Triple operators

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Abstract

In this paper, we introduce a new class of operators acting on a complex Hilbert space $\mathcal{H}$ which is called triple operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is called triple operator if $(TT^*)T = T(TT^*)$, where T^* is the adjoint of the operator T.

We investigate some basic properties of such operators and study the relation between the triple operators and some other operators.

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1- Introduction

Through this paper, $\mathcal{B}(\mathcal{H})$ denoted to the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be isometry if $T^*T = I$, unitary if $T^*T = TT^* = I$ [1] and partial isometry if $TT^*T = T$ [4], where T^* is the adjoint of T. The operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if $TT^* = T^*T$ [3] and quasi-normal if $T(T^*T) = (T^*T)T$ [5].

2- Triple operators

In this section, we will study some properties which are applied for the triple operators.

Definition (2.1): The operator $T \in \mathcal{B}(\mathcal{H})$ is called triple operator if $(TT^*)T = T(TT^*)$, where T^* is the adjoint of the operator T.

Example (2.2): Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be an operator on two-dimensional Hilbert space $\mathbb{C}^2$. Then $(TT^*)T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T(TT^*)$

Therefore T is triple operators.

Example (2.3): If $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is an operator on two-dimensional Hilbert space $\mathbb{C}^2$ .Then $(TT^*)T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \neq T(TT^*) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Thus T is not triple operators .

In the following proposition, we give some properties of the triple operators.

Proposition (2.4): If T is triple operator on a Hilbert space $\mathcal{H}$. Then:

1- αT is triple operator for every complex number α.

2- If S is unitarily equivalence to T, then S is triple operator.

3- If M is closed subspace of $\mathcal{H}$, then (T/M) is triple operator.

Proof:

(1) $[(\alpha T)(\alpha T)^*](\alpha T) = \alpha\bar{\alpha}\alpha(TT^*)T$

$= \alpha\alpha\bar{\alpha}\, T(T\,T^*)$

$= (\alpha T)[(\alpha T)(\alpha T)^*]$

So that (αT) is Triple operator.

(2) since S is unitarily equivalence to T. Then there exists unitary operator U such that $S = UTU^*$, so that

$S^* = UT^*U^*$

$(S\,S^*)S = (UTU^*UT^*U^*)UTU^*$
$= U(TT^*)TU^*$

Since T is triple operator, then $(S\,S^*)S = UT(TT^*)U^*$ … … … … … (1)

On the other hand $S(S\,S^*) = UTU^*(UTU^*UT^*U^*) = UT(TT^*)\,U^*$ … … … (2)

Since (1) = (2) ,Thus S is triple operator.

(3) $[(T/M)(T/M)^*](T/M) = [(T/M)(T^*/M)](T/M)$

$$= (TT^*/M)(T/M)$$
$$= [(TT^*)T]/M$$
$$= [T(TT^*)]/M$$
$$= (T/M)[(TT^*)/M]$$
$$= (T/M)[(T/M)(T/M)^*]$$

Therefore T/M is triple operator.

The following example shows that if S,T are triple operators, then not necessary $(S+T) and (S.T)$ are triple operators.

<u>Example (2.5):</u> Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ are operators on two dimensional Hilbert space $\mathbb{C}^2$.

Since

$(SS^*)S = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = S(SS^*)$ and

$$(TT^*)T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix} = T(TT^*)$$

Then S and T are triple operators.

But $[(S+T)(S+T)^*](S+T) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \neq (S+T)[(S+T)(S+T)^*] = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and

$[(ST)(ST)^*](ST)$
$= \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$
$= \begin{pmatrix} -2 & -2 \\ 16 & -16 \end{pmatrix}$
$\neq (ST)[(ST)(ST)^*]$
$= \begin{pmatrix} -2 & -8 \\ 4 & -16 \end{pmatrix}$

Therefore $(S+T)$ *and* (ST) are not triple operators.

The following theorem show that the condition of S,T are commuting normal operators is very necessary to becomes $(S+T)$ is triple operators.

<u>Theorem(2.6):</u> If S,T are triple operators such that S,T are commuting normal operators, then $(S+T)$ is triple operators.

Proof: by (Putnam –Fuglede theorem)[2] then $ST^* = T^*S$ *and* $TS^* = S^*T$

Thus $[(S+T)(S+T)^*](S+T) = [(S+T)(S^*+T^*)](S+T)$

$$= (SS^*)S + (ST^*)S + (TS^*)S + (TT^*)S + (SS^*)T + (ST^*)T + (TS^*)T + (TT^*)T$$

$$= S(SS^*) + S(ST^*) + S(TS^*) + S(TT^*) + T(SS^*) + T(ST^*) + T(TS^*) + T(TT^*)$$

$$= (S+T)[(S+T)(S+T)^*]$$

The following theorem show that the condition of S,T are normal operators is very necessary to becomes (ST) is triple operators.

Theorem (2.7): Let S, T are triple operators such that S, T are normal operators. Then (ST) is triple operator.

Proof: $[(ST)(ST)^*](ST) = [(ST)(T^*S^*)](ST)$

$= (ST)(T^*SS^*T)$

$= (ST)(ST^*TS^*)$

$= (ST)(STT^*S^*)$

$= (ST)[(ST)(ST)^*]$

Therefore (ST) is triple operator.

Remark (2.8): (1) If T is unitary operator, then T is triple operator.

(2) If T is isometry operator, then T is triple operator.

(3) If T is a partial isometry operator, then T is triple operator.

But the converse of the three above- mentioned points is not true as we saw in the following example:

Example (2.9): If $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is an operator on two-dimensional Hilbert space $\mathbb{C}^2$ .Then T is triple operator as we seen in example (2.5)

But $(TT^*) = (T^*T) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \neq I$ .Thus T is not unitary, isometry operator.

$TT^*T = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} \neq T$ .Therefore T is not partial isometry.

Proposition (2.10): If T is triple operator and normal operator, then T^{-1} is triple operator.

Proof: since T is triple operator, then $(TT^*)T = T(TT^*)$ taking inverse of two sides, we get $T^{-1}(T^{*-1}T^{-1}) = (T^{*-1}T^{-1})T^{-1}$

$T^{-1}(T^{-1}T^{*-1}) = (T^{-1}T^{*-1})T^{-1}$[since T is normal,then T^{-1} is normal operator]

Then T^{-1} is triple operator.

The following two examples show that quasi normal operators and triple operators are independent.

Example(2.11): Let U be a unilateral shift operator on ℓ_2, (i.e. $U\,(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3 \dots)$ .Then $U(U^*U) = U = (U^*U)U$
Thus U is quasi normal operator . But $(UU^*)U = U \neq U(UU^*) = U^2U^*$,Then U is not triple operator.

Example (2.12): Let U^* be the adjoint of the unilateral shift operator on ℓ_2. (i.e. $U^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4 \dots)$)

Since $(U^*U)U^* = U^* = U^*(U^*U)$. Then U^* is triple operator.

But $U^*(UU^*) = U^* \neq (UU^*)U^* = UU^{*2}$ Thus U^* is not quasi normal operator.

Theorem (2.13): Let $T_1, T_2, \ldots, T_m$ be triple operators in $\mathcal{B}(\mathcal{H})$. Then $(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes T_2 \otimes \ldots \otimes T_m)$ are triple operators.

Proof

$$[(T_1 \oplus T_2 \oplus \ldots \oplus T_m)(T_1 \oplus T_2 \ldots \oplus T_m)^*]$$
$$(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$$
$$= [(T_1 \oplus T_2 \oplus \ldots \oplus T_m)(T_1^* \oplus T_2{}^* \oplus \ldots \oplus T_m^*)]$$
$$(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$$
$$= (T_1 T_1^* \oplus T_2 T_2{}^* \oplus \ldots \oplus T_m T_m^*)(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$$
$$= (T_1 T_1^*)T_1 \oplus (T_2 T_2{}^*)T_2 \oplus \ldots \oplus (T_m T_m^*)T_m)$$

Since $T_1, T_2, \ldots, T_m$ are triple operators. Then

$$= (T_1(T_1 T_1{}^*) \oplus T_2(T_2 T_2{}^*) \oplus \ldots \oplus T_m(T_m T_m^*))$$
$$= (T_1 \oplus T_2 \oplus \ldots \oplus T_m)$$
$$[(T_1 \oplus T_2 \oplus \ldots \oplus T_m)(T_1 \oplus T_2 \oplus \ldots \oplus T_m)^*]$$

Also

$$[(T_1 \otimes T_2 \ldots \otimes T_m)(T_1 \otimes T_2 \otimes \ldots \otimes T_m)^*]$$
$$(T_1 \otimes T_2 \otimes \ldots \otimes T_m)$$
$$= [(T_1 \otimes T_2 \otimes \ldots \otimes T_m)(T_1^* \otimes T_2{}^* \otimes \ldots \otimes T_m^*)]$$
$$(T_1 \otimes T_2 \otimes \ldots \otimes T_m) =$$
$$(T_1 T_1^* \otimes T_2 T_2{}^* \otimes \ldots \otimes T_m T_m^*)(T_1 \otimes T_2 \otimes \ldots \otimes T_m)$$
$$= ((T_1 T_1^*)T_1 \otimes (T_2 T_2{}^*)T_2 \otimes \ldots \otimes (T_m T_m^*)T_m)$$

Since $T_1, T_2, \ldots, T_m$ are triple operators. Then

$$= (T_1(T_1 T_1^*) \otimes (T_2(T_2 T_2^*) \otimes \ldots \otimes (T_m(T_m T_m^*))$$
$$= (T_1 \otimes T_2 \otimes \ldots \otimes T_m)[(T_1 \otimes T_2 \otimes \ldots \otimes T_m)$$
$$(T_1 \otimes T_2 \otimes \ldots \otimes T_m)^*]$$

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المؤثرات الثلاثية

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الملخص

الهدف من هذا البحث هو تقديم نوع جديد من المؤثرات على فضاء هلبرت المعقد الذي أطلقنا عليه اسم المؤثرات الثلاثية. المؤثر $T \in \mathcal{B}(\mathcal{H})$ يسمى المؤثر الثلاثي اذا كان $(TT^*)T = T(TT^*)$ حيث T^* هو المؤثر المرافق (المصاحب) للمؤثر **T.**
سوف نقدم في هذا البحث بعض الخواص الأساسية لهذا المؤثر وندرس العلاقة بين المؤثرات الثلاثية وبعض الأنواع الأخرى من المؤثرات.

الكلمات المفتاحية: المؤثرات الثلاثية ، المؤثرات .