*The product of a weighted composition operator $\mathcal{W}^*_{f,\varphi}$ with the adjoint of a weighted composition operator $\mathcal{W}^*_{f,\psi}$ on the Hardy space $\mathbb{H}^2$

Alaa Hussien Mohammed

College of Education , University of Al-Qadisiya

alaa.hussien@qu.edu.iq

Eiman H. Abood

College of science, University of Baghdad

eiman.mat@yahoo.com

Received:- 13/7/2017 Accepted:-28/11/2017

Abstract:

In this paper we study the product of a weighted composition operator $\mathcal{W}_{f,\varphi}$ with the adjoint of a weighted composition operator $\mathcal{W}^*_{f,\psi}$ on the Hardy space $\mathbb{H}^2$. We will try to completely describe when the operator $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is invertible , isometric and unitary .

Keywords: isometric and unitary operator , weighted composition operator , inner function.

Physical Classification QA299.6- 433

* This research is a part of an Ph.D dissertation in the case of the second researcher.

1. Introduction

Let U denote the open unite disc in the complex plan ,let $\mathbb{H}^\infty$ denote the collection of all holomorphic function on U and let $\mathbb{H}^2$ is consisting of all holomorphic self-map on U such that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ whose Maclaurin coefficients are square summable (i.e)

$f(z)=\sum_{n=0}^{\infty}|a_n|^2<\infty$. More precisely $f(z)=\sum_{n=0}^{\infty} a_n z^n$ if and only if $\|f\|=\sum_{n=0}^{\infty}|a_n|^2<\infty$.The inner product inducing the$\mathbb{H}^2$ norm is given by $\langle f,g\rangle=\sum_{n=0}^{\infty} a_n \overline{b_n}$.

Given any holomorphic self-map φ on U, recall that the composition operator

$$C_\varphi(h)=ho\varphi \quad (h\in\mathbb{H}^2)$$

Is called the composition operator with symbol φ, is necessarily bounded. Let

$f\in\mathbb{H}^\infty$, the operator $T_f:\mathbb{H}^2\to\mathbb{H}^2$ defined by

$$T_f(h(z))=f(z)h(z), \quad for\ all\ z\in \mathrm{U}, h\in\mathbb{H}^2$$

is called the Toeplitz operator with symbol f . Since $f\in\mathbb{H}^\infty$, then we call T_f a holomorphic Toeplitz operator. If T_f is a

holomorphic Toeplitz operator, then the operator $T_f C_\varphi$ is bounded and has the form

$$T_f C_\varphi g=f(go\varphi) \qquad (g\in\mathbb{H}^2).$$

We call it the weighted composition operator with symbols f and φ [1] and [3], the linear operator

$$\mathcal{W}_{f,\varphi}\, g=f(go\varphi) \qquad (g\in\mathbb{H}^2).$$

We distinguish between the two symbols of weighted composition operator $\mathcal{W}_{f,\varphi}$, by calling f the multiplication symbol and φ composition symbol.

For given holomorphic self-maps f and φ of U, $\mathcal{W}_{f,\varphi}$ is bounded operator even if $f\notin\mathbb{H}^\infty$ .To see a trivial example, consider $\varphi(z)=p$ where $p\in$ U and $f\in\mathbb{H}^2$, then for all $g\in\mathbb{H}^2$, we have

$$\|\mathcal{W}_{f,\varphi}\, g\|_2=\|g(p)\|\|f\|_2=\|f\|_2|\langle g,K_p\rangle|\le \|f\|_2\|g\|_2\|K_p\|_2.$$

In fact, if $f\in\mathbb{H}^\infty$,then $\mathcal{W}_{f,\varphi}$ is bounded operator on $\mathbb{H}^2$ with norm

$$\|\mathcal{W}_{f,\varphi}\|=\|T_f C_\varphi\|\le\|f\|_\infty\|C_\varphi\|=\|f\|_\infty\sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

2. Basic Concepts

We start this section, by giving the following results which are collect some properties of Toeplitz and composition operators.

Lemma (2.1):[4, 6] Let φ be a holomorphic self-map of U, then

(a) $C_\varphi T_f=T_{fo\varphi}\, C_\varphi.$
(b) $T_g T_f=T_{gf}$.
(c) $T_{f+\gamma g}=T_f+\gamma T_g$.
(d) $T_f^*=T_{\bar f}$.

Proposition (2.2):[1] Let φ and ψ be two holomorphic self-map of U, then

1. $C_\varphi^n=C_{\varphi_n}$ for all positive integer n.
2. C_φ is the identity operator if and only if φ is the identity map.
3. $C_\varphi=C_\psi$ if and only if $\varphi=\psi$.
4. The composition operator cannot be zero operator.

For each $\alpha\in U$,the reproducing kernel at α ,defined by $K_\alpha(z)=\frac{1}{1-\bar\alpha z}$

It is easily seen for each $\alpha\in U$ and $f\in H^2$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$ that

$$\langle f,K_\alpha\rangle=\sum_{n=0}^{\infty} a_n\,\alpha^n=f(\alpha).$$

When $\varphi(z)=(az+b)/cz+d)$ is linear-fractional self-map of U, Cowen in [2] establishes $C_\varphi^*=T_g C_\sigma T_h^*$, where the Cowen auxiliary functions g, σ and h are defined as follows:

$g(z) = \frac{1}{-\bar{b}z+\bar{d}}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ *and* $h(z) = cz + d$.

If φ is linear fractional self-map U, then $W_{f,\varphi}^* = (T_f C_\varphi)^* = C_\varphi^* T_f^* = T_g C_\sigma T_h^*$.

Proposition (2.4):[5] Let each of $\varphi_1, \varphi_2, \dots \varphi_n$ be holomorphic self-maps of U and $f_1, f_2, \dots f_n \in \mathbb{H}^\infty$, then

$$\mathcal{W}_{f_1,\varphi_1} . \mathcal{W}_{f_2,\varphi_2} \dots . \mathcal{W}_{f_n,\varphi_n} = T_h C_\phi$$

Where
$T_h = f_1.(f_2 o\varphi_1).(f_3 o\varphi_2 o\varphi_1). \dots (f_2 o\varphi_{n-1} o\varphi_{n-2} o \dots . o\varphi_1)$
and

$C_\phi = \varphi_n o\varphi_{n-1} o \dots . o\varphi_1$.

Corollary (2.5): Let φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$ then

$$\mathcal{W}_{f,\varphi}^n = T_{f\ \ (f\ \ o\varphi\ \ )(f\ \ o\varphi_2)\dots(f\ \ o\varphi_{n-1})} C_{\varphi_n}$$

Lemma (2.6):[3] If the operator $\mathcal{W}_{f,\varphi}$: $\mathbb{H}^2 \to \mathbb{H}^2$ is bounded, then for each $\alpha \in U$

$$\mathcal{W}_{f,\varphi}^* K_\alpha = \overline{f(\alpha)} K_{\varphi(\alpha)}.$$

3- Main Results

In this section ,we study the product of a weighted composition operator $\mathcal{W}_{f,\varphi}$ with the adjoint of a weighted composition operator $\mathcal{W}_{f,\psi}^*$ on the Hardy space $\mathbb{H}^2$. First we try to obtain some properties of the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$.

Proposition (3. 1): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$, such that 0 is not a fixed point of U then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is self-adjoint if and only if

$\psi(z) = \lambda\varphi(z)$, for all $z \in U$.

Proof : Let $\beta \in U$, then for each $z \in U$, we have

$$(\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*)^* K_\beta(z) = \mathcal{W}_{f,\psi}\mathcal{W}_{f,\varphi}^* K_\beta(z)$$

$$= T_f C_\psi \left(\overline{f(\beta)} K_{\varphi(\beta)}(z) \right)$$

$$= \overline{f(\beta)}\, f(z) K_{\varphi(\beta)}(\psi(z)) \quad .$$

On the other hand , for each $z \in U$, we have

$$\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^* K_\beta(z) = T_f C_\varphi \left(\overline{f(\beta)} K_{\psi(\beta)}(z) \right)$$

$$= \overline{f(\beta)}\, f(z) K_{\psi(\beta)}(\varphi(z)) \quad .$$

Therefore , $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is self-adjoint if and only if for each $z \in U$

$$K_{\varphi(\beta)}(\psi(z)) = K_{\psi(\beta)}(\varphi(z))$$

Hence ,

$$\frac{1}{1 - \overline{\varphi(\beta)}\psi(z)} = \frac{1}{1 - \overline{\psi(\beta)}\varphi(z)} \qquad (1)$$

In particular letting $\beta = 0$ in equation (3.1), we get

$\psi(z) = \lambda\varphi(z)$ *where* $\lambda = \overline{(\frac{\psi(0)}{\varphi(0)})}$ (note that $\varphi(0) \neq 0$) . ■

Recall that [2] an operator T is an isometry if $\|Tx\| = \|x\|$ for all x or equivalently $T^* T = I$.

Nordgren E.M [7] characterized the isometry composition operator as follows .

Theorem (3. 2): A composition operator C_φ is an isometry if and only if φ is an inner function and $\varphi(0) = 0$.

Lemma (3. 3): Suppose that φ be a holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}$ is an isometry, then φ must be inner function and $\|f\| = 1$.

Proof : Let the operator $\mathcal{W}_{f,\varphi}$ is an isometry , then $\mathcal{W}_{f,\varphi}^* . \mathcal{W}_{f,\varphi} = I$. Thus for each $p \in U$, we have

$\|\mathcal{W}_{f,\varphi}K_p\| = \|K_p\|$, then $\|T_f C_\varphi K_p\| = \|K_p\|$.

This implies that $\|f(K_p \circ \varphi)\| = \|K_p\|$ .Hence , by taking $p = 0$, then $K_0 = 1$

and thus $\|f(1 \circ \varphi)\| = \|1\|$ $, then$ $\|f\| = 1$

In addition that, if $g(z) = z$, then it is clear that $\|g\| = 1$ **.** Therefore

$\|\mathcal{W}_{f,\varphi}g\| = \|g\|$, and then $\|T_f C_\varphi g\| = \|g\|$.

Thus , $\|f(go\varphi)\| = \|g\|$, $then$ $\|f.\varphi\| = 1$.

Since $|\varphi(e^{it})| \le 1$ $a.e.$ $t \in [0,2\pi)$

and both $\|f\|$ and $\|f.\varphi\|$ are 1 .Then, by the integral representation of $\|f\|_{\mathbb{H}^2}$

$$\|f\|_{\mathbb{H}^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \quad .$$

So that $|\varphi(e^{it})| = 1$ $a.e.$ $on\ U$, then φ is an inner function **.** ■

Gunatillake G. [5] studied the invertible weighted composition operator on Hardy space $\mathbb{H}^2$. He give the following result .

Theorem (3. 4):[5] The operator $\mathcal{W}_{f,\varphi}$ on $\mathbb{H}^2$ is invertible if and only if f is both bounded and bounded away from zero on the unit disc and φ is an automorphism of the unit disc . the inverse operator is the weighted composition operator $\mathcal{W}_{f,\varphi}^{-1} = \mathcal{W}_{\frac{1}{(f\circ\varphi^{-1})},\varphi^{-1}}$.

Theorem (3. 5): Suppose that $\varphi\ and\ \psi$ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is invertible if and only if each of $\mathcal{W}_{f,\varphi}$ and $\mathcal{W}_{f,\psi}$ is invertible operator .

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is invertible , then the operator $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is one-to-one and onto .Hence, $\mathcal{W}_{f,\varphi}$ is onto . Therefore it is clear that , φ is non- constant map .

Thus , $\mathcal{W}_{f,\varphi}$ is one-to-one . Hence $\mathcal{W}_{f,\varphi}$ is invertible .

Now , since each of $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ and $\mathcal{W}_{f,\varphi}$ is invertible , then we have $\mathcal{W}_{f,\psi}$ must be invertible operator .

The reverse induction follows immediately . ■

A straightforward consequence can obtained from theorem (3. 4).

Corollary (3. 6): Suppose that $\varphi\ and\ \psi$ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is invertible if and only if f is bounded and bounded away from zero on U and each of φ and ψ is an automorphism of U .

Corollary (3. 7): Let $\varphi\ and\ \psi$ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$. If $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is invertible , then $(\mathcal{W}_{f,\varphi}.\mathcal{W}_{f,\psi}^*)^{-1} = C_{\psi^{-1}}^*.\mathcal{W}_{[1/\overline{(fo\psi^{-1})}(fo\varphi^{-1})],\varphi^{-1}}$.

Proof Since by theorem (3.1.3) we have

$\mathcal{W}_{f,\varphi}^{-1} = \mathcal{W}_{\frac{1}{(fo\varphi^{-1})},\varphi^{-1}}$ and $\mathcal{W}_{f,\psi}^{-1} = \mathcal{W}_{\frac{1}{(fo\psi^{-1})},\psi^{-1}}$. Then ,

$$(\mathcal{W}_{f,\psi}^*)^{-1} = (\mathcal{W}_{f,\psi}^{-1})^* = (\mathcal{W}_{\frac{1}{(fo\psi^{-1})},\psi^{-1}})^* = (T_{\frac{1}{(fo\psi^{-1})}}C_{\psi^{-1}})^*$$

$$= C_{\psi^{-1}}^* T_{\frac{1}{\overline{(fo\psi^{-1})}}} \quad .$$

Hence , $(\mathcal{W}_{f,\varphi}.\mathcal{W}_{f,\psi}^*)^{-1} = (\mathcal{W}_{f,\psi}^*)^{-1}(\mathcal{W}_{f,\varphi})^{-1}$

$$= (C_{\psi^{-1}}^* T_{\frac{1}{\overline{(fo\psi^{-1})}}}).(T_{\frac{1}{(fo\varphi^{-1})}}C_{\varphi^{-1}})$$

$$= C^*_{\psi^{-1}}\, T_{\overline{\frac{1}{(fo\psi^{-1})(fo\varphi^{-1})}}}\, C_{\varphi^{-1}}$$

$$= C^*_{\psi^{-1}}.\mathcal{W}_{\overline{\frac{1}{(fo\psi^{-1})(fo\varphi^{-1})}},\varphi^{-1}} .$$

■

Lemma (3. 8):[9] If T is isometry operator and S is unitary operator , then $TS^*$is an isometry .

Theorem (3. 9): Suppose that φ *and* ψ be two holomorphic self-maps of U and $f \in \mathbb{H}^\infty$ such that $\|f\|_{\mathbb{H}^\infty} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry if and only if $\mathcal{W}_{f,\varphi}$ is an isometry and $\mathcal{W}_{f,\psi}$ is an unitary operator .

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry , therefore

$(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})^*\ \mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi} = I$. Thus

$\mathcal{W}_{f,\psi}\mathcal{W}^*_{f,\varphi}\ \mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi} = I$ **.** Hence one can easily see that $\mathcal{W}_{f,\psi}$ is onto .

This it is clear that, ψ is non- constant map . Therefore by lemma (2.4.3) we have $\mathcal{W}_{f,\psi}$ is one-to-one.

Thus $\mathcal{W}_{f,\psi}$ invertible . Therefore by theorem (3.1.5) and corollary (3.1.6) ψ must be an automorphism of U. So that there exists $\eta \in \partial U$ *and* $p \in U$, that for each $z \in U$

$$\psi(z) = \eta\left(\frac{p-z}{1-\bar{p}z}\right), \quad where\ \psi(p) = 0 .$$

But $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry , then for every $p \in U$, we conclude that

$$\|\mathcal{W}_{f,\varphi}\ \mathcal{W}^*_{f,\psi}\ K_p\| = \|K_p\| .$$

Thus , $\|\mathcal{W}_{f,\varphi}(\ \overline{f(p)}K_{\psi(p)})\| = \|K_p\|$.

Hence , $\|T_f C_\varphi(\ \overline{f(p)}K_0)\| = \|K_p\|$.

Then , $\|\overline{f(p)}T_f C_\varphi(K_0)\| = \|K_p\|$ **.**

Therefore , $\|\overline{f(p)}\ f(K_0 \circ \varphi)\| = \|K_p\|$ **.**

But $(K_0 \circ \varphi = 1 \circ \varphi = 1)$, $\|\overline{f(p)}\ f\| = \|K_p\|$ **.**

Hence , $|\overline{f(p)}|\|\ f\| = \|K_p\|$.

Then , $|\langle f, K_p\rangle| = \|K_p\| = \|f\|\|K_p\|$.

Thus, by Cauchy –Schwartz inequality , we have

$$f(z) = \alpha K_p(z) = \frac{\alpha}{1-\bar{p}z} \qquad for\ some\ \alpha \in \mathbb{C}$$

But $\|\ f\| = 1$, then it easily see that $f(z) = r\frac{K_p}{\|K_p\|}$ where $|r| = 1$ *and* $\psi(p) = 0$

Hence by theorem (2. 9) we have $\mathcal{W}_{f,\psi}$ is unitary operator .

Conversely , if $\mathcal{W}_{f,\varphi}$ is an isometry and $\mathcal{W}_{f,\psi}$ is unitary , then

$$\mathcal{W}^*_{f,\varphi}\mathcal{W}_{f,\varphi} = \mathcal{W}^*_{f,\psi}\mathcal{W}_{f,\psi} = \mathcal{W}_{f,\psi}\mathcal{W}^*_{f,\psi} = I \qquad (2)$$

Hence from (3.2) we have

$(\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi})^*\,\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi} = \mathcal{W}_{f,\psi}\ \mathcal{W}^*_{f,\varphi}\ \mathcal{W}_{f,\varphi}\ \mathcal{W}^*_{f,\psi} = I$.

Therefore $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is an isometry ,as desired .

■

Corollary (3. 10): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$ such that $\|f\|_{\mathbb{H}^\infty} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}^*_{f,\psi}$ is unitary if and only if each of $\mathcal{W}_{f,\varphi}$ and $\mathcal{W}_{f,\psi}$

is an unitary operator .

Proof : Suppose that $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is an unitary operator , then it is isometry . Therefore by theorem (3. 9) we have $\mathcal{W}_{f,\psi}$ is unitary operator . But since $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is unitary , then $\mathcal{W}_{f,\psi}\mathcal{W}_{f,\varphi}^*$ is also unitary , thus by theorem (3.9) we have $\mathcal{W}_{f,\varphi}$ is unitary operator .

The converse is clear .

Now , the corollary (3. 9) and theorem (2. 9) we get the following consequence .

Corollary (3. 11): Suppose φ *and* ψ be two holomorphic self-map of U and $f \in \mathbb{H}^\infty$ such that $\|f\|_{\mathbb{H}^\infty} = 1$. Then $\mathcal{W}_{f,\varphi}\mathcal{W}_{f,\psi}^*$ is unitary if and only if each of φ *and* ψ is an automorphism of U and $f(z) = r\frac{K_p}{\|K_p\|}$ such that $p \in U$ where $|r| = 1$ and

$\varphi(p) = \psi(p) = 0$.

References

[1] Abood E. H., **(2003)** " The composition operator on Hardy space H^2", Ph.D. Thesis, University of Baghdad.

[2] Berberian, S.K. **1976** , Introduction to Hilbert Space, *Sec. Ed., Chelesa publishing Com.*, New York, N.Y.

.

[3] Cowen C. C. and Ko E. **(2010)** ,"Hermitian weighted composition operator on H^2 " *Trans.Amer.Math.Soc.*, 362, 5771-5801.

[4] Deddnes J. A. **(1972)** "Analytic Toeplitz and Composition Operators", *Con. J. Math.*, vol(5), 859-865,.

[5] Gunatillake G. **(2011)**,"invertible weighted composition operator ",*J. Funct. Anal.*, 261, 831-860.

[6] Halmos P. R. **,(1974)**,"A Hilbert space problem book", *Sprinrer- Verlag*, NewYork.

[7] Nordgren, E. A. **(1968**), Composition operator, *Can. J. Math*. 20, 442-449.

[8] Shapiro J. H. **(1993),** "Composition Operators and Classical Function Theory", *Springer- Verlage*, New York,.

[9] Clifford, J. H. , Le, T. and Wiggins **2014**, A. ,Invertible composition operators : The product of a composition operators with adjoint of a composition operators,

[10] Akeroyd, J. R. and Ghatage, P. G.,Closed **(2008)** range composition operators on A^2, *Illinois J. Math*. 52,533-549

.

* حاصل الضرب للمؤثر التركيبي الموزون مع الجوينت للمؤثر التركيبي الموزون

ايمان حسن عبود

جامعة بغداد ، كلية العلوم، قسم الرياضيات

eiman.mat@yahoo.com

الاء حسين محمد

جامعة القادسية، كلية التربية، قسم الرياضيات

alaa.hussien@qu.edu.iq

تاريخ الاستلام :- 13/7/ 2017

تاريخ القبول :- 28/ 11/ 2017

الخلاصة

في هذا البحث قمنا بدراسة حاصل الضرب للمؤثر التركيبي الموزون مع الجوينت للمؤثر التركيبي الموزون حول فضاء هاردي كما وحاولنا اعطاء صورة واضحة عن المؤثر التركيبي الموزون عندما يكون قابل للعكس و وحدووي

الكلمات المفتاحية : المؤثر التركيبي .المؤثر التركيبي الموزون.المؤثرالوحدوي .الدالة الداخلية

*البحث مستل من اطروحة دكتوراه للباحث الثاني

.