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A Class of Fuzzy Integral Equations and the Existence and Uniqueness of Their Solutions

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Abstract

The aim of this paper to presented a class of interesting fuzzy integral nonlinear equations involving nonlinear fuzzy function of other multi fuzzy nonlinear functions that presented in fist time, the formula under study is an interesting type of a fuzzy nonlinear Volterra integral equation which defined on fuzzy numbers that has lower and upper functions. The Housdorf fuzzy metric space on fuzzy numbers and a fuzzy metric space on a fuzzy continuous functions and the space of all uniform modulus of absolutely fuzzy continuous functions have been given in details. Also the observer results investigated on parameter cases of fuzzy problem formulation, some necessary and sufficient conditions of boundedness and locally Lipschitz for different fuzzy functions and differential of them and also using Leibnitz rule for presented fuzzy problem formulation. The locally existence and uniqueness solutions in the space of all uniform modulus of absolutely fuzzy continuous functions which is generalized of fuzzy number space and established on contraction condition which proved by using the properties of fuzzy metric space this certain space. Also used concept of fuzzy Riemann integral and normed function to prove interesting results and illustrated in some examples to clear the proposal fuzzy integral nonlinear equations has a solution in certain fuzzy space that in firstly explained with different metrics spaces and normed properties and satisfied all the hypothesis for the component of the equation and obtain the estimations of the suggested conditions on certain fuzzy metrics as a stuffiest condition for grantee the assuming equations has a fuzzy solution with their lower and upper parameters.

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صنف من المعادلات التكاملية الضبابية والوجودية والوحدانية لحلولها

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الخلاصة

الهدف من هذا البحث هو عرض صنف من المعادلات التكاملية الضبابية غير الخطية تحوي على دالة ضبابية غير خطية لدوال أخرى ضبابية غير خطية عرضت لأول مرة، الصيغة التي تحت الدراسة من الأنواع المهمة من معادلات فولتيرا غير الخطية والتي تحوي على أدني واعلى دالة. الفضاء المتري الضبابي لهاوس

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دورف المعرف على الاعداد الضبابية والفضاءات المترية الضبابي على الدوال المستمرة الضبابية وأيضا عرفت على الدوال المستمرة الضبابية المطلقة ذات المعيار المنتظم أعطيت بشكل مفصل. كذلك النتائج الظاهرة حققت على الحالات المعلمية لصيغة المسالة الضبابية، بعض الشروط الضرورية والكافية للتقيد ولبشز لمختلف الدوال ومشتقاتها وكذلك استخدام قاعدة لبنز لصيغة المسالة المعروضة. الوجودية والوحدانية المحلية للحلول على فضاء الدوال المستمرة الضبابية المطلقة ذات المعيار المنتظم والذي هو تعميم لفضاء الاعداد الضبابية والمبنية على شرط الانكماش والذي زود من خلال استخدام خواص الفضاء المتري الفضاء المتري المختار. كذلك استخدام مفهوم تكامل ريمان الضبابي ودالة النورم لبرهان نتائج مهمة ووضحت ببعض من الأمثلة لإيضاح المعادلات التكاملية الضبابية غير الخطية المقترحة وحلها في الفضاء المختار والفضاء النورمي وخواصه ويحقق كل الفرضيات للمركبات صيغة المسالة المقترحة وإظهار المقدرات للشروط المناسبة في الفضاء المتري الضبابي المختار الي يعتبر شرط كافي لضمان وجود الحل الضبابي للمعادلات المفترضة.

1. Introduction

The fuzzy differential equations are considered as one of the basic equations in the field of engineering modeling, medical and physical applications, as most of their models are defined in fuzzy sets, and therefore the solutions are fuzzy, so they are confined between an upper value and a lower value, and these equations may consist of Volterra and Fredholm integral formulations. Sometime these fuzzy integral equations need some initial conditions or boundary conditions.

The interesting issue in analytic of fuzzy integral equations is the constriction of boundedness as well as necessary and sufficient conditions of convergent, so we presented the following researchers such as some of them studied the sufficient conditions for the solution to be of fuzzy integral equations is bounded in [1].The approximate solution of nonlinear Hammerstein fuzzy integral equations that is nonlinear, moreover the convergent of successive approximations method have been presented in [2]. The second kind of linear Fredholm fuzzy integral equations instructed their analytic solution by using parametric form of fuzzy numbers, see [3]. In [4], the fuzzy Volterra integral equations with time lag, also the efficient iterative and equivalent of uniqueness and convergence of the method had been given in details. Sometime studied together analytic and approximate solution and presented reliable numerical algorithm. The convert of linear fuzzy Fredholm integral equation to linear system of second kind integral equation is studied with fuzzy number parametric form, [5].as well as the new interesting results of common fixed-point used in applications on stochastic volterra integral equations achieved contraction prosperity, [6]. In [7], the ordinary approximation technique is used to solve fuzzy Fredholm integral equations with respect fuzzy valued functions is introduced by classical approximation.

The hybrid method for computation solutions of equations with degenerate kernel devoted in [8]. The optimal values of solution appeared by using homotopy analytic method in [9]. The existence and uniqueness are interested in many articles which is established the analytically and approximate solutions in different approaches for different methods where their targets are integrals have types Volterra and Fredholm. [10], [11],[12],

The fuzzy differentiability and integrability studied with a new notion correlated fuzzy processes on time scales was happened in [13], fuzzy Volterra and Fredholm integral equations with the convolution type kernel forced the integral fuzzy problem under Hukuhara differentiability used fuzzy Laplace transform method , [14]. In [6], the mixed of two technical of methods such as fuzzy Lagrange interpolation and the fuzzy Gauss-Legendre quadrature formula for establish collocation method based are interesting to solve linear and nonlinear fuzzy integral equations of two-dimensional and some main results for convergent

There are many numerical methods such as embedding method, Adomian method, Adomian decomposition method, variational iteration method and homotopy analysis method was used to compute a numerical solution for fuzzy differential and integral equations which investigated with different types of kernels functions and some of them used weakly singular kernels where in the domain of integration may change sign, or some time approximated the fuzzy delay integral equations by piecewise fuzzy polynomial interpolation see [15], [16]. Also the numerical algorithm for solving Fredholm and Volterra Riemann integral second kind and two-dimensional nonlinear fuzzy integrals, all have been presented to convert to a nonlinear system in a crisp case, see [17], [18], [19] and analytic of solution for second kind fuzzy Fredholm integral equations, [20]. The optimal homotopy asymptotic method which is efficient algorithm for solution of system of fuzzy Fredholm integral equations with two dimensional, [21].

The existence and uniqueness of solution for the Volterra-Fredholm nonlinear integral equation with fuzzy parameters have been presented in [22].In [23], Sharif et al. solved the one-dimensional fuzzy integral equations studded with fuzzy Laplace transform method. Double fuzzy sumudu transform used for solving Volterra fuzzy integral equations were carried by Alidema and Georgieva in [24]. The existence and uniqueness of nonlinear Volterra Fuzzy Integral Equations solutions have been investigated in[11]. Fuzzy integral equation with nonlinear fuzzy kernels has been studied with analysis of convergence which introduced in [25]. In[26],fuzzy integral equations of product type have been studied their existence and uniqueness solution. Additionally, in[8], the Integro-differential equations are nonlinear fuzzy. An Application for fuzzy Integral Equations by new Contraction studied in[12].

The aim of this paper to study the existence and uniqueness solutions of some fuzzy integral equations in uniform modulus of absolutely continuous functions space and explained in details by some illustrative examples simulated the formulation of fuzzy integral equations.

$$\tilde{u}(t,\alpha) = \tilde{f}(t,\alpha) +_F (FR)\int_a^t \tilde{F}\left(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big),\tilde{F}_2\big(t,s,u(s,\alpha)\big)\right) ds \quad (1)$$

where $\tilde{F}: [a,b] \times [a,b] \times R_F \times R_F \to R_F$, $\tilde{f}(t,\alpha): [a,b] \times [0,1] \to R_F$, $\tilde{F}_1, \tilde{F}_2: [a,b] \times [a,b] \times R_F \to R_F$ and $\tilde{u}(s,\alpha): [a,b] \times [0,1] \to R_F$.

This paper contains four sections: In section 2, some basic preliminaries, namely definitions such as Hausdorff metric on fuzzy number set, the fuzzy space of continuous functions and the first study in this paper is the space of all uniform modulus of absolutely fuzzy continuous as well as the fuzzy matrices defined on theses spaces to satisfy the purpose of the Remain fuzzy integrals. In section 3, studied and presented necessary and sufficient hypotheses to achieve existence and uniqueness for classes of fuzzy integral equations with suitable matrices and norms defined on related fuzzy differentials and integrals. In section 4. We give an illustrative example to prove the conditions proposed for existence and uniqueness of some fuzzy integral equations. Lastly, conclusions are shown in section 5.

2. Preliminaries

In this section interesting definitions and results used as a basic concept of existence and uniqueness for presented fuzzy integral equations.

Definition 1, [26].

If the following properties are satisfied for the function $u: R \to [0,1]$, R is a real number such that

(1) there exist $a_0 \in R$ such that $u(a_0) = 1$. Then u is normal.

(2) $u(\hat{\lambda}a + (1-\hat{\lambda})b) \geq \min\{u(a), u(b)\}$, $\forall\, a, b \in R$, $0 \leq \hat{\lambda} \leq 1$. Then u is convex fuzzy set.

(3) u is the upper semi-continuous on R.

(4) The support $[u]_0 = cl\{a \in R: u(a) > 0\}$ is a compact set .

Then u is a fuzzy number and the set of all fuzzy numbers is denoted by R_F.

Remark 1 [27].

1. Any real number $a \in R$ which can described as $\tilde{a} = \chi_{\{a\}}$ and therefore $R \subset R_F$.
2. The r-level set $[u]_r = \{a \in R: u(a) \geq r\}$ that is a closed interval $[u]_r = [\underline{u}_r, \overline{u}_r]$ for all $0 \leq r \leq 1$.
3. $(\underline{u}(r), \overline{u}(r))$ is a fuzzy number defined as the following $\underline{u}(r)$ is increasing and $\overline{u}(r)$ is decreasing functions $\underline{u}, \overline{u}: [0,1] \to R$.

Definition 2 [27].

For $u, v \in R_F$ and $\lambda \in R$, the sum $u +_F v$ defined by $[u + v]_r = [u]_r + [v]_r$ which represented the addition of two intervals (as sub sets of R) and the product λu defined by $[\lambda u]_r = \lambda [u]_r$, for all $0 \leq r \leq 1$,.

Definition 3 [28].

The algebraic prosperities on R_F defined as follows:

for any $u, v, w \in R_F$, we have that

1. $u +_F (v +_F w) = (u +_F v) +_F w$ and $u +_F v = v +_F u$
2. $u +_F \tilde{0} = \tilde{0} +_F u$ for $\tilde{0} = \chi_{\{0\}}$
3. $(a +_F b) ._F u = a ._F u +_F b ._F u$, for all $a, b \in R$ with $ab \geq 0$.
4. $a ._F (u +_F v) = a ._F u +_F a ._F v$ for all $a \in R$
5. $a ._F (b ._F v) = (a ._F b) ._F u$, for all $a, b \in R$ and $1 ._F u = u$.

Definition 4 [29].

The distance between fuzzy numbers defined by:

$D(u, v) = \sup_{r \in [0,1]} \{\max(|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ for all $u, v \in R_F$ which is called Housdorff metric.

Lemma 1 [29].

The Hausdorff metric has the following properties for all $u, v, w, e \in R_F$

1. (R_F, D) is a complete metric space
2. $D(u +_F w, v +_F w) = D(u, v)$
3. $D(u +_F v, w +_F e) \leq D(u, w) +_F D(v, e)$
4. $D(k ._F u, k ._F v) = |k| D(u, v)$, for all $k \in R$.
5. $(k_1 ._F u, k_2 ._F u) = |k_1 - k_2| D(u, \tilde{0})$, for all $k_1, k_2 \in R$ with $k_1 k_2 \geq 0$.

The property (4) gives the usual norm which is defined by the function $\|.\|: R_F \to R$ such that $\|u\| = D(u, \tilde{0})$ and we have to use it in throughout the article.

Lemma 2 [30].
If $\hat{f}, \hat{g}: [a, b] \subseteq R \to R_F$ are fuzzy functions which is continuous, then the function $F: [a, b] \to R^+$ defined by $F(t) = D\left(\hat{f}(t), \hat{g}(t)\right)$ is continuous on [a,b]. Also $D\left(\hat{f}(t), \tilde{0}\right) = \left\|\hat{f}(t)\right\| \leq M$ for all $x \in [a, b]$, $M > 0$ that is $\hat{f}$ is Globally bounded.

Definition 5 [30].
Let $\tilde{\emptyset} : [a, b] \to [0,1]$ be a real valued function, for every partition p=$[y_0, y_1, \dots, y_m]$ of [a,b] and every $\eta_j \epsilon [y_{i-1}, y_i]$, $1 \leq j \leq m$, we define $s_q = \sum_{i=0}^{m} \tilde{\emptyset}(\eta_j)(y_i - y_{i-1})$
Now let $\Delta_j = \max|y_i - y_{i-1}|$, i=1,2,…,m then for $\tilde{\emptyset}(y)$, over $[a, b]$ the defined integral can be written as $\lim_{\Delta_j \to 0} s_q = \int_a^b \tilde{\emptyset}(y)dy$ and the parametric form may be represented as $\int_a^b \tilde{\emptyset}(y)dy = \left(\int_a^b \underline{\emptyset}(y, k)dy, \int_a^b \overline{\emptyset}(y, k)dy\right)$

Lemma 3 [30].
If (FR) Remain integrable fuzzy functions, and real numbers defined by $\hat{f}, \hat{g}: [a, b] \to R_{F,}$, then
$(FR)\int_a^b (\alpha \cdot_F \hat{f}(t) +_F \beta \cdot_F \hat{g}(t)dt) = \alpha \cdot_F (FR) \int_a^b \hat{f}(t)dt +_F \beta \cdot_F (FR) \int_a^b \hat{g}(t)dt$
The following definition included the different fuzzy metrics on different fuzzy spaces related among them are interesting to prove the existence of solution for proposal fuzzy nonlinear equation (1), and also achieved interested condition is a completeness.

Definition 6.
1. The fuzzy space of continuous functions over $[a, b]$ denoted by $C_F([a, b])$. The metric defined on $C_F([a, b])$ is $D^*(f, g) = \sup_{a \leq t \leq b} D\left(f(t), g(t)\right)$, for all $f, g \in C_F([a, b])$ and complete fuzzy metric space. [13]
2. The function $w_{[a,b]}(f, \delta): R^+ \cup \{0\} \to R^+$ defined by
   $w_{[a,b]}(f, \delta) = \sup\left\{\sum_{i=1}^{n} D\left(f(t_i), f(\acute{t}_i)\right); t_i, \acute{t}_i \in [a, b], \sum_{i=1}^{n} |t_i - \acute{t}_i| \leq \delta\right\}$ is called uniform modulus of absolutely fuzzy continuous of $f \in C_F([a, b])$, $a \leq \acute{t}_i \leq t_i \leq b$.
3. Let $f \in V_F([a, b])$ be the space of all uniform modulus of absolutely fuzzy continuous of f on $[a, b]$ and $\int_a^t \left(D\left(\frac{\partial}{\partial t} f(t), \tilde{0}\right)\right) dt$ is a fuzzy Riemann integral.
4. The fuzzy metric defined on $V_F([a, b])$ as follows:
$$D^{\#}(\Gamma_1, \Gamma_2) = D^*(\Gamma_1(t), \Gamma_2(t)) +_F \int_a^t D^*\left(\frac{\partial}{\partial x}\Gamma_1(x), \frac{\partial}{\partial x}\Gamma_2(x)\right)dx,$$
   for any $\Gamma_1, \Gamma_2 \in V_F([a, b])$ and $V_F([a, b])$ is a complete metric space corresponding to a complete fuzzy metric space $(C_F([a, b], D^*)$.

The main results of this paper is to prove the existence and uniqueness of Fuzzy integral equations in (1) on $V_F([a, b])$ be the space of all uniform modulus of absolutely fuzzy continuous of f on $[a, b]$ by using metric space $D^{\#}(.,.)$ defined in definition(6) (4).

3. Main Results

In this section the fuzzy problem formulation has been presented with Remain fuzzy integral and fuzzy operations and take a class of Volterra fuzzy nonlinear integral formulation which given to study their existence solution.

Consider the following Fuzzy integral nonlinear equation

$$\tilde{u}(t,\alpha) = \tilde{f}(t,\alpha) +_F (FR)\int_a^t \tilde{F}\left(t,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right)ds$$

Where $\tilde{F}: [a,b]\times[a,b]\times R_F\times R_F\to R_F$, $\tilde{f}(t,\alpha): [a,b]\times[0,1]\to R_F$, $\tilde{F}_1,\tilde{F}_2: [a,b]\times[a,b]\times R_F\to R_F$ and $\tilde{u}(s,\alpha): [a,b]\times[0,1]\to R_F$.

And their parameter cases are

$$\begin{cases}\underline{u}(t,\alpha) = \underline{f}(t,\alpha) + (FR)\int_a^t F(t,s,\underline{F_1(t,s,u(s,\alpha))},\underline{F_2(t,s,u(s,\alpha))})ds\\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + (FR)\int_a^t F\left(t,s,\overline{F_1(t,s,u(s,\alpha))}\right),\overline{F_2(t,s,u(s,\alpha))}ds\end{cases} \quad (2)$$

$\tilde{f}(t,\alpha) = (\underline{f}(t,\alpha),\overline{f}(t,\alpha))$, $\tilde{u}(s,\alpha) = (\underline{u}(s,\alpha),\overline{u}(s,\alpha))$.

and

$$\overline{F_1(t,s,u(s,\alpha))} = \begin{cases}G_1(t,s)\overline{u}(s,\alpha), & G_1(t,s)\geq 0\\ G_1(t,s)\underline{u}(s,\alpha), & G_1(t,s)<0\end{cases}$$

$$\overline{F_2(t,s,u(s,\alpha)} = \begin{cases}G_2(t,s)\overline{u}(s,\alpha), & G_2(h,s)\geq 0\\ G_2(t,s)\underline{u}(s,\alpha), & G_2(h,s)<0\end{cases}$$

$$\underline{F_1(t,s,u(s,\alpha))} = \begin{cases}G_1(t,s)\underline{u}(s,\alpha), & G_1(h,s)\geq 0\\ G_1(t,s)\overline{u}(s,\alpha), & G_1(h,s)<0\end{cases}$$

$$\underline{F_2(t,s,u(s,\alpha))} = \begin{cases}G_2(t,s)\underline{u}(s,\alpha), & G_2(h,s)\geq 0\\ G_2(t,s)\overline{u}(s,\alpha), & G_2(h,s)<0\end{cases}$$

$$F(t,s,\overline{F_1(t,s,u(s,\alpha))},\overline{F_2(t,s,u(s,\alpha))}) = \begin{cases}G_3(t,s)\overline{F_1(t,s,u(s,\alpha))}\ \overline{F_2\left(t,s,u(s,\alpha)\right)}, & G_3(t,s)\geq 0\\ G_3(t,s)\underline{F_1(t,s,u(s,\alpha))}\ \underline{F_2(t,s,u(s,\alpha))}, & G_3(t,s)<0\end{cases}$$

$$F(t,s,\underline{F_1(t,s,u(s,\alpha))},\underline{F_2(t,s,u(s,\alpha))}) = \begin{cases}G_3(t,s)\underline{F_1(t,s,u(s,\alpha))}\ \underline{F_2\left(t,s,u(s,\alpha)\right)}, & G_3(t,s)\geq 0\\ G_3(t,s)\overline{F_1(t,s,u(s,\alpha))}\ \overline{F_2(t,s,u(s,\alpha))}, & G_3(t,s)<0\end{cases}$$

$$T(t) = \int_a^t F(t,s,\underline{F_1(t,s,u(s,\alpha))}),\underline{F_2(t,s,u(s,\alpha))}ds$$

$$L(t) = \int_a^t F(t,s,\overline{F_1(t,s,u(s,\alpha))}),\overline{F_2(t,s,u(s,\alpha)}ds$$

Let $T(.)\in C_F([a,b])$, such that

$$T(t) = (FR)\int_a^t \tilde{F}\left(t,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right)ds$$

Consider the following hypothesis:

Hypotheses 1.

1. $|G_3(t,s)|\leq M_3$

2. $\left|\frac{\partial}{\partial t}G_3(t,s)\right|\leq M$

3. $\left|\frac{\partial}{\partial t}G_1(t,s)\right|\leq M_4$

4. $\left|\frac{\partial}{\partial t}G_2(t,s)\right|\leq M_5$

5. $\left\|\tilde{F}_1(t,s,u(s,\alpha))\right\|\leq M_1\|\tilde{u}(s,\alpha)\|$

6. $\left\|\tilde{F}_2(t,s,u(s,\alpha))\right\|\leq M_2\|\tilde{u}(s,\alpha)\|$

7. $\mathbf{B}\big(\tilde{u}(t,\alpha)\big) = \{\, \tilde{u}(s,\alpha) : \|\tilde{u}(s,\alpha)\| \le \hat{r}\}$

8. $D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) \le L_1 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$

9. $\left(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\right) \le L_2 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$

10. $\big(\tilde{F}(t,s,\tilde{F}_1\big(h,s,u_1(s,\alpha)\big), \tilde{F}_2(h,s,u_1(s,\alpha))), \tilde{F}(t,s,\tilde{F}_1\big(h,s,u_2(s,\alpha)\big), \tilde{F}_2\big(h,s,u_2(s,\alpha)\big),\big)$

$$\le L_0|G_3(t,s)|\left(D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) + D\left(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\right)\right)$$

11. $\|G_3(t_i,s) - G_3(\acute{t}_i,s)\| \le \hat{L}_5|t_i - t_i|$

Lemma 4.
Consider the fuzzy integral equation holds the hypotheses (1) as follows:

$$\tilde{u}(t,\alpha) = \tilde{f}(t,\alpha) + \int_a^t F\left(t,s,\tilde{F}_1(t,s,u(s,\alpha)), \tilde{F}_2\big(t,s,u(s,\alpha)\big)\right) ds$$

with parameter cases are

$$\begin{cases} \underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \int_a^t F\left(t,s,\underline{F_1\big(t,s,u(s,\alpha)\big)}\right), \underline{F_2\big(t,s,u(s,\alpha)\big)} ds \\ \bar{u}(x,\alpha) = \bar{f}(x,\alpha) + \int_a^t F\big(t,s,\overline{F_1(t,s,u(s,\alpha)}\big), \overline{F_2(t,s,u(s,\alpha)} ds \end{cases}$$

Then the following statements are satisfied:

1. $D\left(\tilde{F}(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big), \tilde{F}_2\big(t,s,u(s,\alpha)\big)\right) \le M_3 M_1 M_2 \hat{r}^2$
2. $D\big(\tilde{F}(t,s,\tilde{F}_1\big(h,s,u_1(s,\alpha)\big), \tilde{F}_2(h,s,u_1(s,\alpha))), \tilde{F}(t,s,\tilde{F}_1(h,s,u_2(s,\alpha)), \tilde{F}_2(h,s,u_2(s,\alpha))\big)$
   $\le L_0 M_3 (L_1 + L_2) D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$
3. $D\left(\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big), \tilde{F}_2\big(t,s,u(s,\alpha)\big)), \tilde{0}\right) \le \hat{r}^2\big(M M_1 M_2 + M_3\,(M_4 M_2 + M_5 M_1)\big)$
4. $D\left(\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_1(s,\alpha))) - \frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1\big(t,s,u_2(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha))), \tilde{0}\right) \le \big(M M_1 \hat{r} L_2 + M_3 M_1 \hat{r}\widehat{M} + M_3 \widehat{M}_1 \hat{r} M_5 L_2 + M M_2 \hat{r} L_1 + M_3 M_5 \hat{r} L_1\big) D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$

Proof

1. $D\big(\tilde{F}(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big), \tilde{F}_2\big(t,s,u(s,\alpha)\big), \tilde{0}\big) \le$
   $|G_3(t,s)|\left\|\tilde{F}_1\big(t,s,u(s,\alpha)\big)\right\|\left\|\tilde{F}_2\big(t,s,u(s,\alpha)\big)\right\| \le M_3 M_1 M_2 \hat{r}^2$
2. $D(\tilde{F}(t,s,\tilde{F}_1\big(h,s,u_1(s,\alpha)\big), \tilde{F}_2(h,s,u_1(s,\alpha)))$
   $\tilde{F}(t,s,\tilde{F}_1\big(h,s,u_2(s,\alpha)\big), \tilde{F}_2\big(h,s,u_2(s,\alpha)\big), \tilde{0}\big) \le L_0|G_3(t,s)|$
   $\left(D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) + D\left(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\right)\right) \le$
   $$L_0 M_3 (L_1 + L_2) D\big(u_1(s,\alpha), u_2(s,\alpha)\big)$$

3. $D\left(\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1(t,s,u(s,\alpha)), \tilde{F}_2(t,s,u(s,\alpha)), \tilde{0}\right) =$

$$\left\|\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big), \tilde{F}_2(t,s,u(s,\alpha)))\right\|$$

$$\le \left|\frac{\partial}{\partial t}G_3(t,s)\right| \left\|\tilde{F}_1\big(t,s,u(s,\alpha)\big)\tilde{F}_2\big(h,s,u(s,\alpha)\big)\right\|$$

$+|G_3(t,s)| \left\|\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(h,s,u(s,\alpha))\right)\right\|$

$\leq M\left\|\tilde{F}_1(h,s,u(s,\alpha))\tilde{F}_2(h,s,u(s,\alpha))\right\| + |G_3(t,s)| \left\|\frac{\partial}{\partial t}G_1(t,s)u(s,\alpha)\tilde{F}_2(h,s,u(s,\alpha)) + \tilde{F}_1(h,s,u(s,\alpha))\frac{\partial}{\partial t}G_2(t,s)u(s,\alpha)\right\| \leq \left(MM_1M_2 + M_3(M_4M_2 + M_5M_1)\right) \quad \|\tilde{u}(s,\alpha)\|^2 \leq \hat{r}^2\left(MM_1M_2 + M_3(M_4M_2 + M_5M_1)\right)$

for $t,s \in [a,b]$ and $M_k > 0$.

4.

$D\left(\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_1(s,\alpha)))(-1)._F\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1(t,s,u_2(s,\alpha)),\tilde{F}_2(t,s,u_2(s,\alpha))),\tilde{0}\right) =$

$D\left(\frac{\partial}{\partial t}G_3(t,s)\tilde{F}_1(t,s,u_1(s,\alpha))\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\frac{\partial}{\partial t}G_3(t,s)\tilde{F}_1(t,s,u_1(s,\alpha))\right)$

$\tilde{F}_2(t,s,u_2(s,\alpha)) +_F \frac{\partial}{\partial t}G_3(t,s)\tilde{F}_1(t,s,u_1(s,\alpha))\tilde{F}_2(t,s,u_2(s,\alpha)) -$

$\frac{\partial}{\partial t}G_3(t,s)\tilde{F}_1(t,s,u_2(s,\alpha))\tilde{F}_2(t,s,u_2(s,\alpha)),\tilde{0}\Big)$

$= D\left(\frac{\partial}{\partial t}G_3(t,s)\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right.$

$\left. +_F\frac{\partial}{\partial t}G_3(t,s)\left(\tilde{F}_2(t,s,u_2(s,\alpha))\right)\left(\tilde{F}_1(t,s,u_1(s,\alpha)) - \tilde{F}_1(t,s,u_2(s,\alpha))\right),\tilde{0}\right) =$

$D\left(\frac{\partial}{\partial t}G_3(t,s)\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right.$

$\left. +_F G_3(t,s)\frac{\partial}{\partial t}\left(\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha)) - \tilde{F}_2(t,s,u_2(s,\alpha))\right)\right)\right) =$

$D\Big(\frac{\partial}{\partial t}G_3(t,s)\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha)) - \tilde{F}_2(t,s,u_2(s,\alpha))\right) +_F$

$G_3(t,s)\left(\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right)$

$+_F G_3(t,s)\left(\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha)) - \tilde{F}_2(t,s,u_2(s,\alpha))\right)\right)$

$+_F G_3(t,s)\left(\left(\tilde{F}_2(t,s,u_1(s,\alpha))\right)\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right)\right)$

$\frac{\partial}{\partial t}G_3(t,s)\left(\tilde{F}_2(t,s,u_2(s,\alpha))\right)\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right) +_F$

$G_3(t,s)\left(\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right)\right),\tilde{0}\Big) =$

$\Big\|\frac{\partial}{\partial t}G_3(t,s)(\tilde{F}_1(t,s,u_1(s,\alpha))\ )\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)$

$+_F G_3(t,s)\left(\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right)$

$+_F G_3(t,s)\left(\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right)$

$+_F\frac{\partial}{\partial t} +_F G_3(t,s)\left(\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha))\right)\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right)\right)$

$+_F G_3(t,s)\left(\left(\tilde{F}_2(t,s,u_1(s,\alpha))\right)\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right)\right)\Big\|$

$\leq \left\|\frac{\partial}{\partial t}G_3(t,s)\right\| \left\|\tilde{F}_1(t,s,u_1(s,\alpha))\right\| \left\|\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right\| +$

$\|G_3(t,s)\| \left\|\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))\right)\right\| \left\|\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right\| +$

$\|G_3(t,s)\| \left\|\tilde{F}_1(t,s,u_1(s,\alpha))\right\| \left\|\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_2(t,s,u_2(s,\alpha))\right)\right\| +$

$\left\|\frac{\partial}{\partial t}G_3(t,s)\right\| \left\|\tilde{F}_2(t,s,u_1(s,\alpha))\right\| \left\|\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right\| +$

$\|G_3(t,s)\| \left\|\frac{\partial}{\partial t}\left(\tilde{F}_2(t,s,u_1(s,\alpha))\right)\right\| \left\|\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F\tilde{F}_1(t,s,u_2(s,\alpha))\right)\right\| +$

$\|G_3(t,s)\|\|\tilde{F}_2(t,s,u_1(s,\alpha))\| \left\|\frac{\partial}{\partial t}\left(\tilde{F}_1(t,s,u_1(s,\alpha))(-1)._F F_1(t,s,u_2(s,\alpha))\right)\right\| \leq$
$MM_1\hat{r}L_2\, D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha)) +$
$M_3M_5\hat{r}L_2\, D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha)) + M_3M_1\hat{r}L_2\, D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha))$
$+ MM_2\hat{r}L_1 D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha)) + M_3M_5\hat{r}L_1\, D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha)) +$
$M_3M_2\hat{r}M_1L_1\, D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha)) \leq \big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} + M_3\widehat{M}_1\hat{r}M_5L_2 + MM_2\hat{r}L_1 +$
$M_3M_5\hat{r}L_1\big)D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha))$ and for any $\tilde{u}_1(t,s,\alpha),\tilde{u}_2(t,s,\alpha)$ are fuzzy valued continuous functions.

Theorem 1.
If $\widehat{T}(t) \in C_F([a,b])$ such that

$$\widehat{T}(t) = (RF)\int_a^t \tilde{F}\left(t,s,\tilde{F}_1(h,s,u(s,\alpha)),\tilde{F}_2(h,s,u(s,\alpha))\right) ds.$$

Then $\widehat{T}(t)$ is a uniform modulus of absolutely continuous on $[a,b]$.

Proof

$\sum_{i=1}^{n} D(T(t_i),T(\acute{t}_i),\tilde{0})$
$= \sum_{i=1}^{n} D\Big((RF)\int_a^{t_i}\tilde{F}(t,s,\tilde{F}_1(h,s,u(s,\alpha)),\tilde{F}_2(h,s,u(s,\alpha)))ds$
$(RF)\int_a^{\acute{t}_i}\tilde{F}\left(t,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right) ds,\tilde{0}\Big)$

By lemma (2.7), we get that

$\sum_{i=1}^{n}\Big\|(RF)\int_a^{t_i}\tilde{F}\left(t_I,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right) ds\,(-1)._F$
$(RF)\int_a^{\acute{t}_i}\tilde{F}\left(\acute{t}_I,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right) ds\Big\|$
$=\sum_{i=1}^{n}\Big\|(RF)\int_a^{t_i}\tilde{F}\left(t_i,s,F_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right) ds$
$(-1)._F\,(RF)\int_a^{\acute{t}_i}\tilde{F}\left(\acute{t}_i,s,\tilde{F}_1(t,s,u(s,\alpha)),\tilde{F}_2(t,s,u(s,\alpha))\right) ds\Big\|$
$\sum_{i=1}^{n}\Big\|(RF)\int_a^{t_i}G_3(t_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds$
$(-1)._F\,(RF)\int_a^{\acute{t}_i}G_3(\acute{t}_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds\Big\|$
$\leq \sum_{i=1}^{n}\Big\|(RF)\int_a^{\acute{t}_i}G_3(t_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds$
$+_F(RF)\int_{\acute{t}_i}^{t_i}G_3(t_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds$
$(-1)._F\,(RF)\int_a^{\acute{t}_i}G_3(\acute{t}_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds\Big\|$
$\leq \sum_{i=1}^{n}\Big\|(RF)\int_a^{\acute{t}_i}G_3(t_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds$
$(-1)._F\,(RF)\int_a^{\acute{t}_i}G_3(\acute{t}_i,s)\tilde{F}_1(t,s,u(s,\alpha))\tilde{F}_2(t,s,u(s,\alpha))ds\Big\|$
$+\sum_{i=1}^{n}\Big\|(RF)\int_{\acute{t}_i}^{t_i}G_3(t_i,s)\tilde{F}_1(h,s,u(s,\alpha))\tilde{F}_2(h,s,u(s,\alpha))ds\Big\|$
$\leq \sum_{i=1}^{n}\int_a^{\acute{t}_i}\|G_3(t_i,s)-G_3(\acute{t}_i,s)\|\|\tilde{F}_1(t,s,u(s,\alpha))\|\|\tilde{F}_2(t,s,u(s,\alpha))\|ds+\sum_{i=1}^{n}\int_{\acute{t}_i}^{t_i}\|G_3(t_i,s)-$
$G_3(\acute{t}_i,s)\|\|\tilde{F}_1(t,s,u(s,\alpha))\|\|\tilde{F}_2(t,s,u(s,\alpha))\|ds$
$\leq \hat{L}_5\sum_{i=1}^{n}|\acute{t}_i-a||t_i-\acute{t}_i|\,M_1M_2\|\tilde{u}(s,\alpha)\|^2 + \hat{L}_5\sum_{i=1}^{n}|t_i-\acute{t}_i|^2\,M_1M_2\|\tilde{u}(s,\alpha)\|^2 \leq$
$\hat{L}_5\sum_{i=1}^{n}|b-a||t_i-\acute{t}_i|\,M_1M_2\tilde{r}^2$
$\leq 2\hat{L}_5|b-a|^2M_1M_2\tilde{r}^2\left(\frac{\epsilon}{2\hat{L}_5|b-a|^2M_1M_2\tilde{r}^2}\right) \leq \epsilon$

Hence $\widehat{T}(t)$ a uniform modulus of absolutely continuous on $[a,b]$.

Theorem 2.
Let $\widehat{T} \in C_F([a,b])$ such that

$$\frac{\partial}{\partial t}\widehat{\mathrm{T}}(\mathrm{t}) = \frac{\partial}{\partial t}\int_{\mathrm{a}}^{\mathrm{t}} \tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\mathrm{ds} \tag{3}$$

Then $\int_{\mathrm{a}}^{\mathrm{b}}\left(\mathrm{D}\left(\frac{\partial}{\partial t}\widehat{\mathrm{T}}(\mathrm{t}),\tilde{0}\right)\right)\mathrm{dt}$ is fuzzy Riemann integrable .

Proof

We have that $\frac{\partial}{\partial t}\widehat{\mathrm{T}}(\mathrm{t}) = \tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)$
$+\int_{\mathrm{a}}^{\mathrm{t}}\frac{\partial}{\partial t}\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\mathrm{ds}$ then for a fuzzy number $\widetilde{\mathrm{A}}$, we have that

$$\mathrm{D}\left(\int_{\mathrm{a}}^{\mathrm{t}}\left(\mathrm{D}\left(\frac{\partial}{\partial t}\,\widehat{\mathrm{T}}(\mathrm{t}),\tilde{0}\right)\right)\mathrm{dt}\ ,\widetilde{\mathrm{A}}\right)$$

$$=$$

$$\mathrm{D}\left(\int_{\mathrm{a}}^{\mathrm{t}}\left\|\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)+_{\mathrm{F}}\int_{\mathrm{a}}^{\mathrm{t}}\frac{\partial}{\partial t}\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{t},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\mathrm{ds}\right\|\mathrm{dt},\widetilde{\mathrm{A}}\right)$$

$$\leq \mathrm{D}\left(\int_{\mathrm{a}}^{\mathrm{t}}\left\|\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\right\|\mathrm{dt},\widetilde{\mathrm{A}}\right)$$
$$+\,\mathrm{D}\left(\int_{\mathrm{a}}^{\mathrm{t}}\left\|\int_{\mathrm{a}}^{\mathrm{t}}\frac{\partial}{\partial t}\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\mathrm{ds}\right\|\mathrm{dt},\widetilde{\mathrm{A}}\right)$$

Where $\widetilde{\mathrm{A}}$ is a fuzzy Number .
By condition (1) and (2) , we get

$$\mathrm{D}\left(\int_{\mathrm{a}}^{\mathrm{t}}\left(\mathrm{D}\left(\frac{\partial}{\partial t}\,\widehat{\mathrm{T}}(\mathrm{t}),\tilde{0}\right)\right)\mathrm{dt}\ ,\mathrm{A}\right)\leq \mathrm{M}_3\mathrm{M}_1\mathrm{M}_2\hat{\mathrm{r}}^2\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{dt}+$$
$$\int_{\mathrm{a}}^{\mathrm{t}}\left\|\int_{\mathrm{a}}^{\mathrm{t}}\frac{\partial}{\partial t}\tilde{\mathrm{F}}\left(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big),\tilde{\mathrm{F}}_2\big(\mathrm{h},\mathrm{s},\mathrm{u}(\mathrm{s},\alpha)\big)\right)\mathrm{ds}\right\|\mathrm{dt}$$

Hence,

$$\int_{\mathrm{a}}^{\mathrm{t}}\left(\mathrm{D}\left(\frac{\partial}{\partial \mathrm{t}}\mathrm{T}(\mathrm{t}),\tilde{0}\right)\right)\mathrm{dt}\leq \mathrm{M}_3\mathrm{M}_1\mathrm{M}_2\hat{\mathrm{r}}^2(\mathrm{b}-\mathrm{a})+(\mathrm{b}-\mathrm{a})\hat{\mathrm{r}}^2\big(\mathrm{M}\mathrm{M}_1\mathrm{M}_2\ +\mathrm{M}_3\ (\mathrm{M}_4\mathrm{M}_2+\mathrm{M}_5\mathrm{M}_1)\big)<\infty$$

Hence, $\int_{\mathrm{a}}^{\mathrm{t}}\left(\mathrm{D}\left(\frac{\partial}{\partial t}\widehat{\mathrm{T}}(\mathrm{t}),\tilde{0}\right)\right)\mathrm{dt}$ is fuzzy Riemann integrable.

Let a sub-intervals $a \leq x_0 \leq x_1 \dots \leq x_n \leq b$ of $[a,b]$ into N, such that $\Delta x = x_j - x_{j-1}, j = 1,\dots,N,$ and $\Delta x = \frac{b-a}{N}$, this partition are needed in following result.

Theorem 3.

Let $\Gamma \in \mathrm{V_F}([\mathrm{x_j},\mathrm{x_{j+\Delta x}}])$ such that

$$\Gamma\big(\tilde{\mathrm{u}}_1(\mathrm{s},\alpha)\big)=\tilde{\mathrm{f}}(\mathrm{t},\alpha)+_{\mathrm{F}}(\mathrm{FR})\int_{\mathrm{a}}^{\mathrm{t}}\tilde{\mathrm{F}}(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1(\mathrm{t},\mathrm{s},\mathrm{u}_1(\mathrm{s},\alpha))\ ,\tilde{\mathrm{F}}_2(\mathrm{t},\mathrm{s},\mathrm{u}_1(\mathrm{s},\alpha)))\mathrm{ds}$$
$$\Gamma\big(\tilde{\mathrm{u}}_2(\mathrm{s},\alpha)\big)=\tilde{\mathrm{f}}(\mathrm{t},\alpha)+_{\mathrm{F}}(\mathrm{FR})\int_{\mathrm{a}}^{\mathrm{t}}\tilde{\mathrm{F}}(\mathrm{t},\mathrm{s},\tilde{\mathrm{F}}_1(\mathrm{t},\mathrm{s},\mathrm{u}_2(\mathrm{s},\alpha))\ ,\tilde{\mathrm{F}}_2(\mathrm{t},\mathrm{s},\mathrm{u}_2(\mathrm{s},\alpha)))\mathrm{ds}$$

Then $\mathrm{D}^{\#}(\Gamma(\tilde{\mathrm{u}}_1),\Gamma(\tilde{\mathrm{u}}_2))\leq \mathrm{D}^{*}(\tilde{\mathrm{u}}_1,\tilde{\mathrm{u}}_2)$ for all $\tilde{\mathrm{u}}_1,\tilde{\mathrm{u}}_2\in \mathrm{C_F}([\mathrm{a},\mathrm{b}])$, and $\alpha\in[0,1]$, and Γ has locally existence and unique solution.

Proof

We have that

$$\mathrm{D}^{\#}(\Gamma(\tilde{\mathrm{u}}_1),\Gamma(\tilde{\mathrm{u}}_2))=\mathrm{D}^{*}(\Gamma_1(\mathrm{t},\alpha),\Gamma_2(\mathrm{t},\alpha))+_{\mathrm{F}}\int_{\mathrm{t_j}}^{\mathrm{t_{j+\Delta t}}}\mathrm{D}^{*}(\frac{\partial}{\partial t}\Gamma_1(\mathrm{t},\alpha),\Gamma_2(\mathrm{t},\alpha))\mathrm{dt}$$
$$=\sup_{\mathrm{a}\leq\mathrm{t}\leq\mathrm{b}}\mathrm{D}(\Gamma\big(\tilde{\mathrm{u}}_1(\mathrm{s},\alpha)\big)(\mathrm{t}),\Gamma\big(\tilde{\mathrm{u}}_2(\mathrm{s},\alpha)\big)(\mathrm{t}))$$
$$+_{\mathrm{F}}(\mathrm{FR})\int_{\mathrm{t_j}}^{\mathrm{t_{j+\Delta t}}}\sup_{\mathrm{a}\leq\mathrm{t}\leq\mathrm{b}}\mathrm{D}(\frac{\partial}{\partial t}\Gamma\big(\tilde{\mathrm{u}}_1(\mathrm{s},\alpha)\big)(\mathrm{t}),\frac{\partial}{\partial t}\Gamma\big(\tilde{\mathrm{u}}_2(\mathrm{s},\alpha)\big)(\mathrm{t}))\mathrm{dt}$$
$$=(\mathrm{FR})\int_{\mathrm{t_j}}^{\mathrm{t_{j+\Delta t}}}\sup_{\mathrm{a}\leq\mathrm{t}\leq\mathrm{b}}\mathrm{D}(\frac{\partial}{\partial t}\Gamma\big(\tilde{\mathrm{u}}_1(\mathrm{s},\alpha)\big)(\mathrm{t}),\Gamma\big(\tilde{\mathrm{u}}_2(\mathrm{s},\alpha)\big)(\mathrm{t}))\mathrm{dt}$$

$=(FR)\int_{t_j}^{t_j+\Delta t}\big(\sup_{a\le t\le b} D\big(\tilde{F}(t,s,\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_1(s,\alpha)))\big)$
$,\tilde{F}(t,s,\tilde{F}_1(t,s,u_2(s,\alpha)),\tilde{F}_2(t,s,u_2(s,\alpha))))ds +_F (FR)\sup_{a\le t\le b}$
$\int_{t_j}^{t} D(\frac{\partial}{\partial t}\tilde{F}(t,s,\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_1(s,\alpha)))$
$,\frac{\partial}{\partial t}\tilde{F}\Big(t,s,\tilde{F}_1\big(t,s,u_2(s,\alpha)\big),\tilde{F}_2\big(t,s,u_2(s,\alpha)\big)\Big),\Big)ds\Big)dt \le$
$\int_{t_j}^{t_j+\Delta t}\Big(L_0M_3(L_1+L_2)D^*\big(\tilde{u}_1(t,\alpha),\tilde{u}_2(t,\alpha)\big)dt + \int_{t_j}^{t}\big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} + M_3\widehat{M}_1\hat{r}M_5L_2 +$
$MM_2\hat{r}L_1 + M_3M_5\hat{r}L_1\big)D^*\big(\tilde{u}_1(t,\alpha),\tilde{u}_2(t,\alpha)\big)ds\Big)dt$
$\le \Delta t\Big(L_0M_3(L_1+L_2) + (\Delta t L_3\big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} + M_3\widehat{M}_1\hat{r}M_5L_2 + MM_2\hat{r}L_1 +$
$M_3M_5\hat{r}L_1\big)\Big)D^*\big(\tilde{u}_1(t,\alpha),\tilde{u}_2(t,\alpha)\big)$
Set $\Delta t < 1$, $\sigma(t) = \Big(L_0M_3(L_1+L_2) + (\Delta t L_3\big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} + M_3\widehat{M}_1\hat{r}M_5L_2 +$
$MM_2\hat{r}L_1 + M_3M_5\hat{r}L_1\big)\Big)$ and $\Big(L_0M_3(L_1+L_2) + \Big(\Delta t L_3\big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} +$
$M_3\widehat{M}_1\hat{r}M_5L_2 + MM_2\hat{r}L_1 + M_3M_5\hat{r}L_1\big)\Big)\Big) < \Big(L_0M_3(L_1+L_2) + \Big(L_3\big(MM_1\hat{r}L_2 + M_3M_1\hat{r}\widehat{M} +$
$M_3\widehat{M}_1\hat{r}M_5L_2 + MM_2\hat{r}L_1 + M_3M_5\hat{r}L_1\big)\Big)\Big)$
Take $\Delta t < \frac{1}{L_3\left(\mathbf{MM_1\hat{r}L_2} + M_3\mathbf{M_1\hat{r}\widehat{M}} + M_3\widehat{M}_1\mathbf{\hat{r}M_5L_2} + \mathbf{MM_2\hat{r}L_1} + M_3\mathbf{M_5\hat{r}L_1}\right)}$, therefore
$D^{\#}(\Gamma(\tilde{u}_1),\Gamma(\tilde{u}_2)) \le \Delta t\,\sigma(t)D^*\big(\tilde{u}_1(t,\alpha),\tilde{u}_2(t,\alpha)\big) < \Delta t\Big(L_0M_3(L_1+L_2) + (L_3\big(MM_1\hat{r}L_2 +$
$M_3M_1\hat{r}\widehat{M} + M_3\widehat{M}_1\hat{r}M_5L_2 + MM_2\hat{r}L_1 + M_3M_5\hat{r}L_1\big)\Big)D^*\big(u_1(t,\alpha),u_2(t,\alpha)\big) <$
$D^*\big(\tilde{u}_1(t,\alpha),\tilde{u}_2(t,\alpha)\big)$, hence
$D^{\#}(\Gamma(\tilde{u}_1),\Gamma(\tilde{u}_2)) \le D^*(\tilde{u}_1,\tilde{u}_2)$.
Hence $\Gamma \in V_F([x_j, x_{j+\Delta x}])$ for arbitrary Δx is a contractive mapping and since $V_F([x_j, x_{j+\Delta x}])$ is a complete fuzzy metric space m so by Banach fixed point theorem Γ has locally existence and unique solution in $V_F([x_j, x_{j+\Delta x}])$.
The following illustrative examples are presented to show the role of necessary and sufficient conditions to grantee the existence and uniqueness of solution.

4. Illustrative Examples:
Example 1.

Consider the following fuzzy integral nonlinear equation
$\tilde{u}(t,\alpha) = \tilde{f}(t,\alpha) + \int_a^t \tilde{F}\Big(t,s,\tilde{F}_1\big(t,s,u(s,\alpha)\big)\Big),\tilde{F}_2\big(t,s,u(s,\alpha)\big)ds$
Has the following upper and lower parts such as

$$\begin{cases} \underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \displaystyle\int_a^t F\Big(t,s,\underline{F_1\big(t,s,u(s,\alpha)\big)}\Big),\underline{F_2\big(t,s,u(s,\alpha)\big)}ds \\ \bar{u}(x,\alpha) = \bar{f}(x,\alpha) + \displaystyle\int_a^t F\big(t,s,\overline{F_1(t,s,u(s,\alpha))}\big),\overline{F_2(t,s,u(s,\alpha)}ds \end{cases}$$

Where
$\underline{f}(x,\alpha) = \frac{37\,\alpha^2 t^{10}}{224}$,
$\bar{f}(x,\alpha) = \frac{37\,(\frac{\alpha}{2}-2)^2 t^{10}}{504}$, and $G_3(t,s) = st$
$\underline{F_1\big(t,s,u(s,\alpha)\big)} = G_1(t,s)\underline{u}(t,\alpha) = (t+s)\underline{u}(t,\alpha)$

$\underline{F_2(t,s,u(s,\alpha))}=G_2(t,s)\underline{u}(t,\alpha)=(t^2-s^2)\underline{u}(t,\alpha)$, also

$F\left(t,s,\underline{F_1(t,s,u(s,\alpha))}\right),\underline{F_2(t,s,u(s,\alpha))}=st(t+s)\underline{u}(t,\alpha)(t^2-s^2)\underline{u}(t,\alpha)$

and we have the following upper deceptions of functions

$\overline{F_1(t,s,u(s,\alpha)}=(t+s)\overline{u}(t,\alpha)$

$\overline{F_2(t,s,u(s,\alpha)}=(t^2-s^2)\overline{u}(t,\alpha)$, also

$$F\left(t,s,\overline{F_1(t,s,u(s,\alpha))}\right),\overline{F_2(t,s,u(s,\alpha)}=st(t+s)\overline{u}(t,\alpha)(t^2-s^2)\overline{u}(t,\alpha)$$

Which satisfies the following conditions:

1) $|G_3(t,s)|=|st|\le b^2=M_3$
2) $\left|\frac{\partial}{\partial t}G_3(t,s)\right|=|s|\le b=M$
3) $\left|\frac{\partial}{\partial t}G_1(t,s)\right|\le 1=M_4$
4) $\left|\frac{\partial}{\partial t}G_2(t,s)\right|\le 2b=M_5$
5) $\left\|\tilde{F}_1(t,s,u(s,\alpha))\right\|\le 2b\|\tilde{u}(s,\alpha)\|=M_1\|\tilde{u}(s,\alpha)\|$
6) $\left\|\tilde{F}_2(t,s,u(s,\alpha))\right\|\le b^2\|\tilde{u}(s,\alpha)\|=M_2\|\tilde{u}(s,\alpha)\|$

7) $\mathbf{B}(u(t,\alpha))=\{u(t,\alpha):\|u(t,\alpha)\|\le\hat{r}\}$, $\hat{r}$ is a radius fuzzy number.
8) $D\left(\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_1(t,s,u_2(s,\alpha)\right)\le$
$2bD(u_1(s,\alpha),u_2(s,\alpha))=L_1D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha))$
9) $\left(\tilde{F}_2(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_2(s,\alpha)\right)\le$
$b^2D(u_1(s,\alpha),u_2(s,\alpha))=L_2D(\tilde{u}_1(s,\alpha),\tilde{u}_2(s,\alpha))$
10)
$D(\tilde{F}(t,s,\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_1(s,\alpha))),\tilde{F}(t,s,\tilde{F}_1(h,s,u_2(s,\alpha)),$
$\tilde{F}_2(h,s,u_2(s,\alpha)),0)\le 2b^2\left(D\left(\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_1(t,s,u_2(s,\alpha)\right)+\right.$
$\left.D\left(\tilde{F}_2(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_2(s,\alpha)\right)\right)$
$=L_0|G_3(t,s)|\left(D\left(\tilde{F}_1(t,s,u_1(s,\alpha)),\tilde{F}_1(t,s,u_2(s,\alpha)\right)+\right.$
$\left.D\left(\tilde{F}_2(t,s,u_1(s,\alpha)),\tilde{F}_2(t,s,u_2(s,\alpha)\right)\right)$
11) $\|G_3(t_i,s)-G_3(\acute{t}_i,s)\|\le b|t_i-t_i|=\hat{L}_5|t_i-t_i|$.

The fuzzy integral equation has a locally existence and unique solution.

Example 2.

Consider the details components of other fuzzy integral nonlinear equation

$\underline{f}(x,\alpha)=\frac{41\,\alpha^2t^{12}}{140}$,

$\bar{f}(x,\alpha)=\frac{41\,(\frac{\alpha}{2}-2)^2t^{12}}{315}$, and $G_3(t,s)=st^2$

$\underline{F_1(t,s,u(s,\alpha))}=G_1(t,s)\underline{u}(t,\alpha)=(t+s)^2\underline{u}(t,\alpha)$

$\underline{F_2(t,s,u(s,\alpha))}=G_2(t,s)\underline{u}(t,\alpha)=(t^2-s^2)\underline{u}(t,\alpha)$, also

$F\left(t,s,\underline{F_1(t,s,u(s,\alpha))}\right),\underline{F_2(t,s,u(s,\alpha))}=st^2(t+s)^2\underline{u}(t,\alpha)(t^2-s^2)\underline{u}(t,\alpha)$

and we have the following upper deceptions of functions

$\overline{F_1(t,s,u(s,\alpha)}=(t+s)^2\overline{u}(t,\alpha)$

$\overline{F_2(t,s,u(s,\alpha)}=(t^2-s^2)\overline{u}(t,\alpha)$, also

$$F\left(t,s,\overline{F_1(t,s,u(s,\alpha))}\right),\overline{F_2(t,s,u(s,\alpha)}=st^2(t+s)^2\overline{u}(t,\alpha)(t^2-s^2)\overline{u}(t,\alpha)$$

Which satisfies the following conditions:

1) $|G_3(t,s)| = |st^2| \le b^3 = M_3$
2) $\left|\frac{\partial}{\partial t} G_3(t,s)\right| = |2ts| \le 2b^2 = M$
3) $\left|\frac{\partial}{\partial t} G_1(t,s)\right| \le 4b = M_4$
4) $\left|\frac{\partial}{\partial t} G_2(t,s)\right| \le 2b = M_5$
5) $\left\|\tilde{F}_1\big(t,s,u(s,\alpha)\big)\right\| \le \mathbf{4b}\|\tilde{u}(s,\alpha)\| = M_1\|\tilde{u}(s,\alpha)\|$
6) $\left\|\tilde{F}_2\big(t,s,u(s,\alpha)\big)\right\| \le b^2\|\tilde{u}(s,\alpha)\| = M_2\|\tilde{u}(s,\alpha)\|$

7) $\mathbf{B}\big(u(t,\alpha)\big) = \{\, \tilde{u}(t,\alpha) : \|\tilde{u}(t,\alpha)\| \le \hat{r}\}$,$\hat{r}$ is a radius fuzzy number.
8) $D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) \le$
$b^3 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big) = L_1 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$
9) $\left(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\right) \le b^2 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big) = L_2 D\big(\tilde{u}_1(s,\alpha), \tilde{u}_2(s,\alpha)\big)$
10)

$$\big(F(t,s,\tilde{F}_1\big(h,s,u_1(s,\alpha)\big), \tilde{F}_2(h,s,u_1(s,\alpha))), \tilde{F}(t,s,\tilde{F}_1\big(h,s,u_2(s,\alpha)\big), \tilde{F}_2\big(h,s,u_2(s,\alpha)\big),\big)$$
$$\le 2b^3\left(D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) + D\Big(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\Big)\right.$$
$$=L_0|G_3(t,s)|\left(D\left(\tilde{F}_1\big(t,s,u_1(s,\alpha)\big), \tilde{F}_1(t,s,u_2(s,\alpha)\right) + D\Big(\tilde{F}_2\big(t,s,u_1(s,\alpha)\big), \tilde{F}_2(t,s,u_2(s,\alpha)\Big)\right.$$

11) $\|G_3(t_i,s) - G_3(\acute{t}_i,s)\| \le 2b^2|t_i - t_i| = \hat{L}_5|t_i - t_i|$

Hence the fuzzy integral equation has locally existence and uniqueness solution.

5. Conclusion

1. We concluded to establish a fuzzy metric space of uniform modulus of absolutely fuzzy continuous functions need to define a metric space of fuzzy continuous functions and this defined on a set of fuzzy numbers which denoted by R_F .
2. We concluded to cover all the parameters cases of all components of proposal fuzzy integral problem formulation, we need to explain these classes with upper and lower functions and their ordinary kernel functions which all depended on the values of $\alpha \in [0,1]$.
3. We concluded the certain necessary and sufficient conditions was choice it in a good accuracy suitable with the certain hypotheses for proposal problem to investigate the results of existence and uniqueness fuzzy solutions of proposal fuzzy integral classes.
4. We concluded the interesting result is to achieve the metric space of fuzzy continuous functions a fuzzy Riemann integrable which is made a great role for techniques of the proving of main result.
5. We concluded that to be the main results that are interesting for applications in fuzzy theory, we need the illustrative examples and supported that the proposal fuzzy formulation is exist and has a fuzzy solution in suitable fuzzy metric space.
6. We concluded the fuzzy counteractive condition is satisfied by divided of interval solution to grantee the result existence and uniqueness by using fixed point theorem with introduced fuzzy metric space.
7. We concluded that the norm function was defined with respect to the metric space of fuzzy continuous and used in technical of prove some results.
8. We concluded the main result does not satisfy without fixed point theorem on fuzzy numbers that which appearing in all definitions of the proving stricture.

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