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# LOAD DEFLECTION ANALYSIS OF BEAM-COLUMN USING TOTAL POTENTIAL ENERGY (TPE) PRINCIPLE

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## ABSTRACT

A modified TPE approach to perform finite deflection analysis of slender beam-column elements has been developed. The proposed approach utilizes the energy principal method and takes into account the geometric nonlinearity including the effects of axial force on bending stiffness, the end moments on axial stiffness (bowing), and the initial imperfection. A new equation of the deformation curve that approaches to the exact solution is used in the strain-displacement relation to obtain a more accurate beam-column response. The derived formulation of displacement of the beam-column under axial compressive load with single curvature bowing is presented with initial imperfection and different end eccentricities. The Green strain tensor equation is developed to consider higher-order bowing term. Nonlinear analysis of central finite deflection is carried out using Newton-Raphson iteration that includes high order terms of total potential energy (TPE).  The beam-column stability is verified by computing the hessian determinant of the total potential energy. The validity of the new approach is established by comparing the numerical results obtained using the proposed equations against data previously published in the literature. Outputs from the analysis indicate that the proposed approach is capable of capturing the deflection of the beam-column with enhanced accuracy, ranging from 8.5% for e = 0.025 and up to 23.5% when e = 0.125.

## KEYWORDS

Beam-Column, Finite Deflection, Uniaxial Bent, Green Strain Tensor, Initial Imperfection, Bowing Effect, Member Stability.

## 1. INTRODUCTION

Beam-column deflection is a critical aspect to consider in the design and analysis of structural elements such as columns, beams, and beam-columns. It refers to the deformation or bending of a beam-column under applied loads, which can affect the structural integrity and stability of the element. The determination of beam-column deflection is crucial for ensuring that the structure can withstand the anticipated loads and maintain its functionality and safety.

Closed form solution of the beam-column finite deflection problem required dealing with elliptical integrals of a nonlinear differential equation. Due to the difficulty of performing finite deflection closed form solution, several studies suggested more practical approximate and numerical solutions, such as the energy method, finite difference, finite integral, and finite element analysis (Kabir & Aghdam, 2019) (Bert & Malik, 1997) (Wang, et al., 2020). However, it is important to note that the accuracy and applicability of these methods may vary depending on the complexity of the beam-column element and the specific boundary conditions. An example of such a solution can be found in the study conducted by (Oran.C, 1973), in which a tangent stiffness matrix was established for in-plane linear elastic beam-column. The derived tangent stiffness matrix included the effects of axial force on bending stiffness as a major source of geometric nonlinearity. The effect of flexure bowing of deformed strut was also included. Later on, (Oran & Kassimali, 1976) introduced the stability solution method of analysis of elastic skeletal structures under static and dynamic loads. The general solution type of stability functions, bowing functions, and tangent stiffness matrix, presented earlier by (Oran.C, 1973) was used to model the structure geometrical nonlinearity in the elastic range. (Goto & Chen, 1987) established methods of second order analysis of frames, which were derived by virtual work and plane member differential equations. The nonlinear Green strain tensor was formulated using displacement components and stiffness matrices. Also, (Goto, et al., 1991) presented the effect of bowing on the axial displacements of rigid frames by introducing closed form stiffness and tangent stiffness matrices. Nonlinear buckling behavior of portal frames was examined under primary bending moment.

Finite deflection of eccentrically loaded beam-column was introduced by (Kalaga & Alduri, 2000). The derivation of stress equation was investigated using total potential energy (TPE) principle. A central deflection formulation was utilized. In addition, the equilibrium and hessian matrix of TPE was used to check the beam-column member stability. The finite deflection response of uniaxial loaded beam-column using the TPE principles was also derived and examined by (Kalaga & Alduri, 2000). The solution involved an approximate third-order deflection equation, i.e. it neglects the higher order terms of deflection. The bowing and initial imperfection of the member were included. A study of slender beam-column member with bowing effects was conducted by (Kalaga & Alduri, 2001), in which the end moments effect on member bending stiffness was explored. Three types of analysis of the deflection functions were presented. The effective stiffness concept was used and the stiffness degradation due to bowing effects was expressed in terms of load amount and its eccentricity. Geometric nonlinearity of cantilever beam column with large deflection was also presented by (Banerjee, et al., 2008). In this study, a new variation method of nonlinear beam column shortening was investigated. The large deflection of the desired member under arbitrary load was solved and compared with elliptical integral.

In a more recent study, a refined Updated-Lagrangian method (UL) was developed to account for the large deflection. (Areiza-Hurtado, 2019) also presented a structural method for second order analysis of beam column element on elastic foundation. The proposed method takes into account the simultaneous effects of bending, shear deformations, and axial forces at both ends. Fourier series was used to model the transverse loads and the initial deflections, which allows to model of all types of applied loads and initial imperfections. The large deflection analysis was also studied for prismatic and tapered beam-columns by (Areiza-Hurtado, 2020). In this study, a new method based on the Differential Transform Method (DTM), but with more efficiency, was proposed. The governing integro-differential equation of the problem was converted into a polynomial equation. Another study on large deformation analysis was performed by (Chen, et al., 2020), at which a new beam column element was derived. The Gaussian quadrature method was utilized to overcome the difficulty in computing the section stiffness when forming the element stiffness matrix.

The load deflection behavior of the beam-column can be considered as an important aspect in the structural analysis. Accounting for some factors, such as shear deformation and second-order effects in the finite deflection problem may significantly impact the overall deformation. Thus, an accurate prediction of the finite deflection of beam-columns is crucial in assessing structural integrity and ensuring the safety of buildings and other structures. Accordingly, an analytical solution for nonlinear finite deflection analysis of beam-column elements using the TPE principle was developed. The proposed model can be used instead of the elliptical integration method, which involves solving complicated governing equations. The geometric nonlinearity, which includes the effects of axial force on bending stiffness, the end moments effects on axial stiffness, and the initial imperfection was also considered. A new equation of the deformation curve was proposed and used in the strain-displacement relation to obtain a more accurate beam-column response. The Green tensor was formulated to take into account the effect of bowing deformation accurately. A nonlinear analysis of central finite deflection was implemented using Newton-Raphson iterations, which included higher order terms of TPE. Load-displacement behaviors of simply supported beam-column members were presented for various values of load eccentricities and initial imperfections. The proposed model was validated by comparing the predicted displacement with numerical and experimental data.

## 2. ANALYTICAL FORMULATIONS

### 2.1. Member Geometry

A typical prismatic plane frame member is shown in Fig. 1. The member is subjected to an axial force and moments. The problem is described using the rectangular coordinates system (x,y). In the reference state, the x-axis is aligned with the longitudinal axis of the beam member and it passes through the centroid of the cross section at each ends. The element considered in the study is a plane frame element, which is a two dimensional beam column element. The cross sectional properties of the plane frame element include the modulus of elasticity E, cross-sectional area A, moment of inertia I, and length L. The plane frame element is modeled so that it can deform in the longitudinal and transvers directions of the member. Hence, the member can carry axial forces, and moments. Accordingly, the element has 3 degree of freedom (3DOF), rotational DOF at each end, and axial displacement. The study is based on the following assumptions:

1- Plane section perpendicular to the longitudinal axis before deformation remains plane and perpendicular to the longitudinal axis after deformation.

2- Loads are applied in the xy plane only.

3- The material is linear elastic, isotropic, and homogeneous.

4- Deformations are relatively moderate, but the strains are small.

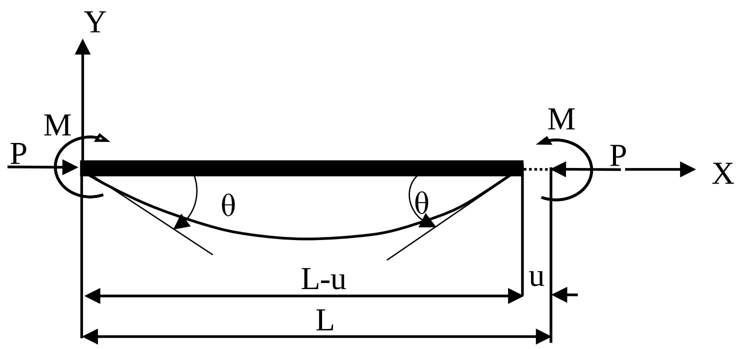



**Fig. 1, Typical plane frame member**

### 2.2. Deformation Formula

In this study, a modified beam-column deformation formula is proposed instead of the well-known approximate sine curve formula. The suggested formula, given by Eq. 1, is an exact fitting of the beam-column element deformation at three points (L/6, L/2, and 5L/6 respectively) multiplied by the sine curve. The main advantage of using the proposed formula is to attain a more accurate response compared to the sine deformation formula used by (Kalaga & Alduri, 2000), (Kalaga & Alduri, 2001), and (Kalaga & Alduri, 2000). The proposed deformation equation can be applied only to simply supported prismatic members, at which the equation may satisfy the end conditions (zero deflection at both ends). In addition, it is well established that multiplying the sine curve by a polynomial, particularly if it was fitted to an exact data, can significantly enhance the accuracy of the solution. This becomes very true when the problem is so complex, such as the cases that involve high geometrical nonlinearity, where a simple sine curve may not be able to fully capture the nonlinear behavior. Accordingly, using the exact curve fitting polynomial will allow introducing additional terms that can adjust the shape of the deflection curve, which in turn leads to a better prediction of the displacement behavior with high nonlinearity problems.

$$V = V_m \left(1.25 - \frac{x}{L} + \frac{x^2}{L^2}\right) sin \left(\frac{\pi\, x}{L}\right) \tag{1}$$

Where $V_m$ is beam-column central deflection, V is the deflection, L is the length of the member, and x is the distance from support to any point in the member.

### 2.3. Improved Expression of the Total Potential Energy (TPE)

An improved expression of the TPE was derived in this study by using a modified and more accurate equation of the Green strain tensor $\epsilon_{xx}$. The Green strain tensor can be defined in terms of axial displacement U and deflection V, which also considers the exact bowing deformation term and as in Eq. 3.

$$\epsilon_{xx} = \frac{dU}{dx} + \left(\sqrt{1 + \left(\frac{dV}{dx}\right)^2} - 1\right) - \frac{y\left(\frac{d^2V}{dx^2}\right)}{\left[1 + \left(\frac{dV}{dx}\right)^2\right]^{1.5}} \tag{2}$$

Where y is the distance measured from the natural axis to the extreme fiber.

The higher order terms in Eq. 2 may be expanded as follows:

For the bowing term $\left(\sqrt{1+\left(\frac{dV}{dx}\right)^2}-1\right)$, let $\frac{dV}{dx}=x$, then the bowing term can be expressed as f $(x)$ = $\left(\sqrt{1+(x)^2}\right)-1)$

knowing that the Maclaurin's series formula is:

f (x) $=\sum_{n=0}^{\infty}\left(\frac{f^n(x_o)}{n!}\right)(x-x_o)^n$, where $x_o=0$ and n = derivative order. The value of n was considered equal to 4, thus

$f'(x_o=0)=\left(\frac{x}{(1+x^2)^{1/2}}\right)=0$ $f''(x_o=0)=\left(\frac{x^2}{(1+x^2)^{3/2}}\right)=0$

$f^3(x_o=0)=\left(\frac{3x}{(1+x^2)^{5/2}}\right)=0$ $f^4(x_o=0)=\left(\frac{12x^2-3}{(1+x^2)^{7/2}}\right)=-3$

Inserting the above derivative terms, and x = $\frac{dV}{dx}$ in Maclaurin's series leads to

$$f\left(\frac{dV}{dx}\right)=\frac{1}{2}\left(\frac{dV}{dx}\right)^2-\frac{1}{8}\left(\frac{dV}{dx}\right)^4$$

The same procedure was used to expand the strain term due to flexural curvature.
Substituting the higher order terms of Eq. 2 yields

$$\epsilon_{xx}=\frac{dU}{dx}+\left(\frac{1}{2}\left(\frac{dV}{dx}\right)^2-\frac{1}{8}\left(\frac{dV}{dx}\right)^4\right) - y\left(\frac{d^2V}{dx^2}\right)\left[1-1.5\left(\frac{dV}{dx}\right)^2+1.875\left(\frac{dV}{dx}\right)^4\right] \quad (3)$$

Including higher-order terms in solving Eq.3 will lead to a more accurate representation of the beam-column's deformation, which in turns increase the accuracy of the solution.
Introducing parameters, B and C and substituting Eq. 1 into Eq. 3 leads to Eq. 4.

$$\epsilon_{xx}=\frac{dU}{dx}+\left(\frac{1}{2}B^2-\frac{1}{8}B^4\right)-y\,C[1-1.5\,B^2+1.875\;B^4] \quad (4)$$

in which,

$$B=\frac{5\,V_m}{4\,L^3}\left[\pi\left(L^2-\frac{4}{5}x\,L+\frac{4}{5}x^2\right)\cos\left(\frac{\pi\,x}{L}\right)-\frac{4}{5}\,L\,(L-2x)\sin\left(\frac{\pi\,x}{L}\right)\right] \quad (5)$$

$$C = \frac{-5\,V_m}{4\,L^4}\left[\left(\left(\pi^2 - \frac{8}{5}\right)L^2 - \frac{4}{5}\pi^2 x\,L + \frac{4}{5}\pi^2 x^2\right) sin\,\left(\frac{\pi\,x}{L}\right) + \frac{8}{5}\,L\pi\,(L - 2x) cos\,\left(\frac{\pi\,x}{L}\right)\right] \quad (6)$$

The general formula of strain energy SE may be expressed as

$$E = \frac{E}{2}\int_V \epsilon_{xx}^2\,dV \quad (7)$$

Where $E$ is the modulus of elasticity.

By inserting Eq. (4) into (7) and performing integration to the whole beam-column volume, the resulted beam-column improved strain energy with higher order displacements terms is given by

$$SE = \frac{EA\,U^2}{2\,L} + \frac{2.6678\,EA\,U\,V_m^2}{L^2} - \frac{6.363\,EA\,U\,V_m^4}{L^4} + \frac{30.943\,EI\,V_m^2}{L^3} + \frac{6.363\,\,EA\,V_m^4}{L^3} - \frac{461.16\,\,EI\,V_m^4}{L^5} - \frac{35.836\,EA\,V_m^6}{L^5} + \frac{8279.5\,\,EI\,V_m^6}{L^7} + \frac{54.622\,EA\,V_m^8}{L^7} - \frac{83572\,\,EI\,V_m^8}{L^9} + \frac{6.1587\times 10^5\,\,EI\,V_m^{10}}{L^{11}} \quad (8)$$

Where A is the cross-sectional area of the member.

The end rotation can be obtained by differentiating Eq. 1 with respect to x and applying x=0. The modified end rotation $\theta$ is given by

$$\theta(0) = \frac{5\,\pi\,V_m}{4\,L} \quad (9)$$

The potential energy, PE, generated from the external loads (axial load, P, and end moments, 2Pe) may be defined as

$$PE = -PU - 2\,Pe\,\theta(0) \quad (10)$$

Application of Eq. 9 into Eq. 10 leads to

$$PE = -PU - \frac{5\,Pe\,\pi\,V_m}{2\,L} \quad (11)$$

The first term of Eq. 11 is due to axial shortening of the beam-column, while the second term is resulted from the end rotations.

The improved formula of the total potential energy (TPE) can then be calculated by adding Eq.11 to Eq. 8, which yields

$$TPE = \frac{EA\,U^2}{2\,L} + \frac{2.6678\,EA\,U\,V_m^2}{L^2} - \frac{6.363\,EA\,U\,V_m^4}{L^4} + \frac{30.943\,EI\,V_m^2}{L^3} + \frac{6.363\;\;EA\,V_m^4}{L^3} - \frac{461.16\;\;EI\,V_m^4}{L^5} - \frac{35.836\,EA\,V_m^6}{L^5} + \frac{8279.5\;\;EI\,V_m^6}{L^7} + \frac{54.622\,EA\,V_m^8}{L^7} - \frac{83572\;\;EI\,V_m^8}{L^9} + \frac{6.1587 \times 10^5\;\;EI\,V_m^{10}}{L^{11}} - PU - \frac{5\,P\,e\,\pi\,V_m}{2\,L} \tag{12}$$

Where U is the axial shortening that can be calculated considering the first derivative of Eq.12 with respect to U $\left(\frac{\partial(TPE)}{\partial U} = 0\right)$, which yields

$$U = \frac{PL}{EA} - \frac{2.6678\,V_m^2}{L} + \frac{6.363\,V_m^4}{L^3} \tag{13}$$

## 2.4. Total Finite Deflection Vm

A modified expression of deflection may be defined based on the improved TPE equation which was derived in the previous section. The TPE can be expressed in terms of central displacement $V_m$ by substituting Eq. 13 into Eq. 12, which leads to

$$TPE = \frac{1}{EA\,L^{11}}\Big[\big(34.378\,{V_m}^8L^4 - 18.861{V_m}^6L^6 + 2.8044\,{V_m}^4L^8\big)E^2A^2 + \Big(\big(2.6678\,{V_m}^2P - 7.854\,P\,e\,V_m\big)L^{10} + \big(30.943\,EI\,{V_m}^2 - 6.363\,{V_m}^4P\big)L^8 - 461.16\,EI\,{V_m}^4L^6 + 8279.5\,EI\,{V_m}^6L^4 - 83572\,EI\,{V_m}^8L^2 + 6.1587 \times 10^5\,EI\,{V_m}^{10}\Big)EA - 0.5\,P^2L^{12}\Big] \tag{14}$$

Differentiating Eq. 15 with respect to $V_m$ yields:

$$\frac{\partial TPE}{\partial V_m} = \frac{1}{L^{11}}\big(49677\,EI\,{V_m}^5L^4 - 7.854\,P\,e\,L^{10} - 1844.6\,EI\,{V_m}^3L^6 - 6.6858 \times\;10^5\,EI\,{V_m}^7L^2 - 61.886\,EI\,V_m\,L^8 - 113.16\,EA\,{V_m}^5L^6 + 275.02\,EA\,{V_m}^7L^4 + 11.218\,EA\,{V_m}^3L^8 - 25.452\,P\,{V_m}^3L^8 + 5.3356\,P\,V_m\,L^{10} + 6.1587 \times 10^6\,EI\,{V_m}^9\big) = 0 \tag{15}$$

$$\frac{\partial^2 TPE}{\partial {V_m}^2} = \frac{1}{L^{11}}\big(2.48385\;\times 10^5\,EI\,{V_m}^4\,L^4 - 5533.8\;\;EI\,{V_m}^2\,L^6 - 4.68\;\times 10^6\,EI\,{V_m}^6\,L^2 + 61.886\;\;EI\;\;L^8 - 565.8\,EA\,{V_m}^4\,L^6 + 1925.14\;\;EA\,{V_m}^6\,L^4 + 33.654\;\;EA\,{V_m}^2\,L^8 - 76.356\,P\,{V_m}^2\,L^8 + 5.3356\,P\,L^{10} + 5.54283 \times 10^7\,EI\,{V_m}^8\big) \tag{16}$$

Vm can be calculated by solving Eq. 15 with the aid of Newton-Raphson iteration and by keeping the higher order of Vm up to 10. The numerical solution of $V_m$ is given by Eq. 17, which continues until $\left|\frac{\frac{\partial TPE}{\partial Vm}}{\frac{\partial^2 TPE}{\partial Vm^2}}\right| \leq tolerance$. In the first iteration, $(V_m)_n$ is the start value, which can be estimated using the proposed approximate Eq. 18. The first and second derivatives of Eq. 14 with respect to Vm are required to obtain the solution of Eq. 17. These derivatives are given in Eqs. 15 and 16. In case of presence of initial imperfection, $V_0$, it is possible to add its value to $V_m$ directly.

$$(V_m)_{n+1} = (V_m)_n - \frac{\frac{\partial TPE}{\partial V_m}}{\frac{\partial^2 TPE}{\partial {V_m}^2}} \tag{17}$$

$$(V_m)_{n=1} = \frac{7.854\, P\, e\, L^2}{61.886\, EI\ + 5.335\, P\, L^2} \tag{18}$$

### 2.5. Stability Hessian Matrix

The stability formulations of elastic frames are a crucial aspect in the design and construction of structures. It ensures that the structure can withstand various loads and environmental conditions without experiencing excessive deformations or failure. The stability is mainly concerned with field displacements ($V_m$ and U) and equilibrium condition of the beam-column, which requires that the hessian determinant $|H| \geq 0$ , where the hessian determinant may be defined as

$$|H| = \frac{\partial^2 TPE}{\partial U^2}\, \frac{\partial^2 TPE}{\partial {V_m}^2} - \left[\frac{\partial^2 TPE}{\partial U\, \partial V_m}\right]^2 \tag{19}$$

The determinant of the hessian matrix is considered as a key parameter in assessing the stability of structures. A positive determinant signifies a stable behavior, which assures the structure's reliability and safety. Conversely, a negative determinant raises concerns about instability, prompting engineers to take necessary measures to rectify the situation. A more refined Hessian equation was developed in this study by differentiating Eq. 12 with respect to the field displacement, along with the substitution of Eqs. 15 and 16 into Eq. 19, which leads to

$$|H| = \frac{EA}{L^{12}}\big(-565.82\ EA\ V_m{}^4L^6 + 1925.2\ EA\ V_m{}^6L^4 + 5.5428 \times 10^7\ EI\ V_m{}^8 - 5533.9\ EI\ V_m{}^2L^6 + 2.4838\ \times 10^5\ EI\ V_m{}^4L^4 - 76.356\ V_m{}^2\ L^8\ P + 33.653\ EA\ V_m{}^2\ L^8 + 5.3356\ \ L^{10}\ P - 4.68\ \times 10^6\ EI\ V_m{}^6L^2 + 61.886\ EI\ L^8\big) \tag{20}$$

## 3. CASE STUDIES

To validate the proposed model, the predicted results computed using the modified formulations were compared with numerical results carried out by (Kalaga & Alduri, 2000), (Oran.C, 1973), , (Oran & Kassimali, 1976), and with (Lin & Hsiao, 2001), as illustrated in the subsections that follow.

### 3.1. Case Study I: simply supported beam subjected to eccentric axial force

A uniaxial simply supported beam-column subjected to axial compressive end loads was studied by (Kalaga & Alduri, 2000), as shown in Fig. 2. The axial loads are applied eccentrically for three different eccentricities, namely, for e = 0.025, 0.05, and 0.125 in. The beam-column data is: Length L= 57.7 in, Area A = 1 $in^2$, Moment of inertia I = 0.0833 $in^4$, Slenderness ratio L/r = 200, and Modulus of elasticity E = 29,000 ksi. The Euler load of the column is Pcr = -7.16 kips. Additionally, the effect of initial imperfection on the deflection was also studied for values range from 0 to 0.1.

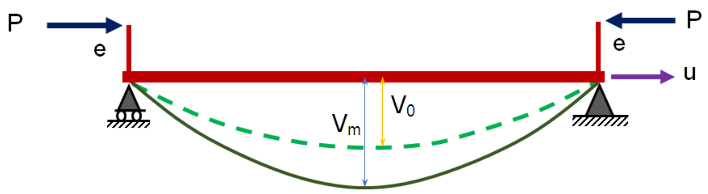



**Fig. 2. Simply supported eccentrically loaded beam-column**

#### 3.1.1. Results and Analysis

Using the deformation formula proposed in this study, the deflection was calculated for three different eccentricities. That is, at eccentricity e = 0.025, 0.05, and 0.125 in. The predicted values of deflection, as a function of distance, were compared with the exact solution and also with numerical results obtained from (Kalaga & Alduri, 2000), as illustrated in Fig. 3. As can be seen in Fig. 3, the predicted values are in good agreement with the exact and the approximate results. The proposed equation, however, seems to yield the best match with the exact solution compared with the approximate one, particularly at certain points along the span. It worth mentioning that the displacement is positive in the upward direction and the force, P, is positive in tension.

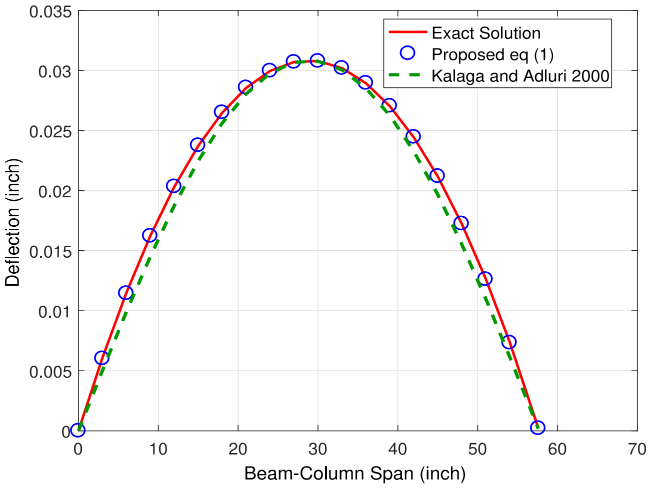
The validity of the proposed expression of the TPE was also investigated by comparing the predicted values from the current study with numerical results obtained from (Kalaga & Alduri, 2000). The values of eccentricity considered in the analysis are e = 0.025, 0.05, and 0.125 in. Fig. 4 shows the predicted TPE values as a function of central deflection obtained using the proposed equation and from (Kalaga & Alduri, 2000). It can be seen that the discrepancy between the modified formula and (Kalaga & Alduri, 2000) solution increases with the increase of the displacement and continues to deviate until it reaches up to 190%, when the displacement is equal to 2 in. Such a difference implies that excluding the higher order terms in the TPE equation has a considerable effect on the TPE values and leads to overly underestimate the results, particularly, at higher values of displacement. Fig. 5 shows the predicted values of axial shortening and the values obtained from (Kalaga & Alduri, 2000). It can be also noted that there is a difference between the axial shortening values computed from the modified formula and (Kalaga & Alduri, 2000) solution, reaches up to 7%.

To establish the validity of the proposed hessian matrix, the determinant of the hessian matrix was calculated and compared with numerical results (Kalaga & Alduri, 2000) for three different values of axial load, which are: P = 0, P = Pcr, and P = 2Pcr. As can be seen in Fig. 6, the desired beam-column uniaxial bent is still stable, since $|H|>0$ under the effects of nonlinear moderately finite deflection system. Also, it may be noted that the state of critical stability, namely at $|H|=0$, is reached when P = Pcr = -7.16 kips. However, when the applied load is equal to 2Pcr = -14.32 kips, the member state becomes unstable, that is $|H|<0$. On close inspection to Fig. 6, It can be noted that (Kalaga & Alduri, 2000) solution overestimates the determinant of the hessian matrix at early stages of loading. The overestimation continues to increase at later stages of loading, i.e. at higher displacements values, up to 132%.

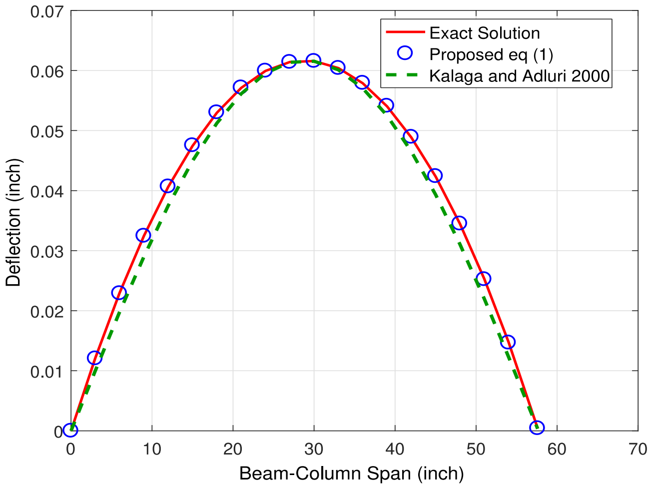
Such a difference implies that excluding the higher order terms in the Hessian matrix has a considerable effect on its values and leads to overly underestimate the results, particularly, at higher values of lateral displacement. This discrepancy in evaluating the determinant of the Hessian matrix can have severe consequences on assessing the structural integrity and may lead to substantially overestimate the overall stability.

To further demonstrate the effectiveness of the proposed model, a nonlinear finite deflection analysis of a beam-column subjected to an eccentric axial load was investigated, in which three different values of eccentricity were adopted, particularly the eccentricities e = 0.025, 0.05, and 0.125. The effect of initial imperfection on the deflection was also studied for values range from

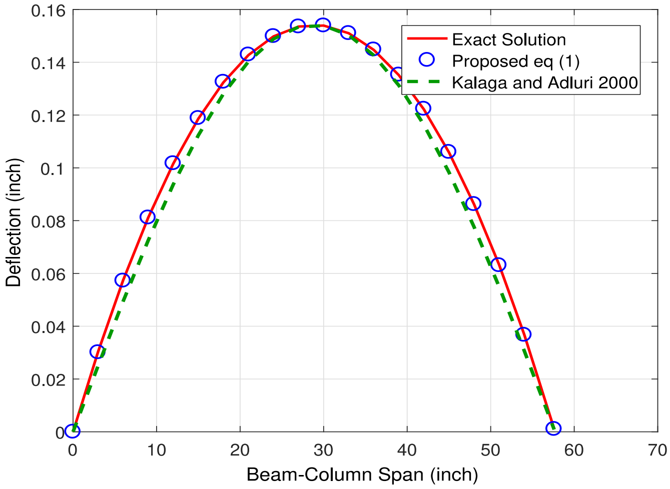
0 to 0.1. The predicted values from the proposed deflection equation were compared with the values obtained from (Kalaga & Alduri, 2000), as illustrated in Fig.7. Moreover, the results were also compared with (Oran.C, 1973) and (Oran & Kassimali, 1976), in which the stiffness method was utilized and the effects of bowing and geometric nonlinearity were included, Fig.7. In addition, it can be seen that there is a notable enhancement of the predicted results, in terms of trends and values, compared to Kalaga solution. This becomes more obvious at larger values of eccentricity and initial imperfection. For example, for load values larger that Pcr, particularly, at P = 8 k, the prediction of deflection is enhanced by 8.5% for e = 0.025 and up to 23.5% when e = 0.125.




(a) e = 0.025 in.

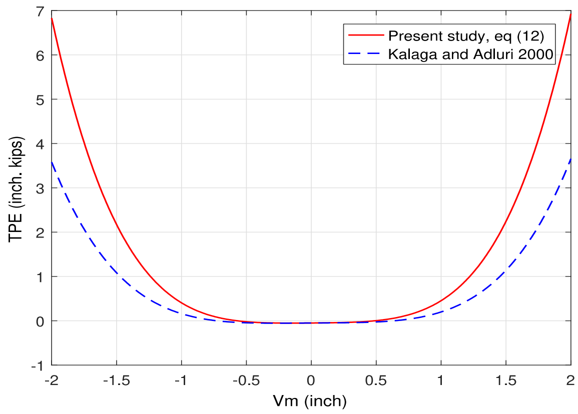



(b) e = 0.05 in.

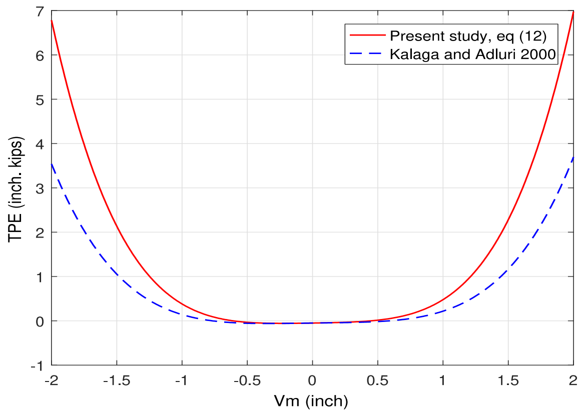



(c) e = 0.125

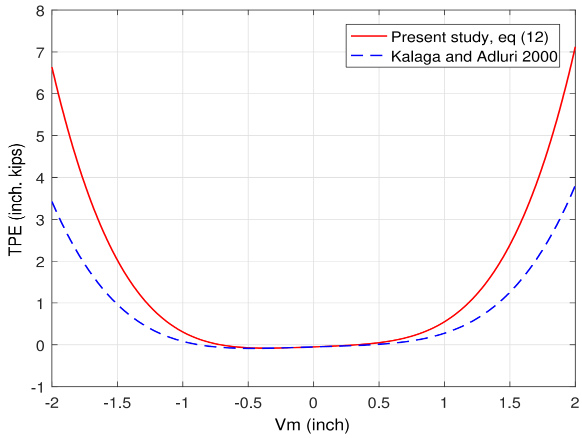
**Fig. 3, Beam-column elastic curves**




(a) e = 0.025 in.




(b) e = 0.05 in.




(c) e = 0.125 in.

**Fig. 4, Beam-column TPE, e = 0.025 in.**

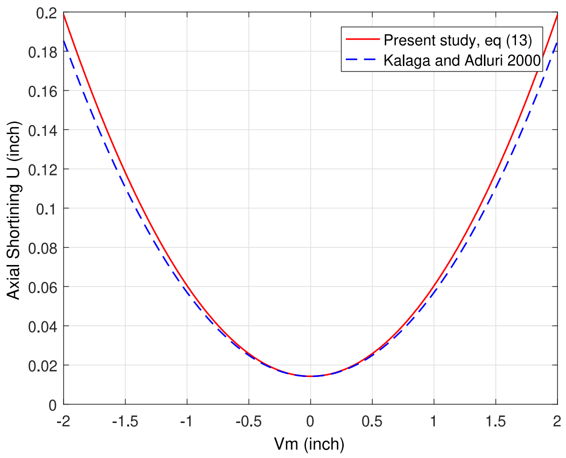



**Fig. 5, Beam-column axial shortening U**

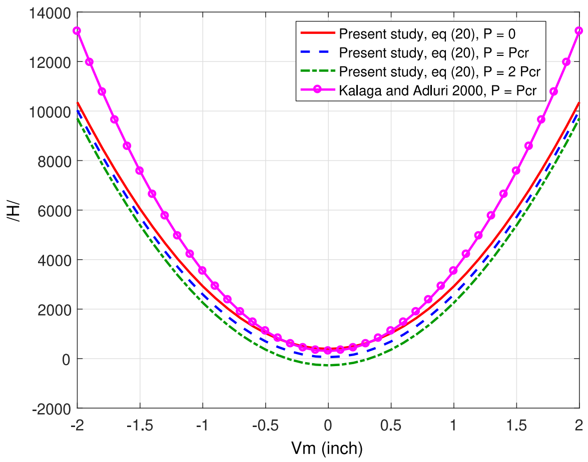



**Fig. 6, Beam-column hessian determinant**

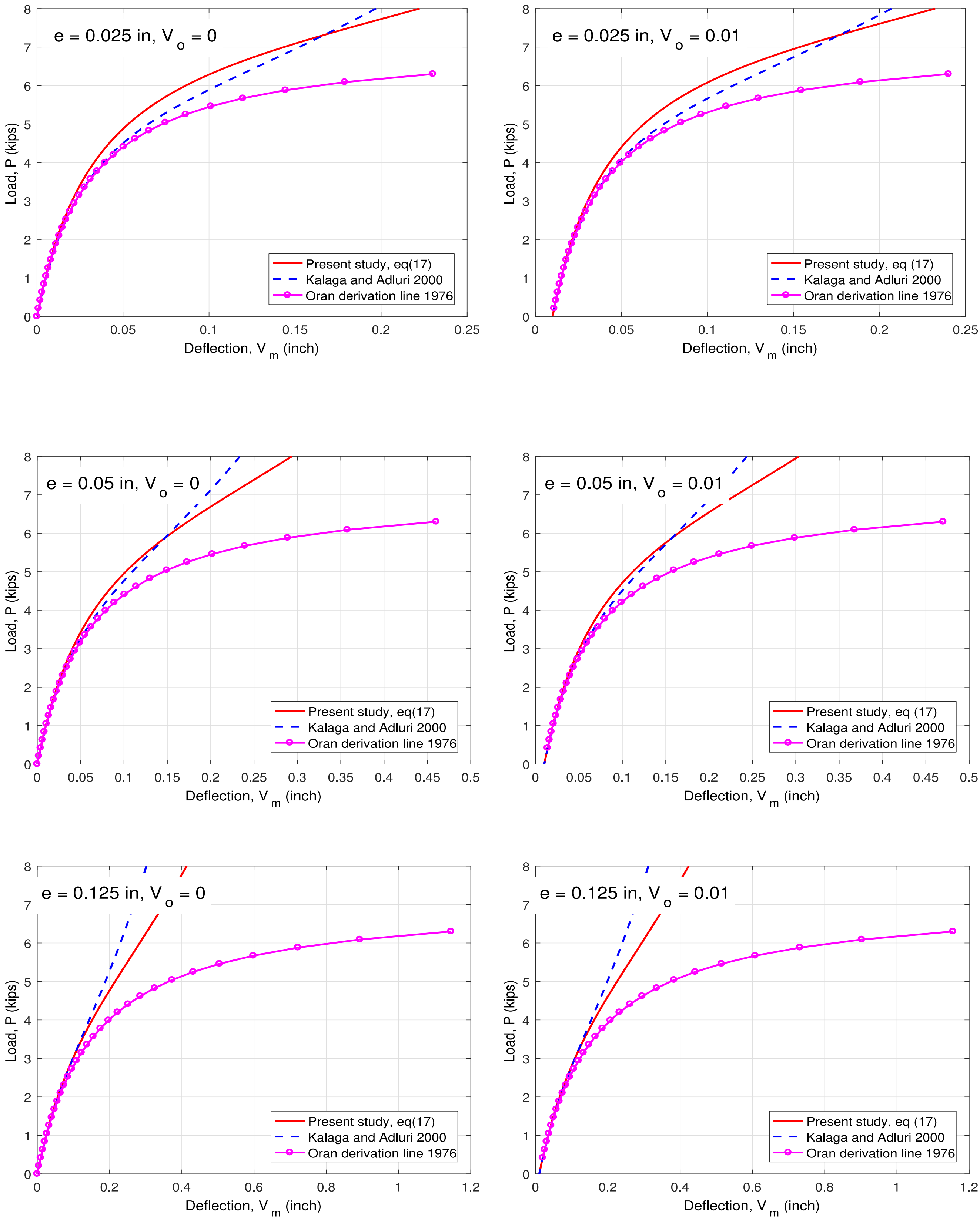



**Fig. 7, Load-central deflection curves for several e and $V_o$ values**

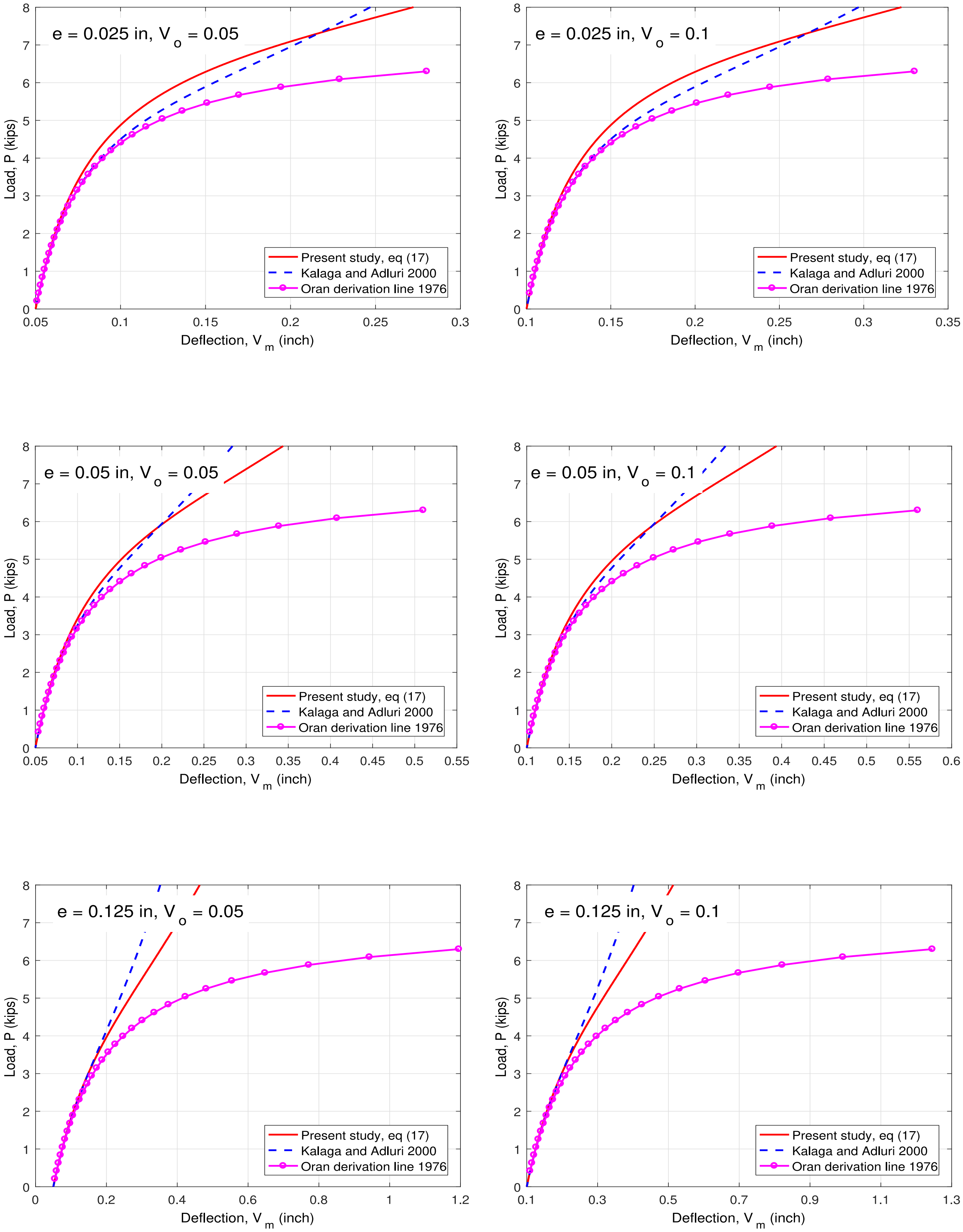



**Fig. 7, Load-central deflection curves for several e and $V_o$ values**

## 3.2. Case Study II: simply supported beam Under Varied Loading Conditions

### 3.2.1. Central concentrated force

In this example, a simply supported beam subjected to a concentrated force at mid-span is considered, as shown in Fig.8 . This example was studied both experimentally and theoretically by (Woolcock & Trahair, 1974). The same example was considered later by (Lin & Hsiao, 2001) to validate their proposed solution. The geometrical and material properties of the beam are L = 143.9 in, b = 0.86 in, $t_f$ = 0.22 in, $t_w$ = 0.085 in, h =2.862 in, Young's modulus E = 9,300,000 psi. The buckling load of the member is 17.65 Ib.

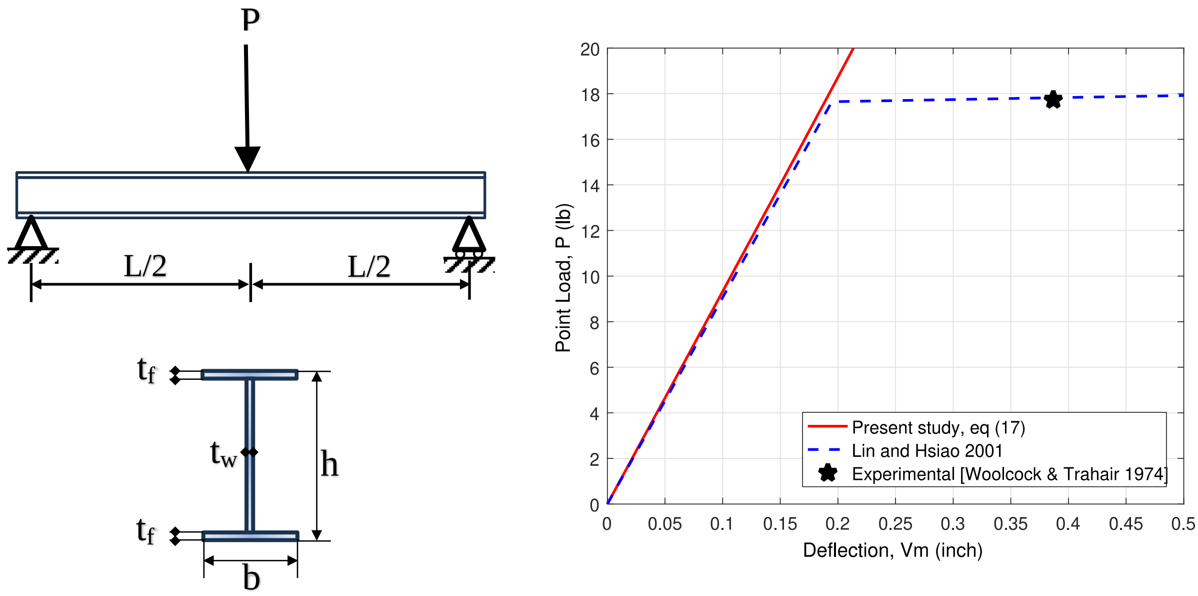



**Fig. 8 Load displacement of simply supported beam subjected to a central point load**

### 3.2.2. Eccentric axial force

An analytical investigation was conducted on a simply supported W14 × 34 beam by (Lawrence A. Soltis,1972), where the beam was subjected to an eccentric axial force at each ends, as shown in Fig.9, with the load deflection curve of the beam presented. Later, (Lin & Hsiao, 2001) examined this case study for validation purposes. In this context, the same simply supported beam column is considered here to validate the displacement prediction obtained from the numerical model. The geometrical and material properties of the beam are L = 264,6 in. and Young's modulus E = 29,000 psi. and the buckling load is 150.1 kip.

### 3.2.3. Results and Analysis

The validity of the proposed model to predict the deflection of beam-column members subjected to axial and transverse forces was investigated by comparing the results obtained from the developed displacement expression with both experimental and numerical data. Figs.8 and 9 show the load deflection-curve of the present study with numerical and measured values. It can be seen that there is a good agreement between the predicted, the numerical, and the measured displacements. However, as the applied load approaches the critical load value, minor discrepancies in the load-deflection curves may be noted. This can be attributed to the fact that the proposed model doesn't account for geometrical nonlinearities and large deformation that occur after buckling, namely, post bucking behavior was not considered in this study. The proposed model, however, is more effective within the elastic range and for loads below the critical threshold.

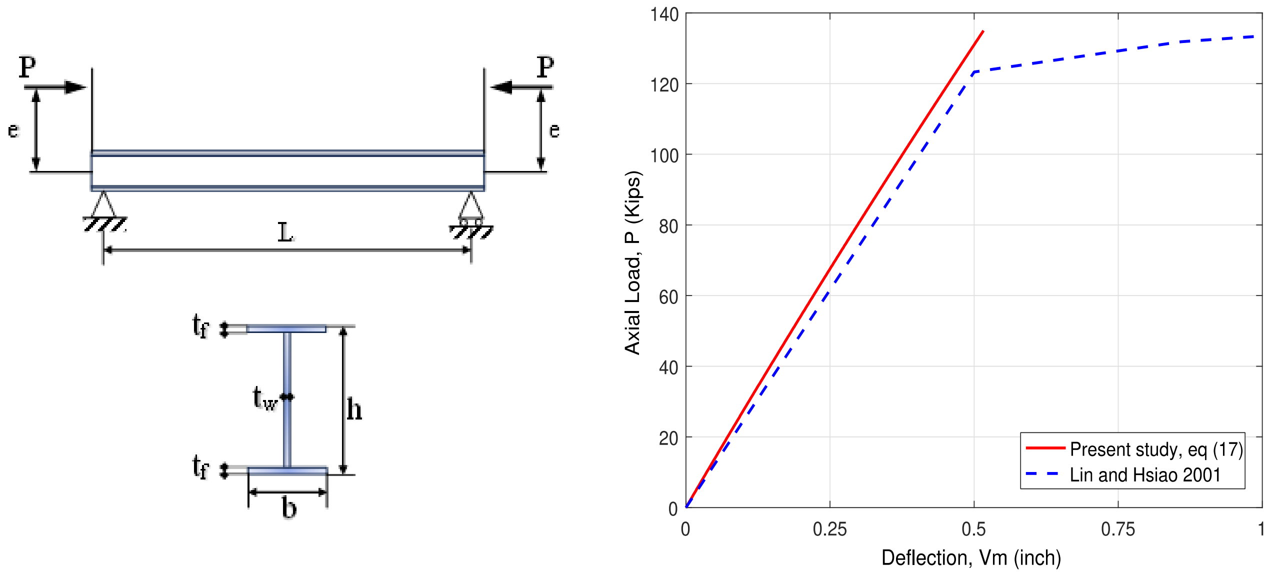



**Fig. 9 Load displacement of simply supported beam subjected to an eccentric axial force**

## 4. CONCLUSIONS

A new numerical model to conduct finite deflection analysis of slender beam column members was developed. The model utilizes the total potential energy approach with enhanced accuracy. Several nonlinearities were taken into account, such as geometric nonlinearity, the effects of axial force on bending stiffness, the end moments on axial stiffness (bowing), and the initial imperfection. A new deformation curve was used in the strain-displacement relation to obtain a more accurate beam-column response. The nonlinear Green strain tensor was also modified by considering more (higher-order) accurate bowing deformation terms. In addition, a modified expression of the total deflection was defined based on an improved TPE equation.

The validity of the proposed model was established by comparing the predicted results against experimental and numerical data. Based on the numerical results, it can be concluded that the developed model is capable of predicting the load-displacement behavior with enhanced accuracy. In addition, it may be noted that the proposed model is more effective within the elastic range and for loads below the critical value. To accurately predict deflections beyond the critical load, more advanced models that account for geometric and material nonlinearities are required. These models can deal with large deformations and provide realistic predictions of post-buckling behavior. Future research should focus on developing more refined models that account for post buckling behavior, in which further sources of nonlinearity can be included. Namely, shear deformation, tapered members, flexible end conditions, and material nonlinearity. Additional experimental studies that deal with different cases of loading conditions are recommended to validate these models.

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