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On Some Properties of Characteristics Polynomials of the Complete Graphs K_n

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ABSTRACT

This paper discusses the properties of the characteristic polynomial of the complete graphs K_n, n=1, 2… respective to the adjacency matrices. Two different types of matrices, the adjacency matrix and the signless Laplacian matrix, are presented. A recurrence relation for computing the characteristic polynomials depending on the adjacency matrix is introduced. We deduce that the coefficients of the polynomials based on the two different matrices have a relationship with Pascal triangle. The coefficients are computed using Matlab program. Many other properties of these coefficients are discussed also.

Keywords: Laplacian matrix, Pascal triangle, Complete graph.

حول بعض الخصائص لمتعدد الحدود المميز للبيانات التامة K_n

الخلاصة

في هذا البحث تمت مناقشة خصائص متعدد الحدود المميز للبيانات التامة K_n,n=1, 2.., على الترتيب لمصفوفات التجاور. وقد استخدم نوعين مختلفين من المصفوفات , مصفوفات التجاور ومصفوفة لابلاس الخالية من الاشارة, هذا بالاضافة الى العلاقة المرتدة لحساب متعددات الحدود المميزة المعتمدة على مصفوفة التجاور. حيث تم استنتاج وجود علاقة بين معاملات متعددات الحدود المعتمدة على النوعين المختلفين من المصفوفات (مصفوفة التجاور ومصفوفة لابلاس الخالية من الاشارة) مع مثلث باسكال . وقد استخدام برنامج Matlab لحساب المعاملات, بالاضافة الى مناقشة خصائصها .

INTRODUCTION

The theory of graphs is one of the few fields of mathematics with a definite birth date. Graph theory is considered to have begun in 1736 with the publication of Euler's solution of the Konigsberg bridge problem. Any

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mathematical object involving points and connections between them may be called a graph [1].

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics and its results have application in many areas of the computing, social, and natural sciences [2]. Depending on the specific problems and personal favor, graph theorists use different kinds of matrices to represent a graph, the most popular ones being the (0, 1)-adjacency matrix and the Laplace matrix. Often, the algebraic properties of the matrix are used as a bridge between different kinds of structural properties of the graph.[3]

The relationship between the structural (combinatorial, topological) properties of the graph and the algebraic ones of the corresponding matrix is therefore a very interesting one. Sometimes the theory even goes further, for example, in theoretical chemistry, where the eigenvalues of the matrix of the graph corresponding to a hydrocarbon molecule are used to predict its stability [3].

The study of specific structural properties of graphs based on algebraic polynomial representations has been a well-known and fruitful concept for several decades. In particular, graph polynomials have been either used for describing combinatorial graph invariants or to characterize chemical structures by using the coefficients or the zeros of a graph polynomial [6].

Indeed, a very important graph polynomial is the characteristic polynomial of a graph which has been intensely studied by Cvetkovic when exploring structural properties of a graph related to its eigenvalues. Several methods to compute the characteristic polynomial explicitly were also developed. Afterwards, other graph polynomials such as the Laplacian polynomial, Matching polynomial, Mühlheim polynomial, Distance polynomial and the Wiener Polynomial etc. have been developed for investigating multifaceted aspects of chemical structures [6].

Many researches are interested in the study of the characteristic polynomials of the graphs and their eigenvalues. For instance, Rehman, and Ipsen [10] were analyze the forward error in the coefficients c_k when they are computed from the eigenvalues of matrix, You and Liu [14] determine the maximum signless Laplacian separators of unicyclic, bicyclic and tricyclic graphs with given order, Astuti and Salman[5] studies the properties of some coefficients of the characteristic and the Laplacian polynomial of a hypergraph, Liu, Wang , Zhang , and Yong [13] studies the spectral characteristic of some unicyclic graphs, Zhu [15] studies the Signless Laplacian spectral radius of bicyclic graphs with a given girth etc..

Some properties of the characteristic polynomials of the path, and the cycles are introduced by Rajab [7]. The properties of the characteristic polynomials of the complete graphs k_n have been focused on and discussed in this paper. These characteristic polynomials are computed according to the adjacency matrix and the signless Laplacian matrix of k_n. We found special properties of characteristic polynomial and proposed an algorithm to compute it, is proposed using matlab language programming. There is a relationship between the coefficients of each characteristic polynomial and Pascal triangle.

BASIC CONCEPTS OF GRAPH THEORY

Let G be a finite undirected simple (no loop or multiple edges) graph. A graph G

(also called a simple graph) is a compose of two type of objects. It has a finite set V(G) of elements called vertices and a set E(G) of pairs of distinct vertices called edges, the number of vertices of the graph is called its order [9].

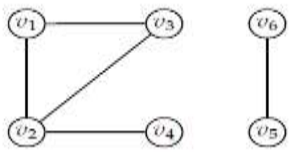

Figure (1) (graph with n=6 and. m=5).

If $V(G)=\{v_1,v_2,\ldots,v_n\}$ is the set of vertices of G, then the adjacency matrix of G, $A(G)=[a_{ij}]$ is an n×n matrix, where $a_{ij}=1$ if v_i and v_j are adjacent , and $a_{ij}=0$ otherwise. Thus A is the symmetric (0,1) matrix with zeros on the diagonal [4].

The general form of the adjacency matrix A(G) with n vertices is as follows:

$$A=\begin{bmatrix} 0 & 1 & 1 & & \cdots & 1 \\ 1 & 0 & 1 & 1 & & \vdots \\ & 1 & 0 & 1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & & 1 \\ & & & & & 1 \\ 1 & \cdots & & & 1 & 0 \end{bmatrix} \qquad ...(1)$$

Let $d_i=d_i(G)=d_G(v_i)$ be the degree of the vertex v_i (the number of incident edges on the vertex v_i). The matrix L(G)=D(G)-A(G) is called the Laplacian matrix of G, where D(G) is the n×n diagonal matrix with $\{d_1,d_2,\ldots,d_n\}$ the diagonal entries. The matrix Q(G)=D(G)+A(G) is called the singless Laplacian matrix of G[11].

The general form of the signless Laplacian matrix with n vertices is:

$$Q=\begin{bmatrix} n & 1 & 1 & & \cdots & 1 \\ 1 & n & 1 & 1 & & \vdots \\ & 1 & n & 1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & & 1 \\ & & & & & 1 \\ 1 & \cdots & & & 1 & n \end{bmatrix} \qquad ...(2)$$

When there exist an edge between every pairs of vertices v_i and v_j in a simple graph, then the graph is said to be a complete graph, such that the number of its edges is n(n-1)/2, and is denoted by K_n[4].

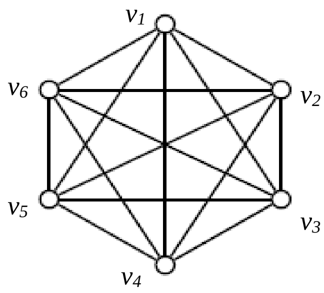

Figure (2) the graph K_6.

PASCAL’S TRIANGLE[9]

One of the most interesting number patterns is Pascal's triangle (named after Blaise Pascal, a famous French mathematician and philosopher).

Most people are introduced to Pascal’s triangle by means of an arbitrary seeming set of rules. Begin with a 1 on the top and with 1’s running down the two sides of a triangle as in Figure (3). Each additional number lies between two numbers and below them, and its value is the sum of the two numbers above it. The theoretical triangle is infinite and continues downward forever, but only the first 14 lines appear in Figure (3).

A different way to describe the triangle is to view the first line as an infinite sequence of zeros except for a single1. To obtain successive lines, add every adjacent pair of numbers and write the sum between and below them. The non-zero part is Pascal’s triangle [12].

```
                           1
                         1   1
                      1    2    1
                   1    3    3    1
                1    4    6    4    1
             1    5    10   10   5    1
          1    6    15   20   15   6    1
        1   7    21    35    35    21   7   1
      1   8    28    56    70    56    28   8   1
   1   9   36    84   126    126   84   36   9   1
1   10   45   120   210   252   210   120   45   10   1
```

Figure (3) Pascal triangle.

The nth row is the set of coefficient in the expansion of binomial $(1+x)^n$, these set of combinatorial coefficients $\binom{n}{r}$, where $0 \le r \le n$, equal to the number of ways to choose r objects from n. A mathlab code to compute Pascal triangle is as follow:

```
p(1,1)=1 {Pascal's triangle}
p(1,2)=1
n=input('n=')
p(n+1,n+1)=zeros
for i=2 :n
    for j= 2:n+1
        p(i,1)=1;
       p(i,j)=p(i-1,j-1)+p(i-1,j);
      if j==i+1
      p(i,j)=1;
      end
  end
end
```

CHARACTERISTIC POLYNOMIAL OF K_n

Characteristic polynomial is one of the earliest known efficiency computable polynomial associated to a graph, it has applications over a large area of mathematical and theoretical chemistry, documentation, computer assisted synthesis design, electronic energy of solids…etc. [8].

The characteristic polynomial of the graph G is defined as follows,

$P_{A(G)}(\lambda)=\det(\lambda I-A(G))$, where I is the identity matrix, it's also can be written as $P_{A(G)}(\lambda)= a_0\lambda^n+a_1 \lambda^{n-1}+\ldots+a_n$, [8,10].

The coefficients properties of the characteristic polynomial based on Laplacian matrix

Let Q be signless Laplacian matrix of the complete graph K_n, shown in equation (2) Then the characteristic polynomial of K_n is

$P_{Q(G)}(\mu)=\det(\mu I-Q(K_n))$,

$= q_0\mu^n+q_1 \mu^{n-1}+\ldots+q_n$

They can be computed for n=1,2,3,…, as follows:

$P_{\varphi(K1)}(\mu)= -\mu$

$P_{\varphi(K2)}(\mu)= \mu^2-2\mu$

$P_{\varphi(K3)}(\mu)= -\mu^3+6\mu^2-9\mu+4$

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$$P_{\varphi(Kn)}(\mu)= \sum_{i=0}^{n} q_i \mu^{n-i}$$

The coefficient of this characteristic polynomial for n=3,4,...,8, is illustrated in Table (1)

Table (1)

n	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8

3	-1	6	-9	4					
4	1	-12	48	-80	48				
5	-1	20	-150	540	-945	648			
6	1	-30	360	-2240	7680	-13824	10240		
7	-1	42	-735	7000	-39375	131250	-240625	187500	
8	1	-56	1344	-18144	151200	-798336	2612736	-4852224	3919104

The relationship between the coefficients (q_i) of the characteristic polynomials and Pascal triangle is shown in Table (2), the element of Pascal triangle is written in bold.

Table (2)

n	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
3	$6*1=2*\mathbf{3}*(3-2)^{0}$	$9*1=3*\mathbf{3}*(3-2)^{1}$	$4*1=4*\mathbf{1}*(3-2)^{2}$					
4	$12*1=3*\mathbf{4}*(4-2)^{0}$	$24*2=4*\mathbf{6}*(4-2)^{1}$	$20*4=5*\mathbf{4}*(4-2)^{2}$	$6*8=6*\mathbf{1}*(4-2)^{3}$				
5	$20*1=4*\mathbf{5}*(5-2)^{0}$	$50*3=5*\mathbf{10}*(5-2)^{1}$	$60*9=6*\mathbf{10}*(5-2)^{2}$	$35*27=7*\mathbf{5}*(5-2)^{3}$	$8*81=8*\mathbf{1}*(5-2)^{4}$			
6	$30*1=5*\mathbf{6}*(6-2)^{0}$	$90*4=6*\mathbf{15}*(6-2)^{1}$	$140*16=7*\mathbf{20}*(6-2)^{2}$	$120*64=8*\mathbf{15}*(6-2)^{3}$	$54*256=9*\mathbf{6}*(6-2)^{4}$	$10*1024=10*\mathbf{1}*(6-2)^{5}$		
7	$42*1=6*\mathbf{7}*(7-2)^{0}$	$147*5=7*\mathbf{21}*(7-2)^{1}$	$280*25=8*\mathbf{35}*(7-2)^{2}$	$315*125=9*\mathbf{35}*(7-2)^{3}$	$210*625=10*\mathbf{21}*(7-2)^{4}$	$77*3125=11*\mathbf{7}*(7-2)^{5}$	$12*15625=12*\mathbf{1}*(7-2)^{6}$	
8	$56*1=7*\mathbf{8}*(8-2)^{0}$	$224*6=8*\mathbf{28}*(8-2)^{1}$	$504*36=9*\mathbf{56}*(8-2)^{2}$	$700*216=10*\mathbf{70}*(8-2)^{3}$	$616*1296=11*\mathbf{56}*(8-2)^{4}$	$336*7776=12*\mathbf{28}*(8-2)^{5}$	$104*46656=13*\mathbf{8}*(8-2)^{6}$	$14*279936=14*\mathbf{1}*(8-2)^{7}$

From above tables we conclude the following properties for the coefficients q_i:-

1- The coefficient of μ^{n} is $q_0 = (-1)^{n}$.

2- The coefficient of μ^{n-1} is $q_1 = n(n-1)$.

3- The coefficient of μ^{n-2} is $q_2 = nm(n-2)$.

4- The coefficient of μ^{n-i} is $q_{n-i} = (n+1)\binom{n}{i}(n-2)^{i-1}$. i=1,2, …, n. which can be calculated using the following algorithm :

```
for i=3 : n  {the coefficients qi in table 2}
   for j =1 :i
      q(i,j)=(i+j-2)*p(i,j+1)*(i-2)^(j-1)
   end
end
```

5- The coefficients of $P_{\varphi(K_n)}(\mu)$ are increasing for i=1,2, …, n-1, i.e. $q_1<q_2<q_3<\ldots<q_{n-1}$ but $q_n<q_{n-1}$

6- The coefficients are alternating in sign.

The coefficients Properties of the characteristic polynomial based on adjacency matrix:

Let A be an adjacency matrix of complete graphs K_n then the characteristic polynomial of K_n with respect to this matrix are computed as follows:

$P_{A(K1)}(\lambda) = -\lambda$

$P_{A(K2)}(\lambda) = \lambda^{2}-1$

$P_{A(K3)}(\lambda) = -\lambda^3+3\lambda+2$ and so on

$$P_{A(Kn)}(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$$

Table (3) represent the coefficients of the characteristic polynomials $P_{A(Kn)}(\lambda)$ with respect to A for n=3, …, 8 .

Table (3)

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	-1	0	3	2					
4	1	0	-6	-8	-3				
5	-1	0	10	20	15	4			
6	1	0	-15	-40	-45	-24	-5		
7	-1	0	21	70	105	84	35	6	
8	1	0	-28	-112	-210	-224	-140	-48	-7

This table can be rewritten in Table (4) to show the relationship between the coefficients a_i of the characteristic polynomials and Pascal triangle, where the bold numbers represent the element of Pascal triangle.

Table (4)

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	-1	0	3 1***3**	2 2***1**					
4	1	0	-6 -1***6**	-8 -2***4**	-3 -3***1**				
5	-1	0	10 1***10**	20 2***10**	15 3***5**	4 4***1**			
6	1	0	-15 -1***15**	-40 -2***20**	-45 -3***15**	-24 -4***6**	-5 -5***1**		
7	-1	0	21 1***21**	70 2***35**	105 3***35**	84 4***21**	35 5***7**	6 6***1**	
8	1	0	-28 -1***28**	-112 -2***56**	-210 -3***70**	-224 -4***56**	-140 -5***28**	-48 -6***8**	-7 -7***1**

From the above table we conclude the following properties:

1- The coefficient of λ^n is $(-1)^n$.

2- The coefficient of λ^{n-1} is 0.

3- The coefficient of λ^{n-2} is ($(-1)^{n+1}$m), which is the number of edges of K_n,

n=1, 2, … .

4- In general the coefficients of λ^{n-1} is $(-1)^{n+1}\binom{n}{i}(i-1); i = 2,\ldots,n$, they are computed as follows

```
for i=3 : n
   for j =1 :i
      q(i,j)=(j-1)*p(i,j+1)*(-1)^(i+1)
   end
End
```

5- The coefficient of λ^1 is $(-1)^{n+1}n(n-2)$.
6- The coefficient of λ^0 is $(-1)^{n+1}(n-1)$.

7- $|a_2| < |a_3| < \ldots < |a_{i+1}| > |a_{i+2}| > \ldots > |a_n|$ where $i = \begin{Bmatrix} \frac{n}{2} & if\ n\ even \\ \frac{n-1}{2} & if\ n\ odd \end{Bmatrix}$

8- $a_0 = \begin{Bmatrix} -1 & if\ \ n\ odd \\ 1 & if\ \ n\ even \end{Bmatrix}$

Proposition

The characteristic polynomial of k_n with adjacency matrix has a recurrence relation of the form:

$$\varphi(k_n) = -\lambda\varphi(k_{n-1},\lambda) + (-1)^{n+1}(n-1)(1+\lambda)^{n-2} \qquad \ldots (3)$$

Proof

We prove this proposition by using mathematical induction on the number of the vertices of k_n.

Since k_1 has a single vertex, its adjacency matrix has only one element, and the characteristic polynomial is $\varphi(k_1) = -\lambda$.

When n=2, the characteristic polynomial is

$$\varphi(k_2) = (\lambda)^2 - 1$$

$$= (-\lambda)(-\lambda) + (-1)^3(2-1)(1+\lambda)^{2-2}.$$

The proposition is true for n=2.

Assume that the proposition is true for all complete graphs of order less than n. To prove the proposition for the complete graph of order n, let v be any vertex in k_n, removing this vertex and all incidence edges on it, to obtain a complete graph with order n-1, which is satisfies the recurrence relation by induction hypothesis. By adding the removal vertex v, and n-1 edges, to k_{n-1} such that v is adjacent to each vertex in k_{n-1}, with n-1degree, we obtained the complete graphk_n.
The adjacency matrix of k_n is illustrated in equation (1) so the characteristic polynomial can be computed as follow:

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & 1 & \cdots & 1 \\ 1 & -\lambda & 1 & \vdots \\ \vdots & 1 & \ddots & 1 \\ 1 & \dots & 1 & -\lambda \end{vmatrix}_{n\times n}$$

Calculate this determinant to obtain equation (3).

CONCLUSIONS

From this research we conclude that the graph in (graph theory) can be dealt with as polynomials; such that any polynomial with its coefficients the structure of the graph may be determined (in order and size) also the Pascal Triangle is a special case for the complete graphs, also both the adjacency and Laplacian matrices of the complete graph have a relationship with Pascal Triangle.

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