Completeness of M-fuzzy metric spaces

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ABSTRACT

In this paper, the definition and basic concepts of M-fuzzy metric is introduced with some of its basic properties and some fundamental results concerning this concept, and then these aspects are used to introduce and prove the completeness of M-fuzzy metric spaces.

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كمال الفضاءات المترية الضبابية باستخدام الدوال المترية M-

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العراق

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**قسم الرياضيات وتطبيقات الحاسوب

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الملخص

في هذه الورقة التعريف والمفاهيم الاساسية في الفضاءات المتريةالضبابية-M مقدم بالبعض من خواصه الاساسية وبعض النتائج الاساسية يتعلقان بهذه المفاهيم ثم هذه السمات تستعمل لتقديم واثبات كمال الفضاءات الضبابية باستخدام الدوال المترية-M

جزء من أطروحة الباحث الاول

Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [1]. Since then, this concept is used in topology and some branches of analysis, many authors have expensively developed the theory of fuzzy sets and application.

We start with the obvious definition of fuzzy sets "A fuzzy set, termed by $\tilde{A}$, in a space of objects X is a class of events with a continuous grade of membership and is characterized by a membership function, termed as $\mu_{\tilde{A}}$, which associates for each x $\in$ X a real number in the interval [0, 1]". The value of $\mu_{\tilde{A}}(x)$ represents of grade of membership of x in $\tilde{A}$, i.e., denotes the degree to which an element or event x may be a member of $\tilde{A}$ or belong to $\tilde{A}$, [2].

Chang, C. L. in 1968 used the fuzzy set theory for defining and introducing fuzzy topological spaces, while Wong, C. K. in 1973, discussed the covering properties of fuzzy topological spaces, [3].

Ercey, M. A. in 1979, studied fuzzy metric spaces and its connection with statistical metric spaces, Ming P. P. and Ming L. Y. in 1980, used fuzzy topology to define neighborhood structure of fuzzy point and Moore-Smith convergence, Zike Deng in 1982, studied the fuzzy point and discussed the fuzzy metric spaces with the metric defined between two fuzzy points, [4].

George and Veermamani [5], Kramosil and Michalek have introduced the concept of M-fuzzy topological spaces induced by M-fuzzy metric spaces which have very important applications in quantum practical physics particularly in connections which were given and studied by El-Naschie

The main objective of this paper is to study and prove the completeness of fuzzy metric spaces using M-fuzzy metric spaces.

1- Basic concepts and preliminaries:

we start with the following definition of D-metric spaces , which plays an important role in this paper.

Definition (1.1), [6]:

Let X be a non empty set. A generalized metric (or D-metric) on X is a function $D : X \times X \times X \longrightarrow \mathbb{R}^+$, that satisfies the following conditions for each x, y, z, a $\in$ X:

(1) $D(x, y, z) \geq 0$.

(2) $D(x, y, z) = 0$ if and only if $x = y = z$.

(3) $D(x, y, z) = D(p\{x, y, z\})$, where p is the permutation function.

(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

The pair (X, D) is called the generalized metric or D-metric space.

Definition (1.2), [7]:

A D-metric space (X, D) is said to be D-bounded if there exists a positive real number N, such that $D(x, y, z) \leq N$, for all x, y, z $\in$ X.

In such a case N is said to be the D-bound for X. Moreover, if

$E \subseteq X$, then E is said to be D-bounded subspace of X if there exists a positive real number M, such that $D(x, y, z) \leq M$, for all $x, y, z \in E$.

Now, an illustrative examples are considered for completeness purpose.

Example (1.3), [1], [6]:

Let X = □ and $D(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Then (X, D) is unbounded D-metric space.

Since if we let $x, y, z, a \in X$, then:

i- $D(x, y, z) = |x - y| + |y - z| + |z - x| > 0$ if and only if two of x, y, z are distinct and if $x = y = z$ then $|x - y| + |y - z| + |z - x| = 0$ and hence $D(x, y, z) = 0$

if $D(x, y, z) = 0$, then $|x - y| + |y - z| + |z - x| = 0$, which is true iff $|x - y| = 0$, $|y - z| = 0$, $|z - x| = 0$ and therefore $x = y = z$.

ii- $D(x, y, z) = |x - y| + |y - z| + |z - x|$

$= |x - z| + |z - y| + |y - x|$

$= D(x, z, y)$

Similarly, $D(x, y, z) = D(x, z, y) = D(z, x, y) = \ldots$

i.e., $D(x, y, z) = D(p\{x, y, z\})$, where p is the permutation function.

iii- $D(x, y, z) = |x - y| + |y - z| + |z - x|$

$\leq |x - y| + |y - a| + |a - x| + |a - z| + |z - z| + |z - a|$

$= D(x, y, a) + D(a, z, z)$

Therefore (X, D) is a D-metric space.

But if there is no positive real number N, such that $D(x, y, z) \leq N$, for all $x, y, z \in X$, then (X, D) is unbounded D-metric space.

Lemma (1.4),[8]:

Let (X, D) be a D-metric space, then $D(x, x, y) = D(x, y, y)$.

The next definition introduces the fuzzy metric spaces by using M-distance functions

Definition (1.5), [6]:

A 4-tuple $(X, M_D, *)$ is called M-fuzzy metric space if X is an arbitrary (nonempty) set, * is M-continuous T-norm and M is a fuzzy subset of $X \times X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$:

1. $M_D(x, y, z, t) > 0$.
2. $M_D(x, y, z, t) = 1$ if and only if $x = y = z$.
3. $M_D(x, y, z, t) = M_D(p\{x, y, z\}, t)$, where p is a permutation function of x, y and z.
4. $M_D(x, y, a, t) * M_D(a, z, z, s) \leq M_D(x, y, z, t + s)$.
5. $M_D(x, y, z, *) : (0, \infty) \longrightarrow [0, 1]$ is a continuous.

Lemma (1.6),[8]:

Let (X, d) be a metric space and consider the M-fuzzy metric space $(X, M_d, *)$ and define M_d by:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $M_d(x, y, t + s) \geq M_d(x, y, t)$, $\forall$ s, t, > 0.

The next lemma shows that condition (4) of definition (1.5) may be proved in all cases and thus this condition may be violated from the definition.

Lemma (1.7),[8]:

Let (X, M_D, *) be an M-fuzzy metric space. Define $M_D : X \times X \times X \times (0, \infty) \longrightarrow [0, 1]$, by:

$$M_D(x, y, z, t) = M_d(x, y, t)*M_d(y, z, t)*M_d(z, x, t)$$

Then:

$$M_D(x, y, z, t + s) \geq M_D(x, y, a, t)*M_D(a, z, z, s)$$

for every t, s > 0 and x, y, z $\in$ X.

Proof:

Since:

$$M_D(x, y, z, t) = M_d(x, y, t)*M_d(y, z, t)*M_d(z, x, t)$$

Then:

$$M_D(x, y, a, t) = M_d(x, y, t)*M_d(y, a, t)*M_d(a, x, t) \quad \text{.....................(1.1)}$$

and

$$M_D(a, z, z, s) = M_d(a, z, s)*M_d(z, z, s)*M_d(z, a, s) \quad \text{..................(1.2)}$$

and hence by definition (2.3.2):

$$M_D(x, y, z, t + s) = M_d(x, y, t + s)*M_d(y, z, t + s)*M_d(z, x, t + s)$$

$\geq M_d(x,y,t)*M_d(y,a,t)*M_d(a,z,s)*M_d(z,a,s)*M_d(a,x,t)$

$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, a, s)$ (using eq.(3.1))

$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, a, s)*1$

$= M_D(x, y, a, t)*M_d(a, z, s)*1*M_d(z, a, s)$

$= M_D(x, y, a, t)*M_d(a, z, s)*M_d(z, z, s)*M_d(z, a, s)$

$= M_D(x, y, a, t)*M_D(a, z, z, s)$

which follows from definition (1.6) and eq.(1.2). ■

Among the main results in this paper are the following two lemmas:

Lemma (1.8),[8]:

Let (X, D) be a D-Metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \; t > 0$$

where:

$D(x, y, z) = |x - y| + |y - z| + |z - x|$

Then $(X, M_D, *)$ is a fuzzy metric space.

Lemma (1.9),[8]:

Let X = □ and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \; t > 0$$

where:

$$D(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}, \forall\ x, y, z \in X$$

Then $(X, M_D, *)$ is M-fuzzy metric space.

The next lemmas are given for completeness without proof ,since their proofs are given in their correspondence references further more, these results gives and details properties of M-fuzzy metric spaces

Lemma (1.10), [6]:

Let $(X, M_D, *)$ be an M-fuzzy metric space. Then $M_D(x, y, z, t)$ is nondecreasing with respect to t, for all $x, y, z \in X$, where:

$$M_D(x, y, z, t) = M_d(x, y, t)*M_d(y, z, t)*M_d(z, x, t)$$

Lemma (1.11),[9]:

Let $\{U_n, n \in W\}$, be a sequence of subsets of $X\times X\times X$, such that $U_0 = X\times X\times X$, where W is any index set and U_n contains the diagonal (the identity relation is called the diagonal), and $U_{n+1} o U_{n+1} o U_{n+1} \subset U_n$, $\forall$ n, where o denotes the composition of three uniformly M-continuous functions is a given

uniformly M-continuous. Then there is a non-negative real valued function d on $X\times X$, such that:

a. $d(x, z) \le d(x, y) + d(y, z)$.
b. $U_n \subset \{(x, y) \mid d(x, y) < 2^{-n} = f(x, y)\} \subseteq U_{n-1}$, for each $n \in \mathbb{N}$ and if each U_n is symmetric (i.e., $U = U^{-1}$).

Then there is a pseudo-metric d satisfying condition (b).

Lemma (1.12), [9]:

If X is a uniform space which has a countable base, then X is pseudo-metrizable.

Proof:

If X has a uniformity U with countable base $\{U_n\}$, then by the principle of mathematical induction, we can construct a subsequence $\{U_n\}$, such that:

1. Each U_n is symmetric.
2. $U_n o U_n o U_n \subseteq U_{n-1}$.
3. $U_n \subseteq V_n$, $\forall$ n $\in$ $\mathbb{N}$.

Hence $\{U_n\}$ form a base for U and hence by the metrization lemma (1.11), we have a uniform space (X, U) is a pseudo-metrizable. ■

Lemma (1.13), [9]:

A regular T_1-space whose topology has a σ-locally finite base is metrizable

2-Main results

In this section, the statement and the proof of the completeness of M-fuzzy metric spaces are given which seems to be a normal results up to our know lodge. But first we need to start and prove some other results

Lemma (2.1), [10]:

Let $(X, M_D, *)$ be a M-fuzzy metric space. Then τ_M is a Hausdorff topological space and for each $x \in X$, $\{B(x, 1/n, 1/n) \mid n \in \mathbb{N}\}$ is a neighborhood base at x for the topology τ_M.

From the above lemma one can note that every fuzzy metric space indeed is a fuzzy Hausdorff space.

Theorem (2.2):

Let $(X, M_D, *)$ be an M-fuzzy metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \quad t > 0$$

where:

$$D(x, y, z) = |x - y| + |y - z| + |z - x|$$

then (X, τ_M) is metrizable fuzzy topological space.

Proof:

For each $n \in \mathbb{N}$, define:

$$U_n = \{(x, y, z) \in X \times X \times X \mid M_D(x, y, z, 1/n) > 1 - \frac{1}{n}\}$$

where:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}$$

$$= \frac{\frac{1}{n}}{\frac{1}{n} + |x - y| + |y - z| + |z - x|}$$

It is sufficient to prove that the sequence $\{U_n\}$ is a base for a uniformity U on X, whose induced topology coincides with τ_M.

First, for each $n \in \mathbb{N}$, to prove that:

$$\{(x, x, x) \mid x \in X\} \subseteq U_n,\ U_{n+1} \subseteq U_n \text{ and } U_n = U_n^{-1}$$

Since:

$$M_D(x, x, x, t) = \frac{t}{t+|x-x|+|x-x|+|x-x|}$$

$$= \frac{t}{t} = 1$$

Hence $M_D \geq 1$ and therefore $\{(x, x, x) \mid x \in X\} \subseteq U_n$, i.e., the diagonal is contained in U_n.

Now, to prove that $U_{n+1} \subseteq U_n$, $\forall\, n \in \mathbb{N}$, and since n + 1 > n, hence $\frac{1}{n+1} < \frac{1}{n}$ and so:

$$1 - \frac{1}{n+1} > 1 - \frac{1}{n}$$

Therefore $U_{n+1} \subseteq U_n$ and $U_n = U_n^{-1}$

If $U = U^{-1}$, then U is called symmetric.

On the other hand, for each $n \in \mathbb{N}$, there is, by the M-continuity of *, $m \in \mathbb{N}$ such that m > 3n. Hence:

$$\frac{1}{m} < \frac{1}{3n}$$

and with * to be the usual product, gives:

$$\left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right] = \left[1-\frac{1}{m}\right]\left[1-\frac{1}{m}\right]\left[1-\frac{1}{m}\right]$$

$$< \left[1-\frac{1}{3n}\right]\left[1-\frac{1}{3n}\right]\left[1-\frac{1}{3n}\right]$$

$$< \left[1-\frac{1}{n}\right]\left[1-\frac{1}{n}\right]\left[1-\frac{1}{n}\right] < 1-\frac{1}{n}, \forall n \in \mathbb{N}$$

Therefore $U_m o U_m o U_m \subseteq U_n$ (by lemma (1.12))

Indeed, let $(x, y) \in U_m$, $(y, y) \in U_m$ and $(y, a) \in U_m$

Since $M_D(x, y, z, *)$ is non decreasing (by lemma (1.10))

Then $M_d(x, a, 1/n) \geq M_d(x, a, 3/m)$, and so:

$$M_d(x, a, 1/n) \geq M_d(x, y, 1/m)* M_d(y, y, 1/m)*M_d(y, a, 1/m)$$

$$\geq \frac{\frac{1}{m}}{\frac{1}{m}+|x-y|} * \frac{\frac{1}{m}}{\frac{1}{m}+|y-y|} * \frac{\frac{1}{m}}{\frac{1}{m}+|y-a|}$$

$$\geq \frac{\frac{1}{m}}{\frac{1}{m}+|x-y|} * \frac{\frac{1}{m}}{\frac{1}{m}+|y-a|}$$

$$\geq \left[1-\frac{1}{m}\right]*\left[1-\frac{1}{m}\right]$$

$$\geq 1-\frac{1}{n}$$

Therefore, $(x, a) \in U_n$ and thus $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity U on X.Since for each $x \in X$ and each $n \in \mathbb{N}$

$$U_n(x) = \{y \in X : M_D(x, y, y, 1/n) > 1 - \frac{1}{n}\}$$

$$= B(x, 1/n, 1/n)$$

Hence from lemma (2.1), that the fuzzy topology induced by U coincide with τ_M.

By lemma (1.13), (X, τ_M) is a metrizable fuzzy topological space. ■

Definition (2.3), [11]:

An M-fuzzy metric space is said to be completely M-fuzzy metrizable if every M-fuzzy Cauchy sequence is M-fuzzy convergent.

Theorem (2.4), [10]:

Let (X, M_D, *) be a M-complete fuzzy metric space, and let:

$$M_D(x, y, z, t) = \frac{t}{t + D(x, y, z)}, \; t > 0$$

where:

$$D(x, y, z) = |x - y| + |y - z| + |z - x|$$

Then (X, τ_M) is M-completely fuzzy metrizable.

Definition (2.5), [10]:

An M-fuzzy metric space (X, M_D, *) is called precompact if
for each r, with 0 < r < 1 and each t > 0, there is a finite subset A of X, such that:

$$X = \bigcup_{a \in A} B(a, r, t)$$

In this case, M is called a precompact M-fuzzy metric space on X.

Definition (2.6),[3,12,13]:

A fuzzy topological space is compact if and only if each open cover of the space has a finite subcover.

Theorem (2.7):

Compact M-fuzzy metric space is M-complete.

Proof:

Suppose that $(X, M_D, *)$ is a compact fuzzy metric space, for each r, with $0 < r < 1$ and each $t > 0$ the open cover $\{B(x, r, t) : x \in X\}$ of X, has a finite subcover by definition (2.6)

Hence $(X, M_D, *)$ is precompact (by definition (2.5)

On the other hand, every M-Cauchy sequence $\{x_n\}$ in $(X, M_D, *)$ has a limit point $y \in X$

Let $\{x_n\}$ be a fuzzy M-Cauchy sequence in $(X, M_D, *)$ having a limit point $x \in X$, then there is a subsequence $\{x_{kn}\}$ of $\{x_n\}$ that M-converges to y with respect to τ_M.

Thus, given r, with $0 < r < 1$ and $t > 0$, there is $n_0 \in$ □ , such that for each $n \geq n_0$

$$M_D(x, x, x_{kn}, \frac{t}{3}) > 1 - s, \text{ where } s > 0$$

Which satisfies $(1 - s)*(1 - s) > 1 - r$

Also, there exists $n_1 \geq k(n_0)$, such that for each $n, m \geq n_1$

$$M_D(x_n, x_n, x_m, \frac{t}{3}) > 1 - s$$

Therefore, for each $n \geq n_1$

$$M_D(x, x, x_n, t) \geq M_D(x, x, x_{kn}, \frac{t}{3})*M_D(x_{kn}, x_n, x_n, \frac{t}{3})$$

$$\geq (1 - s)*(1 - s)$$

$$> 1 - r$$

Hence the fuzzy M-Cauchy sequence $\{x_n\}$ M-converges to x.

Thus $(X, M_D, *)$ is an M-complete fuzzy metric space. ■

References

1. Dubois,D. and Prade, H., "Fuzzy Sets and Systems: Theory and Applications", Academic Press, Inc., 1980.
2. Zadeh, L. A., "Fuzzy Sets", 1965. In Fuzzy Sets and Applications: Selected Papers by L. A. Zadeh, Edited by Yager R. R., Ovchinnikos S., Tong R. M., and Ngnyen W. T., John Wiely and Sons, Inc., 1987
3. Munir, M. A. K., "On Separation Axioms of Fuzzy Topological Spaces", Ph.D. Thesis, College of Education, Al-Mustansiriah University, 2006.
4. Fadhel, F. S., "About Fuzzy Fixed Point Theorem", Ph.D. Thesis, College of Science, Al-Nahrain University, 1998.
5. George, A. and Veeramani, P., "On Some Result in Fuzzy Metric Space", Fuzzy Sets and System, 64, pp.395-399, 1994.
6. Sedghi, S. and Shobe, N., "Fixed Point Theorem in M-Fuzzy Metric Spaces with Properties", ISSN, Vol.1, No.1, pp.55-65, 2006.
7. Ahmed, H. A., "Some Results About D-Metric Spaces and Fixed Point Theory in D-Metric Spaces", M.Sc. Thesis, College of Education, Ibn Al-Haitham, University of Baghdad, 2007.

8. Amani ,A.K., ''About the completeness of fuzzy metric spaces'', M.SC. thesis, college of science,AL-Nahrain University, Baghdad, Iraq.2008.

9. Kelley, J. L., “General Topology”, Van Nostrand, 1955.

10. Valentin, G. and Salvador, R., “Some Properties of Fuzzy Metric Spaces”, Fuzzy Sets and Systems, 115, pp.485-489, 2002.

11. George, A. and Veeramani, P., “On Some Results of Analysis for Fuzzy Metric Spaces”, Fuzzy Sets and System, 90, pp.365-368, 1997.

12. Mary, M. G., “Further Results About Fuzzy Metric Spaces”, M. Sc. Thesis, College of Education, Al-Mustansiriah University, 2004.

13. John, D., “Introduction to Real Analysis”, John Wiley and Sons, 1988.