$ Under Squared Error Loss Function

The risk functions of the estimator under squared error loss is denoted by $R_s(\hat{\theta}_{Mix\,I},\theta)$ is given by:

$$R_s(\hat{\theta}_{Mix\,I},\theta) = \int_0^{\infty} (\hat{\theta}_{Mix\,I} - \theta)^2 f(x_1, x_2, \dots, x_n|\theta) dx_1 dx_2 \dots . dx_n$$

Let $S = \sum X_i$, because $X_i, i = 1,2, \dots, n$ are identically distributed and independent from gamma distribution with parameters (α, θ) then $S = \sum X_i \sim Gamma(n\alpha, \theta)$, so that:

$$R_s(\hat{\theta}_{Mix\,I},\theta)=\int_0^\infty\left(\hat{\theta}_{Mix\,I}{}^2-2\hat{\theta}_{Mix\,I}\theta+\theta^2\right)\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$=\int_0^\infty\hat{\theta}_{Mix\,I}{}^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds-2\int_0^\infty\hat{\theta}_{Mix\,I}\,\theta\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$+\int_0^\infty\theta^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

By substituting equation (13) and simplifying the results; Then, the risk efficiency of $\hat{\theta}_{Mix\,I}$ is:

$$R_s(\hat{\theta}_{Mix\,I},\theta)=\frac{4n\alpha+1}{(2n\alpha+1)^2}\theta^2\qquad\cdots(18)$$

7.5: The Risk Efficiency of $\hat{\theta}_{Mix\,II}$ Under Squared Error Loss Function

The risk functions of the $\hat{\theta}_{Mix\,II}$ estimator under squared error loss is denoted by $R_s(\hat{\theta}_{Mix\,II},\theta)$ is given by:

$$R_s(\hat{\theta}_{Mix\,II},\theta)=\int_0^\infty(\hat{\theta}_{Mix\,II}-\theta)^2 f(x_1,x_2,\dots,x_n|\theta)dx_1dx_2\dots dx_n$$

Let $S=\sum X_i$,then $S=\sum X_i\sim Gamma(n\alpha,\theta)$, so that:

$$R_s(\hat{\theta}_{Mix\,II},\theta)=\int_0^\infty\left(\hat{\theta}_{Mix\,II}{}^2-2\hat{\theta}_{Mix}\theta+\theta^2\right)\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$=\int_0^\infty\hat{\theta}_{Mix\,I}{}^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds-2\int_0^\infty\hat{\theta}_{Mix\,II}\,\theta\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

$$+\int_0^\infty\theta^2\frac{1}{\Gamma n\alpha\ \theta^{n\alpha}}s^{n\alpha-1}e^{-\frac{s}{\theta}}ds$$

By substituting equation (14) and simplifying the results; Then the risk efficiency of $\hat{\theta}_{Mix\,II}$ is:

$$R_s(\hat{\theta}_{Mix\,II},\theta)=\frac{n\alpha+(n\alpha+1)^2}{(n\alpha+1)^3}\theta^2\qquad\cdots(19)$$

8: Relative Efficiency

The relative efficiencies to the square error loss function of Pitman estimator $\hat{\theta}_P$ relative to $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{Mix\ I}$, $\hat{\theta}_{Mix\ II}$ are denoted by $RE_S(\hat{\theta}_{MLE}, \hat{\theta}_P)$, $RE_S(\hat{\theta}_{SB}, \hat{\theta}_P)$, $RE_S(\hat{\theta}_{Mix\ I}, \hat{\theta}_P)$, and $RE_S(\hat{\theta}_{Mix\ II}, \hat{\theta}_P)$, respectively, and are given by:

$$1\cdot\ RE_S(\hat{\theta}_{MLE}, \hat{\theta}_P) = \frac{R_s(\hat{\theta}_{MLE}, \theta)}{R_s(\hat{\theta}_p, \theta)} = \frac{\frac{\theta^2}{n\alpha}}{\frac{\theta^2}{n\alpha}+1}$$

$$= \frac{n\alpha+1}{n\alpha} \quad \cdots (20)$$

$$2\cdot\ RE_S(\hat{\theta}_{SB}, \hat{\theta}_P) = \frac{R_s(\hat{\theta}_{SB}, \theta)}{R_s(\hat{\theta}_p, \theta)} = \frac{\frac{\theta^2(n\alpha+1)}{(n\alpha-1)^2}}{\frac{\theta^2}{n\alpha+1}}$$

$$= \frac{(n\alpha+1)^2}{(n\alpha-1)^2} \quad \cdots (21)$$

$$3\cdot\ RE_S(\hat{\theta}_{Mix\ I}, \hat{\theta}_P) = \frac{R_s(\hat{\theta}_{Mix\ I}, \theta)}{R_s(\hat{\theta}_p, \theta)} = \frac{\frac{\theta^2(n4\alpha+1)}{(2n\alpha+1)^2}}{\frac{\theta^2}{n\alpha+1}}$$

$$= \frac{(n\alpha+1)(n4\alpha+1)}{(2n\alpha+1)^2} \quad \cdots (22)$$

$$4\cdot\ RE_S(\hat{\theta}_{Mix\ II}, \hat{\theta}_P) = \frac{R_s(\hat{\theta}_{Mix\ II,\theta})}{R_s(\hat{\theta}_p, \theta)} = \frac{\frac{\theta^2(n\alpha+(n\alpha+1)^2)}{(n\alpha+1)^3}}{\frac{\theta^2}{n\alpha+1}}$$

$$= \frac{n\alpha+(n\alpha+1)^2}{(n\alpha+1)^2} \quad \cdots (23)$$

9: Application Part

In this section, the simulation method was used to generate random data for the two parameter Gamma distribution in order to compare Pitman method (P) with some estimation methods to estimate the scale parameter of this distribution, which are (the Maximum Likelihood Estimator method (MLE), Standard Bayesian method (SB), first Mixture method between MLE and P method (Mix I); and second Mixture method between SB and P method (Mix II)). The study was conducted using different sample sizes and also different parameter values. The comparison between the estimation methods for estimating the scale parameter was made by utilizing the relative efficiency.

9.1: Simulation

Simulation is defined as a process of representation and the tradition of the true reality by using certain models, we often find in actual fact is that there the processes with complex understanding and analysis therefore it is better that we describe these processes in similar to its real form certain models, understanding the model will give us a sense to original process or actual fact through its simulation (model), and it is normal that degree of similarly between simulation experiment and actual fact depends on matching or similarity range for the simulation procedure to actual fact.

There are different methods of simulation are:

1. Analogy Procedure.
2. Mixed Procedure.
3. Monte Carlo Procedure.

The Monte Carlo procedure, which is the most famous procedure and frequently been used based on the idea of generating random samples from virtual theoretical population which has similarity to real population that is used to generate observation of the most known of probability distributions. (Gentle, 2003) (Rubinstein & Kroese, 2008)

9.1.1: Stages of Building Simulation Experiment

Building the simulation experiment includes four stages which are essential to estimate the scale parameter of the gamma distribution (α,θ). The stages are as follows:

a. First Stage (set default values):

This is the most important stage of the basic stages; the other stages depend on it in the program of building simulation experience; it sets the default values, namely:

1. Specify default values for the parameters (α,θ). In this research nine models have been considered, which are arranged as follows:

Table (1)

The default value of the scale parameter

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9
α	1	1	1	2	2	2	3	3	3
θ	0.5	1	1.5	0.5	1	1.5	0.5	1	1.5

2. Choosing the sample size (n): choosing different sizes of the sample to determine the effect of sample size in deciding the accuracy and bitterness of the results obtained from the estimation methods used in this study. The samples have taken volumes characterized by being small (n = 30), medium (n = 50) and large sizes (n = 75, 100).
3. Choosing the number of sample replicated size (L): the number of sample replicated size (L = 1000).

b. The Second Stage (data generation):

Generating random numbers for the Gamma distribution with two parameters (α, θ), According to the available generation function in (Matlab-R2011a) language:

t = gamrnd (α, θ,[n 1])

c. The Third Stage (find estimators):

At this stage, estimating the scale parameter of the Gamma distribution through estimation methods are dealt with in the theoretical part of this research, according to the methods: - maximum likelihood method (MLE), Standard Bayesian method(SB), Pitman method (P), First mixture method between MLE and Pitman method (Mix I) and second mixture method between SB and Pitman method (Mix II).

d. The Fourth Stage (comparison):

Methods of comparing estimators for scale parameter:

After finding estimators in the third stage, at this stage the comparison between the estimation methods has been made using the relative efficiency of the risk efficiency under squared error loss function.

9.2: Analysis of Simulation Results

In this section the results of the simulation and analysis to get the best methods for estimating the scale parameters and reliability functio for the gamma distribution with two parameters are presented.

The results were obtained utilizing a program written in a Matlab language (Matlab-R2011a) by researcher.

3.4.1: Methods of Estimating the Scale Parameter θ:

The results are explained in tables (2) as follow:

Table (2)

Estimates of scale parameter estimation methods, for different sample size and all models

Model	n	$\hat{\theta}_{MLE}$	$\hat{\theta}_{SE}$	$\hat{\theta}_{Pitman}$	$\hat{\theta}_{Mix-I}$	$\hat{\theta}_{Mix-II}$
1	30	0.49451	0.51157	0.47856	0.48641	0.494
	50	0.49755	0.5077	0.48779	0.49262	0.49736
	75	0.50003	0.50679	0.49346	0.49672	0.49995
	100	0.50032	0.50537	0.49537	0.49783	0.50027
2	30	0.98903	1.02313	0.95713	0.97282	0.988
	50	0.9951	1.0154	0.97558	0.98524	0.99471
	75	1.00007	1.01358	0.98691	0.99345	0.9999
	100	1.00064	1.01075	0.99073	0.99566	1.00054
3	30	1.48354	1.5347	1.43569	1.45922	1.482
	50	1.49264	1.52311	1.46338	1.47787	1.49207
	75	1.5001	1.52038	1.48037	1.49017	1.49984
	100	1.50096	1.51612	1.4861	1.49349	1.50081
4	30	0.49814	0.50658	0.48997	0.49402	0.498
	50	0.49976	0.50481	0.49481	0.49727	0.49971
	75	0.50087	0.50423	0.49756	0.49921	0.50085
	100	0.50061	0.50312	0.49812	0.49936	0.50059
5	30	0.99627	1.01316	0.97994	0.98804	0.99601
	50	0.99952	1.00961	0.98962	0.99455	0.99942
	75	1.00175	1.00847	0.99511	0.99842	1.0017
	100	1.00121	1.00624	0.99623	0.99871	1.00119
6	30	1.49441	1.51974	1.46991	1.48206	1.49401
	50	1.49928	1.51442	1.48443	1.49182	1.49913
	75	1.50262	1.5127	1.49267	1.49763	1.50255
	100	1.50182	1.50936	1.49435	1.49807	1.50178
7	30	0.49879	0.50439	0.49331	0.49603	0.49873
	50	0.50009	0.50345	0.49678	0.49843	0.50007
	75	0.50086	0.5031	0.49865	0.49975	0.50085
	100	0.50052	0.50219	0.49885	0.49968	0.50051
8	30	0.99757	1.00878	0.98661	0.99206	0.99745
	50	1.00018	1.00689	0.99356	0.99686	1.00014
	75	1.00173	1.0062	0.9973	0.99951	1.00171
	100	1.00103	1.00438	0.99771	0.99937	1.00102
9	30	1.49636	1.51317	1.47992	1.48809	1.49618
	50	1.50027	1.51034	1.49034	1.49529	1.5002
	75	1.50259	1.5093	1.49595	1.49926	1.50256
	100	1.50155	1.50657	1.49656	1.49905	1.50153

From table (2) we notice that the estimated scale parameters θ by using all studied estimation methods revealed that the mean is close to default values for this parameter in all models and samples sizes, in addition we notice the following:

1. For all models we noticed that most of averages for estimated scale parameter θ in all estimation methods are close to default values for this parameter by increasing of the samples sizes.
2. Increasing the shape parameter value α leads to increase the averages of estimated parameter θ for all methods except of SB method which the results are low.

Table (3)

The risk efficiencies and the relative efficiency for estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{P}$, $\hat{\theta}_{Mix\,I}$ and $\hat{\theta}_{Mix\,II}$ in a sample *n*=30

α	θ	$R_s(\hat{\theta}_{MLE},\theta)$	$R_s(\hat{\theta}_{SB},\theta)$	$R_s(\hat{\theta}_{P},\theta)$	$R_s(\hat{\theta}_{Mix\,I},\theta)$	$R_s(\hat{\theta}_{Mix\,II},\theta)$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{MLE})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{SB})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\,I})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\,II})$
1	**0.5**	0.00833	0.00922	0.00806	0.00813	0.00832	1.03333	1.14269	1.00806	1.03122
1	**1**	0.03333	0.03686	0.03226	0.03252	0.03327	1.03333	1.14269	1.00806	1.03122
1	**1.5**	0.075	0.08294	0.07258	0.07317	0.07485	1.03333	1.14269	1.00806	1.03122
1.5	**0.5**	0.00556	0.00594	0.00543	0.00546	0.00555	1.02222	1.09298	1.00543	1.02127
1.5	**1**	0.02222	0.02376	0.02174	0.02186	0.0222	1.02222	1.09298	1.00543	1.02127
1.5	**1.5**	0.05	0.05346	0.04891	0.04918	0.04995	1.02222	1.09298	1.00543	1.02127
2	**0.5**	0.00417	0.00438	0.0041	0.00412	0.00416	1.01667	1.06895	1.0041	1.01612
2	**1**	0.01667	0.01752	0.01639	0.01646	0.01666	1.01667	1.06895	1.0041	1.01612
2	**1.5**	0.0375	0.03943	0.03689	0.03704	0.03748	1.01667	1.06895	1.0041	1.01612
2.5	**0.5**	0.00333	0.00347	0.00329	0.0033	0.00333	1.01333	1.05478	1.00329	1.01298
2.5	**1**	0.01333	0.01388	0.01316	0.0132	0.01333	1.01333	1.05478	1.00329	1.01298
2.5	**1.5**	0.03	0.03123	0.02961	0.0297	0.02999	1.01333	1.05478	1.00329	1.01298
3	**0.5**	0.00278	0.00287	0.00275	0.00275	0.00278	1.01111	1.04545	1.00275	1.01087
3	**1**	0.01111	0.01149	0.00989	0.01102	0.01111	1.01111	1.04545	1.00275	1.01087
3	**1.5**	0.025	0.02585	0.02473	0.02479	0.02499	1.01111	1.04545	1.00275	1.01087

Table (4)

The risk efficiencies and the relative efficiency for estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{P}$, $\hat{\theta}_{Mix\ I}$ and $\hat{\theta}_{Mix\ II}$ in a sample *n*=50

α	θ	$R_s(\hat\theta_{MLE},\theta)$	$R_s(\hat\theta_{SB},\theta)$	$R_s(\hat\theta_{P},\theta)$	$R_s(\hat\theta_{Mix\ I},\theta)$	$R_s(\hat\theta_{Mix\ II},\theta)$	$RE_s(\hat\theta_{P},\hat\theta_{MLE})$	$RE_s(\hat\theta_{P},\hat\theta_{SB})$	$RE_s(\hat\theta_{P},\hat\theta_{Mix\ I})$	$RE_s(\hat\theta_{P},\hat\theta_{Mix\ II})$
1	**0.5**	0.005	0.00531	0.0049	0.00493	0.005	1.02	1.0833	1.0049	1.01922
1	**1**	0.02	0.02124	0.01961	0.0197	0.01998	1.02	1.0833	1.0049	1.01922
1	**1.5**	0.045	0.04779	0.04412	0.04433	0.04497	1.02	1.0833	1.0049	1.01922
1.5	**0.5**	0.00333	0.00347	0.00329	0.0033	0.00333	1.01333	1.05478	1.00329	1.01298
1.5	**1**	0.01333	0.01388	0.01316	0.0132	0.01333	1.01333	1.05478	1.00329	1.01298
1.5	**1.5**	0.03	0.03123	0.02961	0.0297	0.02999	1.01333	1.05478	1.00329	1.01298
2	**0.5**	0.0025	0.00258	0.00248	0.00248	0.0025	1.01	1.04081	1.00248	1.0098
2	**1**	0.01	0.01031	0.0099	0.00993	0.01	1.01	1.04081	1.00248	1.0098
2	**1.5**	0.0225	0.02319	0.02228	0.02233	0.0225	1.01	1.04081	1.00248	1.0098
2.5	**0.5**	0.002	0.00205	0.00198	0.00199	0.002	1.008	1.03252	1.00198	1.00787
2.5	**1**	0.008	0.00819	0.00794	0.00795	0.008	1.008	1.03252	1.00198	1.00787
2.5	**1.5**	0.018	0.01844	0.01786	0.01789	0.018	1.008	1.03252	1.00198	1.00787
3	**0.5**	0.00167	0.0017	0.00166	0.00166	0.00167	1.00667	1.02703	1.00166	1.00658
3	**1**	0.00667	0.0068	0.00662	0.00663	0.00667	1.00667	1.02703	1.00166	1.00658
3	**1.5**	0.015	0.0153	0.0149	0.01493	0.015	1.00667	1.02703	1.00166	1.00658

Table (5)

The risk efficiencies and the relative efficiency for estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{P}$, $\hat{\theta}_{Mix\ I}$ and $\hat{\theta}_{Mix\ II}$ in a sample *n*=75

α	θ	$R_s(\hat{\theta}_{MLE},\theta)$	$R_s(\hat{\theta}_{SB},\theta)$	$R_s(\hat{\theta}_{P},\theta)$	$R_s(\hat{\theta}_{Mix\ I},\theta)$	$R_s(\hat{\theta}_{Mix\ II},\theta)$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{MLE})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{SB})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\ I})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\ II})$
1	0.5	0.00333	0.00347	0.00329	0.0033	0.00333	1.01333	1.05478	1.00329	1.01298
1	1	0.01333	0.01388	0.01316	0.0132	0.01333	1.01333	1.05478	1.00329	1.01298
1	1.5	0.03	0.03123	0.02961	0.0297	0.02999	1.01333	1.05478	1.00329	1.01298
1.5	0.5	0.00222	0.00228	0.0022	0.00221	0.00222	1.00889	1.0362	1.0022	1.00873
1.5	1	0.00889	0.00913	0.00881	0.00883	0.00889	1.00889	1.0362	1.0022	1.00873
1.5	1.5	0.02	0.02054	0.01982	0.01987	0.02	1.00889	1.0362	1.0022	1.00873
2	0.5	0.00167	0.0017	0.00166	0.00166	0.00167	1.00667	1.02703	1.00166	1.00658
2	1	0.00667	0.0068	0.00662	0.00663	0.00667	1.00667	1.02703	1.00166	1.00658
2	1.5	0.015	0.0153	0.0149	0.01493	0.015	1.00667	1.02703	1.00166	1.00658
2.5	0.5	0.00133	0.00135	0.00133	0.00133	0.00133	1.00533	1.02156	1.00133	1.00528
2.5	1	0.00533	0.00542	0.00531	0.00531	0.00533	1.00533	1.02156	1.00133	1.00528
2.5	1.5	0.012	0.01219	0.01194	0.01195	0.012	1.00533	1.02156	1.00133	1.00528
3	0.5	0.00111	0.00113	0.00111	0.00111	0.00111	1.00444	1.01794	1.00111	1.00441
3	1	0.00444	0.0045	0.00442	0.00443	0.00444	1.00444	1.01794	1.00111	1.00441
3	1.5	0.01	0.01013	0.00996	0.00997	0.01	1.00444	1.01794	1.00111	1.00441

Table (6)

The risk efficiencies and the relative efficiency for estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{P}$, $\hat{\theta}_{Mix\ I}$ and $\hat{\theta}_{Mix\ II}$ in a sample *n*=100

α	θ	$R_s(\hat{\theta}_{MLE},\theta)$	$R_s(\hat{\theta}_{SB},\theta)$	$R_s(\hat{\theta}_{P},\theta)$	$R_s(\hat{\theta}_{Mix\ I},\theta)$	$R_s(\hat{\theta}_{Mix\ II},\theta)$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{MLE})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{SB})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\ I})$	$RE_s(\hat{\theta}_{P},\hat{\theta}_{Mix\ II})$
1	**0.5**	0.0025	0.00258	0.00248	0.00248	0.0025	1.01	1.04081	1.00248	1.0098
1	**1**	0.01	0.01031	0.0099	0.00993	0.01	1.01	1.04081	1.00248	1.0098
1	**1.5**	0.0225	0.02319	0.02228	0.02233	0.0225	1.01	1.04081	1.00248	1.0098
1.5	**0.5**	0.00167	0.0017	0.00166	0.00166	0.00167	1.00667	1.02703	1.00166	1.00658
1.5	**1**	0.00667	0.0068	0.00662	0.00663	0.00667	1.00667	1.02703	1.00166	1.00658
1.5	**1.5**	0.015	0.0153	0.0149	0.01493	0.015	1.00667	1.02703	1.00166	1.00658
2	**0.5**	0.00125	0.00127	0.00124	0.00125	0.00125	1.005	1.0202	1.00124	1.00495
2	**1**	0.005	0.00508	0.00498	0.00498	0.005	1.005	1.0202	1.00124	1.00495
2	**1.5**	0.01125	0.01142	0.01119	0.01121	0.01125	1.005	1.0202	1.00124	1.00495
2.5	**0.5**	0.001	0.00101	0.001	0.001	0.001	1.004	1.01613	1.001	1.00397
2.5	**1**	0.004	0.00405	0.00398	0.00399	0.004	1.004	1.01613	1.001	1.00397
2.5	**1.5**	0.009	0.00911	0.00896	0.00897	0.009	1.004	1.01613	1.001	1.00397
3	**0.5**	0.00083	0.00084	0.00083	0.00083	0.00083	1.00333	1.01342	1.00083	1.00331
3	**1**	0.00333	0.00337	0.00332	0.00332	0.00333	1.00333	1.01342	1.00083	1.00331
3	**1.5**	0.0075	0.00758	0.00748	0.00748	0.0075	1.00333	1.01342	1.00083	1.00331

From tables (3), (4), (5) and (6) we notice the followings:

1. The increase of the value of the shape parameter α decreases the ratio of each of risk efficiency and relative efficiency for all estimation methods of scale parameter θ and for all samples sizes.
2. The increase of the value of scale parameter θ leads to increase the value of the Risk efficiency but the value of the relative efficiency is constant by change scale parameter θ.
3. Through the comparison between the methods of estimation the scale parameter θ depending on the risk efficiency, in general, the Pitman estimator method gives the lowest risk efficiency for all different samples sizes.
4. By increasing the sample size, the ratio of each of relative efficiency and risk efficiency decrease for all of the estimation methods of scale parameter θ and for all models.
5. The relative efficiency of Pitman estimator to other estimators $RE_s(\hat{\theta}_P, \hat{\theta}_{MLE})$, $RE_s(\hat{\theta}_P, \hat{\theta}_{SB})$, $RE_s(\hat{\theta}_P, \hat{\theta}_{Mix\,I})$ and $RE_s(\hat{\theta}_P, \hat{\theta}_{Mix\,II})$ are greater than 1, which means that the proposed estimator $\hat{\theta}_P$ is preferable to other estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{SB}$, $\hat{\theta}_{Mix\,I}$ and $\hat{\theta}_{Mix\,II}$ for all sample sizes; or the pitman estimator is gives admissible estimator.

4.1: Conclusions

The following conclusions has been drawn:

1. For all models, we notice that most of the averages of estimated scale parameter θ in all estimation methods are very close to the default values for this parameter when sample size increases. Also; we noticed that most of the averages of estimated reliability function will be very close to the real value of the reliability function when sample size increases except the SB method to estimate the reliability function constant.
2. Through the comparison between the estimation methods of the scale parameter θ utilizing risk efficiency, and in general the Pitman estimator method gives less risk efficiency for different sample sizes.
3. The ratio of each of relative efficiency and risk efficiency decreases when the value of the sample size increases for all models.

4. When the shape parameter α increase the ratio of relative efficiency and the ratio of risk efficiency decrease in all estimation methods of the scale parameter θ and to all sample sizes.
5. The relative efficiencies are greater than 1, demonstrating the bitterness of Pitman estimator to other estimators.

References

1. Agarwal, B. (2003) Programmed Statistics (Question-Answers), Second Edition. New Age International.
2. Akahira, M., Ohyauchi, N. & Takeuchi, K. (2007) On the Pitman Estimator for a Family of Non-Regular Distributions. METRON-International Journal of Statistics. 65 (1). p.113-127.
3. Casella, G. & Berger, R. L. (2002) Statistical Inference, Second Edition. Duxbury.
4. Choi, S. C. & Wette, R. (1969) Maximum Likelihood Estimation of the Parameters of the Gamma Distribution and Their Bias. Journal of American Statistical Association and American Society for Quality. 11 (4). p.683-690.
5. Coit, D. W. & Jin, T. (2000) Gamma Distribution Parameter Estimation for Field Reliability Data with Missing Failure Times. IIE Transaction. 32 (12). p.1161-1166.
6. Freue, G. C. (2007) The Pitman Estimator of the Cauchy Location Parameter. Journal of Statistical Planning and Inference, 137 (6). p.1900-1913.
7. Gentle, J. E. (2003) Random Number Generation & Monte Carlo Methods. Springer Sciences Business Media. Inc., New York, USA.
8. Jasim, W. A. (2010) Bayes Estimator of One Parameter Gamma Distribution Under Quadratic and LINEX Loss Function. Iraqi Journal of Statistical Science. 9 (16). p.13-28.
9. Kishan, R. (2014) Comparison Between MLE and Bayes Estimation of Scale Parameter of Generalized Gamma Distribution with Known Shape Parameters Under Squared Error Loss Function. Journal of Reliability and Statistical Studies. 7 (1). p.43-50.
10. Miller, R. B. (1980) Bayesian Analysis of the Two-Parameter Gamma Distribution. Journal of American Statistical Association and American Society for Quality, 22 (1). p.65-69.
11. Mishra, R. C., & Sandilya, A. (2009) Reliability and Quality management. New Age International.
12. Robert, C. P. (2007) The Bayesian Choice: From Decision-Theoretic Foundations to Computational Implementation, Second Edition. Springer Science & Business Media.
13. Rubinstein, R. Y. & Kroese, D. P. (2008) Simulation & the Monte Carlo Method. John Wiley & Sons, Inc., Canada.
14. Särkkä, S. (2013) Bayesian Filtering and Smoothing. Cambridge University Press.