Right n-derivations in prime near – rings

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Abstract

In this paper, we introduce the notions of right derivations and right n-derivations in near-ring N and investigate prime near – rings with right n-derivations satisfying certain differential identities.

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Introduction

A right near – ring (resp. left near ring) is a set N together with two binary operations (+) and (.) such that (i) (N,+) is a group (not necessarily abelian). (ii) (N,.) is a semi group. (iii) For all a,b,c ∈ N ; we have (a + b).c = a.c + b.c (resp. a.(b + c) = a.b + b.c . Through this paper, N will be a zero symmetric left near – ring (i.e., a left near-ring N satisfying the property 0.x = 0 for all x ∈ N). We will denote to the product of any two elements x and y in N ,i.e.; x.y by xy . The symbol Z will denote the multiplicative centre of N, that is Z = {x ∈ N | xy = yx for all y ∈ N} . Let σ and τ be two endomorphisms, for any x, y ∈ N the symbol [x, y] = xy - yx and (x , y) = x + y – x - y stand for multiplicative commutator and additive commutator of x and y respectively. while the symbol $[x, y]_{\sigma,\tau}$ will denote xσ(y) - τ(y)x. N is called a prime near-ring if xNy = {0} implies either x = 0 or y = 0. For terminologies concerning near-rings ,we refer to Pilz **[1]**.

An additive mapping d :N ⟶N is said to be a derivation if d (xy) = d (x)y + x d(y), (or equivalently d(xy) = x d(y) + d(x)y for all x, y ϵ N, as noted in **[2]**.

The concept of right derivation has been already introduced in a ring R by S. Ali **[3]**. An additive mapping d :R ⟶R is said to be right derivation if d(xy) = d(x)y + d(y)x, for all x, y∈ R. In this work and by the same way as in the classical ring theory, we define the concept of right derivation in near-rings and we explore the commutativity of addition and multiplication of near-rings satisfying certain identities involving right derivation on prime near-rings. The concept of derivation has been generalized in several ways by various authors. Mohammad Ashraf and Mohammad Aslam Siddeeque defined the notions of n-derivation, (σ,τ)-n-derivation and generalized n-derivation in near-ring in **[4]**, **[5]** and **[6]** respectively and examined some properties of these derivations. In this work, motivated by these concepts we define the concept of right n-derivation. Also we investigate the commutativity of addition and multiplication of near-rings satisfying certain identities involving right n-derivation.

Definition 1.1 Let N be a near-ring. An additive mapping d :N ⟶N is said to be right derivation of N if d(xy) = d(x)y + d(y)x, for all x, yϵ N.

Definition 1.2 Let N be a near-ring. Let n be a fixed positive integer. An n-additive(i.e.; additive in each argument) mapping

$d:\underbrace{N \times N \times \ldots \times N}_{n-\text{times}} \longrightarrow N$ is said to be right n-derivation if the relations

$$d(x_1\, x_1', x_2, \ldots, x_n) = d(x_1\, , x_2, \ldots, x_n)x_1' + d(x_1', x_2, \ldots, x_n)x_1$$

$$(x_1\, , x_2x_2', \ldots, x_n) = d(x_1\, , x_2, \ldots, x_n)x_2' + d(x_1\, , x_2', \ldots, x_n)x_2$$

$$\vdots$$

$$d(x_1\, , x_2, \ldots, x_nx_n') = d(x_1\, , x_2, \ldots, x_n)x_n' + d(x_1\, , x_2, \ldots, x_n')\, x_n$$

hold for all $x_1, x_1', x_2, x_2', \ldots, x_n, x_n' \in N$.

Example 1.3 Let S be a 2-torsion free zero-symmetric left near-ring. Let us define :

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x, y, 0 \in S \right\}$$

It is clear that N is a zero symmetric near-ring with respect to matrix addition and matrix multiplication.

Now we define $d : N \to N$ by

$$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Define $d_1 : \underbrace{N \times N \times \ldots \times N}_{n-\text{times}} \longrightarrow N$ by

$$d_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1x_2\ldots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It can be easily seen that d is a nonzero right derivation of near-ring N and d_1 is a nonzero right n-derivations of near-ring N.

2. Preliminary results

We begin with the following lemmas which are essential for developing the proofs of our main results.

Lemma 2.1 [7] Let N be a near-ring . If there exists a non-zero element z of Z such that $z + z \in Z$, then (N , +) is an abelian group.

Lemma 2.2 [8] Let N be a prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.

Lemma 2.3 [8] Let N be a prime near-ring and Z contains a nonzero semigroup left ideal or nonzero semigroup left ideal, then N is a commutative ring.

Lemma 2.4.[9] Let N be a prime near-ring, d a nonzero n-derivation of N and $x \in N$.

(i) If $d(N, N, \ldots, N)x = \{0\}$, then $x = 0$.

(ii) If $xd(N, N, \ldots, N) = \{0\}$, then $x = 0$.

Lemma 2.5. Let N be a prime near-ring, d is a nonzero right n-derivation of N and $a \in N$. If $d(N, N, \ldots, N)a = \{0\}$, then $a = 0$.

Proof . Given that

$d(N, N, \ldots, N)a = \{0\}$,i.e.;

$d(x_1, x_2, \ldots, x_n)a = 0,$

$$\text{for all } x_1, x_2, \ldots, x_n \in N \quad (1)$$

Putting x_1a in place of x_1 in relation (1), we get

$$0 = d(x_1a, x_2, \ldots, x_n)a$$

$$= (d(x_1, x_2, \ldots, x_n)a + d(a, x_2, \ldots, x_n)\, x_1)a$$

$$= d(a, x_2, \ldots, x_n)\, x_1a$$

So we get $d(a, x_2, \ldots, x_n)\, Na = \{0\}$ for all $x_2, \ldots, x_n \in N$. Primeness of N implies that

either $a = 0$ or $d(a, x_2, \ldots, x_n) = 0$ for all $x_2, \ldots, x_n \in N$.

If $d(a, x_2, \ldots, x_n) = 0$

for all $x_2, \ldots, x_n \in N$ (2)

Since $d(x(ay), x_2, \ldots, x_n) = d((xa)y, x_2, \ldots, x_n)$ for all $x, y, x_2, \ldots, x_n \in N$, we get

$d(x, x_2, \ldots, x_n)ay + d(ay, x_2, \ldots, x_n)x = d(xa, x_2, \ldots, x_n)y + d(y, x_2, \ldots, x_n)xa$, i.e.;

$d(x, x_2, \ldots, x_n)ay + (d(a, x_2, \ldots, x_n)y + d(y, x_2, \ldots, x_n)a)\, x = (d(x, x_2, \ldots, x_n)a + d(a, x_2, \ldots, x_n)x)y + d(y, x_2, \ldots, x_n)xa$

Using relations (1) and (2) in previous relation we get $d(y, x_2, \ldots, x_n)Na = \{0\}$ for all $y, x_2, \ldots, x_n \in N$. Since $d \neq 0$, primeness of N implies that $a = 0$.

As a result of Lemma 2.5 we can prove the following Lemma:

Lemma 2.6 Let N be a prime near-ring, d is a nonzero right derivation of N and $a \in N$. If $d(N)a = \{0\}$, then $a = 0$.

3.Main Results

Theorem 3.1 Let N be a prime near-ring and d be a nonzero right n-derivation of N. If $d(N,N,\ldots,N) \subseteq Z$, then N is a commutative ring.

Proof. Since $d(N,N,\ldots,N) \subseteq Z$ and d is a nonzero right n-derivation, there exist nonzero elements $x_1, x_2, \ldots, x_n \in N$, such that $d(x_1, x_2, \ldots, x_n) \in Z\backslash\{0\}$. We have

$d(x_1 + x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n) \in Z$. By Lemma 2.1 we obtain that $(N , +)$ is an abelain group.

By hypothesis we get

$d(y_1, y_2, \ldots, y_n)y = yd(y_1, y_2, \ldots, y_n)$ for all $y, y_1, y_2, \ldots, y_n \in N$. (3)

Now replacing y_1 by $y_1 y_1'$ where $y_1' \in N$ in (3) we have

$(d(y_1, y_2, \ldots, y_n)\, y_1' + d(y_1', y_2, \ldots, y_n)y_1)y$

$= y(d(y_1, y_2, \ldots, y_n)\, y_1' +$

$d(y_1', y_2, \ldots, y_n)y_1)$

for all $y, y_1, y_1', y_2, \ldots, y_n \in N$. (4)

By definition of d we get. for all $y_1, y_1', y_2, \ldots, y_n \in N$ that

$d(y_1 y_1', y_2, \ldots, y_n) = d(y_1, y_2, \ldots, y_n)y_1' +$

$d(y_1', y_2, \ldots, y_n)y_1$ (5)

and

$d(y_1' y_1, y_2, \ldots, y_n) = d(y_1', y_2, \ldots, y_n)y_1 +$

$d(y_1, y_2, \ldots, y_n)y_1'$ (6)

Since(N, +) is an abelaian group, from (5) and (6) we conclude that

$d(y_1 y_1', y_2, \ldots, y_n) = d(y_1' y_1, y_2, \ldots, y_n)$ for all $y_1, y_1', y_2, \ldots, y_n \in N$.

So we get

$d([y_1 , y_1'], y_2, \ldots, y_n) = 0$

for all $y_1, y_1', y_2, \ldots, y_n \in N$. (7)

Replacing y_1' by $y_1 y_1'$ in (7) we get

$0 = d([y_1 , y_1 y_1'], y_2, \ldots, y_n)$

$= d(y_1[y_1 , y_1'], y_2, \ldots, y_n)$

$= d(y_1, y_2, \ldots, y_n)[y_1 , y_1'] + d([y_1, y_1'], y_2, \ldots, y_n)y_1$

$= d(y_1, y_2, \ldots, y_n)[y_1, y_1']$

we conclude that $d(y_1, y_2, \ldots, y_n)N[y_1 , y_1'] = \{0\}$ for all $y_1, y_1', y_2, \ldots, y_n \in N$. Primeness of N implies that for each $y_1 \in N$ either $d(y_1, y_2, \ldots, y_n) = 0$ for all $y_2, \ldots,$

$y_n \in N$ or $y_1 \in Z$. If $d(y_1,y_2,...,y_n) = 0$, then equation (4) takes the form $d(y_1',y_2,...,y_n)N[y, y_1] = \{0\}$. Since $d \neq 0$, primeness of N implies that $y_1 \in Z$. Hence we find that N = Z, and N is a commutative ring.

Corollary 3.2 Let N be a prime near-ring and d is a nonzero right derivation of N. If $d(N) \subseteq Z$, then N is a commutative ring.

Theorem 3.3 Let N be a prime near-ring and d_1 and d_2 be any two nonzero right n-derivations. If $[d_1(N,N,...,N) , d_2(N,N,...,N)] = \{0\}$ then (N,+) is an abelian group.

Proof. Assume that $[d_1(N,N,...,N) , d_2(N,N,...,N)] = \{0\}$. If both z and z + z commute element wise with $d_2(N, N, ..., N)$, then for all $x_1,x_2,...,x_n \in N$ we have

$$zd_2(x_1,x_2,...,x_n) = d_2(x_1,x_2,...,x_n)z \quad (8)$$

and

$$(z + z)d_2(x_1,x_2,...,x_n) = d_2(x_1,x_2,...,x_n)(z + z) \quad (9)$$

Substituting $x_1 + x_1'$ instead of x_1 in (9) we get

$(z + z)d_2(x_1 + x_1',x_2,...,x_n) = d_2(x_1 + x_1',x_2,...,x_n)(z + z)$ for all $x_1,x_1',x_2,\ldots,x_n \in N$.

From (8) and (9) the previous equation can be reduced to

$zd_2(x_1 + x_1' - x_1 - x_1',x_2,...,x_n) = 0$ for all $x_1,x_1',x_2,\ldots,x_n \in N$., i.e.;

$zd_2((x_1 , x_1'), x_2, ..., x_n) = 0$ for all $x_1, x_1', x_2, ..., x_n \in N$.

Putting $z = d_1(y_1, y_2, ..., y_n)$ we get

$d_1(y_1, y_2, ..., y_n)d_2((x_1 , x_1'), x_2, ..., x_n) = 0$ for all $x_1, x_1', x_2, ..., x_n , y_1 , y_2, ..., y_n \in N$.

By Lemma 2.5 we conclude that

$$d_2((x_1 , x_1'), x_2, ..., x_n) = 0 \text{ for all } x_1, x_1', x_2, ..., x_n \in N \quad (10)$$

Since we know that for each $w \in N$,

$w(x_1 , x_1') = w(x_1 + x_1' - x_1 - x_1')$

$= wx_1 + wx_1' - wx_1 - wx_1' = (wx_1 , wx_1')$

Which is again an additive commutator, putting $w(x_1 , x_1')$ instead of (x_1 , x_1') in (10) we get

$d_2(w(x_1 , x_1'), x_2, ..., x_n) = 0$ for all $w, x_1, x_1', x_2, ..., x_n \in N$. i.e.;

$d_2(w, x_2, ..., x_n)(x_1 , x_1') + d_2((x_1 , x_1'), x_2, ..., x_n)w = 0$, using (10) in previous equation yields $d_2(w, x_2, ..., x_n)(x_1 , x_1') = 0$. Using Lemma 2.5 we conclude that $(x_1 , x_1') = 0$. Hence (N, +) is an abelain group.

Corollary 3.4 Let N be a prime near-ring and d_1 and d_2 be any two nonzero right derivations. If $[d_1(N),d_2(N)] = \{0\}$ then (N,+) is an abelian group.

Theorem 3.5 Let N be a prime near-ring and d_1 and d_2 be any two nonzero right n-derivations. If $d_1(x_1,x_2,...,x_n)d_2(y_1,y_2,...,y_n) + d_2(x_1,x_2,...,x_n)d_1(y_1,y_2,...,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$, then (N,+) is an abelian group.

Proof. By our hypothesis we have,

$d_1(x_1,x_2,...,x_n)d_2(y_1,y_2,...,y_n) + d_2(x_1,x_2,...,x_n)d_1(y_1,y_2,...,y_n) = 0$

for all $x_1 ,x_2,\ldots,x_n, y_1 ,y_2,\ldots,y_n \in N$. (11)

Substituting $y_1 + y_1'$ instead of y_1 in (11) we get

$$d_1(x_1,x_2,...,x_n)d_2(y_1 + y_1',y_2,...,y_n) + d_2(x_1,x_2,...,x_n)d_1(y_1 + y_1',y_2,...,y_n) = 0$$

for all $x_1,x_2,\ldots,x_n,y_1 ,y_1',y_2,\ldots,y_n \in N$.

Therefore

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(y_1',y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1',y_2,\ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n, y_1,y_1',y_2,\ldots,y_n \in N$.

using (11) again in preceding equation we get

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(y_1',y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(-y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(-y_1',y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Which means that

$d_1(x_1, x_2, \ldots, x_n)\, d_2((y_1, y_1'),y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$. By Lemma 2.5 we obtain

$d_2((y_1,y_1'),y_2,\ldots,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$.

Now putting $w(y_1,y_1')$ instead of (y_1,y_1'), where $w \in N$ in previous equation we get $d_2(w(y_1,y_1'),y_2,\ldots,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$. So we have $d_2(w, y_2,\ldots, y_n)(y_1, y_1') = 0$, using Lemma 2.5; as used in the Theorem 3.3 we conclude that (N, +) is abelain.

Corollary 3.6 Let N be a prime near-ring and d_1 and d_2 be any two nonzero right derivations. If $d_1(x)d_2(y) + d_2(x)d_1(y) = 0$ for all $x, y \in N$, then (N,+) is an abelian group.

Theorem 3.7 Let N be a prime near-ring, let d_1 be a nonzero right n-derivation and d_2 be a nonzero n-derivation.

(i) If $d_1(x_1,x_2,\ldots,x_n)d_2(y_1, y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$, then (N,+) is an abelian group.

(ii) If $d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)\, d_2(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$, then (N,+) is an abelian group.

Proof. (i) By our hypothesis we have,

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$. (12)

Substituting $y_1 + y_1'$, where $y_1' \in N$, for y_1 in (12) we get

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1 + y_1',y_2,\ldots,y_n) +$

$d_2(x_1,x_2,\ldots,x_n)d_1(y_1 + y_1',y_2,\ldots,y_n) = 0$

for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

So we have

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(y_1',y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1,y_2,\ldots,y_n) + d_2(x_1,x_2,\ldots,x_n)d_1(y_1',y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Using (12) in previous equation implies

$d_1(x_1,x_2,\ldots,x_n)d_2(y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(y_1',y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(-y_1,y_2,\ldots,y_n) + d_1(x_1,x_2,\ldots,x_n)d_2(-y_1',y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1, y_1',y_2,\ldots,y_n \in N$.

Which means that

$d_1(x_1, x_2, \ldots, x_n)d_2((y_1,y_1'),y_2,\ldots,y_n) = 0$ for all $x_1,x_2,\ldots,x_n,y_1,y_1',y_2,\ldots,y_n \in N$.

Now using Lemma 2.5 we conclude that $d_2((y_1, y_1'),y_2,\ldots,y_n) = 0$ for all $y_1,y_1',y_2,\ldots,y_n \in N$.

Now putting $w(y_1, y_1')$ instead of (y_1, y_1'), where $w \in N$ in previous equation we get $d_2(w(y_1, y_1'),y_2,\ldots,y_n) = 0$ for all $w,y_1,y_1',y_2,\ldots,y_n \in N$, so we have $d_2(w, y_2,\ldots, y_n)(y_1, y_1') = 0$, using Lemma 2.4(i); we conclude that $(y_1, y_1') = 0$ for all $y_1,y_1' \in N$. Thus (N, +) is an abelain group.

(ii)If N satisfies

$d_2(x_1,x_2,\ldots,x_n)\, d_1(y_1,y_2,\ldots,y_n) +$

$$d_1(x_1,x_2,\ldots,x_n)\, d_2(y_1,y_2,\ldots,y_n) = 0$$

for all $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in N$, then again using the same arguments as in (i) we get the required result.

Corollary 3.8 Let N be a prime near-ring, let d_1 be a nonzero right derivation and d_2 be a nonzero derivation.

(i) If $d_1(x)d_2(y) + d_2(x)d_1(y) = 0$ for all $x,y \in N$, then $(N,+)$ is an abelian group.

(ii) If $d_2(x)d_1(y) + d_1(x)d_2(y) = 0$ for all $x,y \in N$, then $(N,+)$ is an abelian group.

Theorem 3.9 Let N be a prime near-ring, then N admit no nonzero right n-derivation d such that

$x_1 d(y_1,y_2,\ldots,y_n) = d(x_1,x_2,\ldots,x_n)y_1$ for all $x_1,x_2,\ldots,x_n, y_1,y_2,\ldots,y_n \in N$.

Proof. Assume that there is a nonzero right n-derivation d of N such that,

$$x_1 d(y_1, y_2, \ldots, y_n) = d(x_1, x_2, \ldots, x_n)y_1$$

for all $x_1 ,x_2,\ldots,x_n, y_1 ,y_2,\ldots,y_n \in N$. (13)

Substituting y_1z_1, where $z_1 \in N$, for y_1 in (13), we get

$$x_1 d(y_1z_1,y_2,\ldots,y_n) = d(x_1,x_2,\ldots,x_n)y_1z_1$$

$x_1 d(y_1,y_2,\ldots,y_n)z_1 + x_1 d(z_1,y_2,\ldots,y_n)y_1 = d(x_1,x_2,\ldots,x_n)y_1z_1$, using (13) in previous equation we get

$x_1 d(z_1, y_2, \ldots, y_n)y_1 = 0$ for all $x_1,z_1,y_1,y_2,\ldots,y_n \in N$, primeness of N implies that $d(z_1, y_2, \ldots, y_n)y_1 = 0$ for all $z_1,y_1,y_2,\ldots,y_n \in N$. Hence we get

$d(z_1,y_2,\ldots, y_n)y_1 d(z_1, y_2, \ldots, y_n) = 0$ for all $z_1,y_1,y_2,\ldots,y_n \in N$, primeness of N implies that $d = 0$. which contradicts our original assumption that $d \neq 0$.

Corollary 3.10 Let N be a prime near-ring, then N admit no nonzero right derivation d such that $xd(y) = d(x)y$ for all $x,y \in N$.

Theorem 3.11 Let N be a prime near-ring, let d be a nonzero right n-derivation d. If $d([x , y],x_2,\ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$, then N is a commutative ring.

Proof. By hypothesis, we have

$d([x , y],x_2,\ldots,x_n) = 0$

for all $x,y,x_2,\ldots,x_n \in N$. (14)

Replace y by xy in (14) to get

$d([x, xy],x_2,\ldots,x_n) = 0$

for all $x,y,x_2,\ldots,x_n \in N$.

Which implies that

$d(x[x, y],x_2,\ldots,x_n) = 0$

for all $x,y,x_2,\ldots,x_n \in N$.

Therefore

$d(x,x_2,\ldots,x_n)[x , y] + d([x, y],x_2,\ldots,x_n)x = 0$ for all $x,y,x_2,\ldots,x_n \in N$. Using (14) in previous equation we get

$d(x,x_2,\ldots,x_n)[x, y] = 0$ for all $x,y,x_2,\ldots,x_n \in N$, or equivalently

$d(x, x_2, \ldots, x_n)xy = d(x, x_2, \ldots, x_n)yx$

for all $x, y, x_2,\ldots,x_n \in N$. (15)

Replacing y by yz in (15) and using it again, we get

$d(x,x_2,\ldots,x_n)y[x , z] = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Hence we get

$d(x,x_2,\ldots,x_n)N[x, z] = \{0\}$

for all $x,z,x_2,\ldots,x_n \in N$ (16)

This yields that

for each fixed $x \in N$

either $d(x,x_2, \ldots,x_n) = 0$

for all $x_2,\ldots,x_n \in N$ or $x \in Z$. (17)

If $d(x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ and for each fixed $x \in N$. We get $d = 0$, leading to a contradiction as d is a nonzero right n-derivation of N. Therefore there exist $x_1,x_2,\ldots,x_n \in N$, all being nonzero such that $d(x_1,x_2,\cdots,x_n) \neq 0$ such that $x_1 \in Z$. Since $x_1 \in Z$, we conclude that $[x_1y,z] = x_1[y, z]$, where $y,z \in N$, by hypothesis we get

$d([x_1y, z] , x_2, \ldots, x_n) = 0$. This implies that

$$0 = d(x_1[y, z],x_2, \ldots,x_n)$$

$$= d(x_1,x_2, \ldots,x_n)[y, z] + d([y, z],x_2, \ldots,x_n)x_1$$

$$= d(x_1,x_2, \ldots,x_n)[y, z]$$

for all $y,z,x_2,\ldots,x_n \in N$.

which implies that

$d(x_1, x_2, \ldots, x_n)yz = d(x_1, x_2, \ldots, x_n)zy$ for all $y,z,x_2,\ldots,x_n \in N$. Replacing z by zt, where $t \in N$, in previous equation, and using it again, we get

$$d(x_1, x_2, \ldots, x_n)z[y, t] = 0$$

for all $y,t,z,x_2,\ldots,x_n \in N$, i.e.;

$d(x_1,x_2, \ldots,x_n)N[y, t] = \{0\}$ for all $y,t,x_2,\ldots,x_n \in N$, since $d(x_1,x_2,\cdots,x_n) \neq 0$, primeness of N implies that $N = Z$. By Lemma 2.3, we conclude that N is a commutative ring.

Corollary 3.12 Let N be a prime near-ring and let d be a nonzero right derivation d. If $d[x , y] = 0$ for all $x, y \in N$, then N is a commutative ring.

Theorem 3.13 Let N be a 2-torsion free prime near-ring. Then N admits no nonzero right n-derivation d such that $d(xoy, x_2, \ldots, x_n) = 0$ for all $x, y, x_2,\ldots,x_n \in N$.

Proof. Assume that

$d(xoy,x_2, \ldots,x_n) = 0$

for all $x,y,x_2,\ldots,x_n \in N$. (18)

Replace y by xy in (18) to get

$d(xoxy,x_2, \ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

which implies that $d(x(xoy),x_2, \ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

$d(x,x_2, \ldots,x_n)(xoy) + d(xoy,x_2, \ldots,x_n)x = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

Using (18) in previous equation we get

$d(x, x_2, \ldots, x_n)(xoy) = 0$

for all $x, y, x_2,\ldots,x_n \in N$, or equivalently,

$d(x,x_2, \ldots,x_n)yx = - d(x,x_2, \ldots,x_n)xy$

for all $x,y,x_2,\ldots,x_n \in N$. (19)

Replacing y by yz, where $z \in N$, in (19) we get

$$d(x,x_2, \ldots,x_n)yzx = - d(x,x_2, \ldots,x_n)xyz$$

$$= d(x,x_2, \ldots,x_n)xy(- z)$$

$$= d(x, x_2, \ldots, x_n)y(- x)(- z)$$

for all $x,y,z,x_2,\ldots,x_n \in N$.

Using the fact

$- d(x,x_2, \ldots,x_n)yzx = d(x,x_2, \ldots,x_n)yz(- x)$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Hence we get

$d(x, x_2, \ldots, x_n)yz(- x) - d(x, x_2, \ldots, x_n)y(- x)z = 0$

for all $x,y,z,x_2,\ldots,x_n \in N$, which implies that

$d(x,x_2, \ldots,x_n)y[-x, z] = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Replacing x by $-x$ in previous equation we get

$d(-x,x_2, \ldots,x_n)y[x, z] = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Hence we get

$d(-x,x_2, \ldots,x_n)N[x, z] = \{0\}$

for all $x,z,x_2,\ldots,x_n \in N$. (20)

This yields that

for each fixed $x \in N$ either $d(-x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ or $x \in Z$.

Since $d(x, x_2, \ldots, x_n) = -d(-x, x_2, \ldots, x_n) = 0$, so we get

for each fixed $x \in N$ either $d(x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ or $x \in Z$.

If $d(x, x_2, \ldots, x_n) = 0$ for all $x_2,\ldots,x_n \in N$ and for each fixed $x \in N$, we get $d = 0$, leading to a contradiction as d is a nonzero right n-derivation of N. Therefore there exist $x_1,x_2,\ldots,x_n \in N$, all being nonzero such that $d(x_1,x_2,\cdots,x_n) \neq 0$ and $x_1 \in Z$. Since $x_1 \in Z$, we conclude that $(x_1 yoz) = x_1(yo z)$, where $y,z \in N$. By hypothesis we get

$d(x_1 yoz,x_2, \ldots,x_n) = 0$

for all $x_1,y,z,x_2,\ldots,x_n \in N$.

Therefore

$$0 = d(x_1(yoz),x_2, \ldots,x_n)$$

$$= d(x_1,x_2, \ldots,x_n)(yoz) + d(yoz,x_2, \ldots,x_n)x_1$$

$$= d(x_1,x_2, \ldots,x_n)(yoz)$$

for all $y, z, x_2,\ldots,x_n \in N$.

which implies that

$d(x_1,x_2, \ldots,x_n)yz = -d(x_1,x_2, \ldots,x_n)zy$ for all $y,z,x_2,\ldots,x_n \in N$. Replacing z by zt, where $t \in N$, in previous equation and using it again, we get $d(x_1, x_2, \ldots, x_n)z[y, t] = 0$ for all $x_1,x_2,\ldots,x_n,y,z,t \in N$. i.e.; $d(x_1, x_2, \ldots, x_n)N[y, t] = \{0\}$ for all $x_1,x_2,\ldots,x_n,y,z,t \in N$. Since $d(x_1,x_2,\cdots,x_n) \neq 0$, primeness of N implies that $N = Z$. So we conclude that N is a commutative ring in view of Lemma 2.3. In this case, return to hypothesis we find that $2d(xy,x_2,\ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$. Since N is 2-torsion free we get $d(xy,x_2,\ldots,x_n) = 0$ for all $x,y,x_2,\ldots,x_n \in N$.

Hence we get

$d(x,x_2,\ldots,x_n)y + d(y,x_2,\ldots,x_n)x = 0$ for all $x,y,x_2,\ldots,x_n \in N$, replacing x by zx, where $z \in N$, in previous equation yields

$d(zx,x_2,\ldots,x_n)y + d(y,x_2,\ldots,x_n)zx = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Which implies that

$d(y,x_2,\ldots,x_n)zx = 0$ for all $x,y,z,x_2,\ldots,x_n \in N$.

Which means that

$d(y,x_2,\ldots,x_n)Nx = \{0\}$ for all $x,y,x_2,\ldots,x_n \in N$. Since N is prime and $d \neq 0$, we conclude that $x = 0$ for all $x \in N$, a contradiction.

Corollary 3.14 Let N be a 2-torsion free prime near-ring. Then N admits no nonzero right derivation d such that $d(xoy) = 0$ for all $x,y \in N$.

Theorem 3.15 Let N be a prime near-ring, let d be a right n-derivation. If $[d(x,x_2, \ldots,x_n), y] \in Z$ for all $x,y,x_2,\ldots,x_n \in N$, then N is a commutative ring.

Proof. Assume that

$[d(x, x_2, \ldots, x_n), y] \in Z$

for all $x,y,x_2,\ldots,x_n \in N$. (21)

Hence $[[d(x, x_2, \ldots, x_n), y], t] = 0$

for all $x,y,t, x_2,\ldots,x_n \in N$. (22)

Replacing y by $d(x,x_2, \ldots,x_n)y$ in (22), we get

$[d(x,x_2, \ldots,x_n)[d(x,x_2, \ldots,x_n), y], t] = 0$

for all $x,y,t,x_2,\ldots,x_n \in N$. (23)

In view of (21), equation (23) assures that

$[d(x,x_2, \ldots,x_n), y]\ N\ [d(x,x_2, \ldots,x_n), t] = \{0\}$ for all $x,y,t,x_2,\ldots,x_n \in N$. (24)

Primeness of N implies that

$[d(x, x_2, \ldots, x_n), y] = 0$ for all $x, y, x_2,\ldots,x_n \in N$.

Hence $d(N,N,\ldots,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring.

Corollary 3.16 Let N be a prime near-ring, let d be a right derivation of N. If $[d(x), y] \in Z$ for all $x, y \in N$, then N is a commutative ring.

Theorem 3.17 Let N be a prime near-ring, let d be a right n-derivation. If $d(x,x_2, \ldots,x_n)oy \in Z$ for all $x,y,x_2,\ldots,x_n \in N$, then N is a commutative ring.

Proof. Assume that

$d(x,x_2, \ldots,x_n)oy \in Z$

for all $x,y,x_2,\ldots,x_n \in N$. (25)

(a) If Z = 0, then equation (25) reduces to

$yd(x,x_2, \ldots,x_n) = -(d(x,x_2, \ldots,x_n)y) = d(x,x_2, \ldots,x_n)(-y)$

for all $x,y,x_2,\ldots,x_n \in N$. (26)

Substituting zy for y in (26), we obtain

$zyd(x,x_2, \ldots,x_n) = -(d(x, x_2, \ldots, x_n)zy)$

$= d(x, x_2, \ldots, x_n)z(-y)$

$= zd(-x, x_2, \ldots, x_n))(-y)$

for all $x,y,z,x_2,\ldots,x_n \in N$.

Using the fact

$-zyd(x, x_2, \ldots, x_n) = zyd(-x, x_2, \ldots, x_n)$, implies that

$zyd(-x,x_2, \ldots,x_n) = zd(-x,x_2, \ldots,x_n))y$

for all $x,y,z,x_2,\ldots,x_n \in N$.

Which implies that

$z(yd(-x,x_2, \ldots,x_n) - d(-x,x_2, \ldots,x_n)y) = 0$

for all $x,y,z,x_2,\ldots,x_n \in N$. (27)

Taking -x instead of x in (27), we get

$zN(yd(x, x_2, \ldots, x_n) - d(x, x_2, \ldots, x_n)y) = \{0\}$ for all $x, y, z,x_2,\ldots,x_n \in N$. (28)

Primeness of N implies that

$d(N,N,\ldots,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring.

(b) Suppose that $Z \neq 0$, then there exists $0 \neq z \in Z$ and by hypothesis we have $d(x,x_2, \ldots,x_n)oz \in Z$ for all $x,x_2,\ldots,x_n \in N$.

Thus we get $d(x,x_2, \ldots,x_n)z + zd(x,x_2, \ldots,x_n) \in Z$ for all $x,x_2,\ldots,x_n \in N$. Since $z \in Z$, we get

$z(d(x, x_2, \ldots, x_n) + d(x, x_2, \ldots, x_n)) \in Z$ for all $x,x_2,\ldots,x_n \in N$.

By Lemma 2.2 we conclude that

$d(x,x_2, \ldots,x_n) + d(x,x_2, \ldots,x_n) \in Z$

for all $x,x_2,\ldots,x_n \in N$. (29)

By (25), we get

$d(x+ x,x_2, \ldots,x_n)y + yd(x + x,x_2, \ldots,x_n) \in Z$ for all $x,x_2,\ldots,x_n \in N$. (30)

Using equation (29) in (30) we conclude that

$y(d(x+x,x_2,...,x_n)+d(x+x,x_2,...,x_n)) \in Z$ for all $x,y,z,x_2,\dots,x_n \in N$. (31)

Therefore, for all $x,y,t,x_2,\dots,x_n \in N$, we get

$ty(d(x+x,x_2,...,x_n)+d(x+x,x_2,...,x_n)) = y(d(x+x,x_2,...,x_n)+d(x+x,x_2,...,x_n))t = (d(x+x,x_2,...,x_n)+d(x+x,x_2,...,x_n))yt$.

Which implies that

$$(d(x+x,x_2,...,x_n)+d(x+x,x_2,...,x_n))N[t,y]=\{0\}$$

for all $x, y, t, x_2,\dots,x_n \in N$. (32)

Primeness of N implies that

either $d(x+x, x_2, ..., x_n) + d(x+x, x_2, ..., x_n) = 0$ and thus $d = 0$, a contradiction, or $N = Z$, hence $d(N,N,...,N) \subseteq Z$ and application of Theorem 3.1 assures that N is a commutative ring.

Corollary 3.18 Let N be a prime near-ring, let d be right derivation. If $d(x)oy \in Z$ for all $x, y \in N$, then N is a commutative ring.

The following example demonstrates that N to be prime is essential in the hypothesis of our results

Example 3.19 Let S be a 2-torsion free zero-symmetric left near-ring. Let us define :

$$N=\left\{\begin{pmatrix}0&x&y\\0&0&0\\0&0&0\end{pmatrix}, x,y,0\in S\right\}.$$ Thus is a zero symmetric left near-ring with regard to matrix addition and matrix multiplication .

Define $d_1, d_2 : \underbrace{N\times N\times\ldots\times N}_{n-times} \longrightarrow N$ such that

$$d_1\left(\begin{pmatrix}0&x_1&y_1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&0\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&x_1x_2\dots x_n&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$d_2\left(\begin{pmatrix}0&x_1&y_1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&x_2&y_2\\0&0&0\\0&0&0\end{pmatrix},\dots,\begin{pmatrix}0&x_n&y_n\\0&0&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&y_1y_2\dots y_n\\0&0&0\\0&0&0\end{pmatrix}$$

It can be easily seen that d_1, d_2 are nonzero right n-derivations of near-ring N which is not prime. We also have

(i) Let $A \in N$, $A=\begin{pmatrix}0&x&y\\0&0&0\\0&0&0\end{pmatrix}$ such that $x, y \neq 0$, we can see that $d_1(N,N,...,N)A = 0$. However $A \neq 0$.

(ii) $d_1(N,N,...,N) \subseteq Z$. However (N,+) is not abelian group.

(iii) $[d_1(N, N, ..., N), d_2(N, N, ..., N)] = \{0\}$,

(iv) $d_1(A_1,A_2,...,A_n)d_2(B_1,B_2,...,B_n) + d_2(A_1,A_2,...,A_n)d_1(B_1,B_2,...,B_n)$ for all $A_1,A_2,\dots,A_n, B_1,B_2,\dots,B_n \in N$.

(v) $A_1d_1(B_1,B_2,...,B_n) = d_1(A_1,A_2,...,A_n)B_1$ for all $A_1,A_2,\dots,A_n,B_1,B_2,\dots,B_n \in N$.

(vi) $d_1([A, B],A_2,...,A_n) = 0$ for all $A,B,A_2,\dots,A_n \in N$.

(vii) $d_1(AoB,A_2,...,A_n) = 0$ for all $A,B,A_2,\dots,A_n \in N$. However (N , +) is not abelain group.

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الاشتقاقات-n اليمنى على الحلقات المقتربة الأولية

تاريخ الاستلام 2015/9/27 **تاريخ القبول 2015/12/13**

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الخلاصة : في هذه الورقة قدمنا تعريفا للاشتقاقات اليمنى والاشتقاقات-n اليمنى على الحلقة المقتربة الأولية N وبحثنا في الحلقات المقتربة الأولية والتي تحقق بعض الفرضيات على الاشتقاقات-n اليمنى

الكلمات المفتاحية : الحلقة المقتربة الأولية، الاشتقاقات، الاشتقاقات اليمنى، الاشتقاقات-n اليمنى.