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Strongly t-continuous and Strongly t-semisimple Modules

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Abstract

We introduce and investigate strongly t-continuous modules. A module M is called strongly t-continuous if M is strongly t-extending, and every submodule of M which contains $Z_2(M)$ and is isomorphic to a direct summand is a fully invariant direct summand of M. It is shown that, while a direct summand of strongly t-continuous inherits the property, a direct sum of strongly t-continuous modules don't. M is strongly t-continuous if and only if M is strongly t-extending and the endomorphism ring of $\frac{M}{Z_2(M)}$ is Von Neumann regular, if and only if $M = Z_2(M)\oplus M'$, where M' is a strongly t-continuous module. We have shown that strongly t-continuous module and t-continuous module are coinciding under certain conditions. Many other properties and example are given.

KEYWORDS: STRONGLY CONTINUOUS MODULES; T-CONTINUOUS MODULES; STRONGLY T-EXTENDING MODULES AND STRONGLY T-CONTINUOUS.

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1. Introduction

Through this paper all rings are associative with unity and all modules are unitary right modules. We recall some relevant notions and results. A submodule N of an R-module M is essential in M (briefly $N \leq_{ess} M$) if $N \cap W = (0), W \leq M$ implies $W = (0)$[Goodearl ,1976] . A submodule N of M is called closed in M (briefly $N \leq_c M$) if N has no proper essential extension in M, that is if $N \leq_{ess} W \leq M$, then $N = W$[Gangyong ,2010]. The set $\{x \in M: xI = (0)$ for some essential ideal I of $R\}$ is called the singular submodule of M and denoted by $Z(M)$[Goodearl,1976]. Equivalently $Z(M) = \{x \in M: ann(x) \leq_{ess} R\}$ and $ann(x) = \{r \in M: xr = 0\}$. M is called singular (nonsingular) if $Z(M) = M(Z(M) = 0)$. The second singular submodule of M (denoted by $Z_2(M)$) is defined by $Z\left(M/_{Z(M)}\right) = Z_2(M)/_{Z(M)}$ [Goodearl,1976]. As a generalization of essential submodule, Asgari in [Asgari,2011], introduced the notion of t-essential submodule, where a submodule N of M is called t-essential (denoted by $N \leq_{tes} M$) if whenever $B \leq M$, $N \cap B \leq Z_2(M)$ implies $B \leq Z_2(M)$. Also, Asgari in [Asgari ,2011] introduced t-closed submodule, where a submodule N is called t-closed (denoted by $N \leq_{tc} M$) if N has no proper t-essential extension in M[Asgari,2011]. It is known that a module M is called extending (or CS-module or module has C_1-condition) if for every submodule N of M then there exists a direct summand $W(W \leq^{\oplus} M)$ such that $N \leq_{ess} W$ [Dung,1994]. Equivalently M is extending module if and only if every closed submodule is a direct summand.

Saad in [Saad ,2007] defined the following: an R-module is called strongly extending if every submodule is essential in fully invariant direct summand, that is M is strongly extending if and only if every closed submodule is fully invariant direct summand.

Asgari and Haghany in [Asgari ,2011] generalized the concept of extending module in to t-extending if every submodule is t-essential in a direct summand. Equivalently, an R-module M is t-extending if and only if every t-closed is a direct summand.

Elabrhiamy in [Ebrahimi,2016], defined strongly t-extending module, where an R-module is strongly t-extending if every submodule is t-essential in fully invariant direct summand. Equivalently, M is strongly t-extending if and only if every t-closed submodule is fully invariant direct summand.

A submodule N of an R-module M is called fully invariant if for each $f \in \mathrm{End}(M)$, $f(N) \leq N$. A submodule N of M is called stable if for each R-homomorphism $f: N \mapsto M$, $f(N) \leq N$.[Abbas ,1991]. It is clear that every stable submodule is fully invariant, but not conversely. However, every fully invariant direct summand is stable [Patrick,2015]. An R-module M is called duo if every submodule is fully invariant [Özcan ,2006], and M is called fully stable if every submodule is stable [Abbas ,1991].

It is known that an R-module M is called continues if M satisfies the following conditions: (1) C_1-condition M is extending. (2) C_2-condition, for each $N \leq M$, $N \simeq W$ and W is a direct summand of M ($W \leq^{\oplus} M$), then $N \leq^{\oplus} M$.

Saad in [Saad, 2007], introduced and studied strongly continuous module**,** where an R-module M is said to be strongly continuous if it satisfies the following conditions:

(SC_1) Every submodule of M is essential in stable direct summand of M; that is M is strongly extending module which is equivalent to every closed submodule is stable direct summand.

(SC_2) Every submodule of M which is isomorphic to a direct summand of M is stable direct summand.

In 2017, Asgari [Asgari,2017] introduced and investigated t-continuous modules, where an R-module M is called t-continuous if M satisfies the following: (1) M is t-extending, (2) every submodule of M which contains $Z_2(M)$ and is isomorphic to direct summand of M is itself a direct summand. A ring R is called t-continuous if R_R is t-continuous.

It is clear that every Z_2-torsion module is t-continuous module.

The concept of strongly continuous modules and t-continuous modules lead us to introduce strongly t-continuous module, we say that an R-module is strongly t-continuous if it satisfies the following: (1) M is strongly t-extending.

(2) For each submodule N of M containing $Z_2(M)$ if $N \simeq W \leq^{\oplus} M$, then N is stable direct summand.

It is clear that every strongly t-continuous module is t-continuous and every strongly continuous is strongly t-continuous, but the converses are not true.

The properties of strongly continuous and strongly t-continuous modules are coincide in the class of nonsingular modules.

Asgari introduced t-endoregular module where an R-module is called t-endoregular if $M/_{Z_2(M)}$ is endoregular; that is $End(\frac{M}{Z_2(M)})$ is a Von Neumann regular ring[Asgari,2017]. Clearly R is a right t-endoregular if and only if $R/_{R_2(R)}$ is Von Neumann regular ring[Asgari,2017].

Recall that a module M is t-semisimple if for every submodule N of M there exists a direct summand K of M such that $K \leq_{tes} N$[Asgari,2013].

In this paper, we study the strongly t-continuous property for modules. Many results concerned with these concepts are presented. Also, many connections between it and other related concepts are introduced.

In the first we list some results about t-continuous modules from [Asgari,2017].

Theorem (1.1): The following statements are equivalent for a module M.

(1) M is t-continuous.

(2) M is t-extending and t-endoregular.

(3) $M = Z_2(M) \oplus M'$, where M' is a (nonsingular) continuous module.

Theorem (1.2): The following statements are equivalent for a ring R.

(1) Every free (projective) R module is t-continuous.

(2) Every nonsingular R module is continuous.

(3) Every R module is t-continuous.

(4) $R^{(N)}$is t-continuous.

(5) R is right t-semisimple.

Corollary (1.3): The following statements are equivalent for a ring R.

(1) Every free (projective) R module is t-endoregular.

(2) Every nonsingular R module is endoregular.

(3) Every R module is t-endoregular.

(4) $R^{(N)}$ is t-endoregular.

(5) R is right t-semisimple.

Theorem (1.4): The following statements are equivalent for a ring R.

(1) Every nonsingular finitely generated R module is projective and continuous.

(2) Every nonsingular finitely generated R module is projective and injective.

(3) Every finitely generated free (projective) R module is t-continuous.

(4) Every finitely generated R module is t-continuous.

(5) $R = Z_2(R_R) \oplus R'$, where R' is injective.

(6) $R^{(2)}$ is t-continuous.

2. Strongly t-continuous and strongly t-semisimple modules.

In this section, we introduce and investigate the concept of strongly t-continuous module as a proper subclass of the class of t-continuous modules.

Definition (2.1): A module M is called strongly t-continuous if it satisfies the following conditions:

(1) M is strongly t-extending;
(2) For each submodule N of M containing $Z_2(M)$ and N isomorphic to a stable direct summand of M, N is stable direct summand.

Recall that an R-module M is Rickart if $kerf$ is a direct summand of M for all $f \in End(M)$[Gangyong ,2010]".

The following Theorem gives characterizations of strongly t-continuous module.

Theorem (2.2): The following statements are equivalent for a module M.

(1) M is strongly t-continuous;
(2) M is strongly t-extending and M is t-endoregular ;
(3) $M = Z_2(M) \oplus M'$, where M' is nonsingular, strongly continuous module.

Proof (1) $\Rightarrow$(2) By definition of strongly t-continuous module, M is strongly t-extending. Hence $M = Z_2(M) \oplus M'$, where M' is strongly extending (that is every submodule of M is essential stable direct summand [Ebrahimi,2016, Theorem 3.2]. It follows that M' is extending and hence M' is Rickart, since for any $\varphi \in End(M')$, $\varphi(M') \leq M'$, so $\varphi(M')$ is nonsingular. But $M'/_{Ker\varphi} \simeq \varphi(M')$, hence $M'/_{Ker\varphi}$ is nonsingular; that is $Ker\varphi$ is t-closed in M', thus $Ker\varphi$ is closed in M' and since M' is extending, $Ker\varphi \leq^{\oplus} M'$; that is M is Rickart. To show that M' satisfies C_2-condition. Let $A \leq M'$ and $A \simeq D \leq^{\oplus} M'$. Then $Z_2(M) \oplus A \supseteq Z_2(M)$, and $Z_2(M) \oplus A \simeq Z_2(M) \oplus D$. But $D \oplus D' = M'$ for some $D' \leq M'$, so $Z_2(M) \oplus D \oplus D' = M$, so

$Z_2(M)\oplus D \leq^{\oplus} M$. Thus $Z_2(M)\oplus A$ is isomorphic to a direct summand of M, so $Z_2(M)\oplus A \leq^{\oplus} M$. Hence $Z_2(M)\oplus A\oplus B = M$. Now $M' = (Z_2(M)\oplus A\oplus B) \cap M' = A\oplus[(Z_2(M)\oplus B) \cap M']$ by modular law , then $A \leq^{\oplus} M'$. Thus M' has C_2-condition. Thus by [Gangyong, 2010, Theorem 4.1.1] $M/_{Z_2(M)} \simeq M'$ is endoregular.

(2)⇒(3) Since M is strongly t-extending we have $M = Z_2(M)\oplus M'$, M' is nonsingular and strongly extending [Ebrahimi ,2016, Theorem 3.2]. As M is t-endoregular,then $M' \simeq M/_{Z_2(M)}$, is endoregular and so M' has C_2-condition by [Gangyong ,2010, Theorem 4.1.1]. It follows that if $N \simeq W \leq^{\oplus} M'$, then $N \leq^{\oplus} M'$; that is N is closed in M'. But M' is strongly extending, so N is stable direct summand of M'. Thus M' is strongly continuous by definition of strongly continuous.

(3)⇒(1) $M = Z_2(M)\oplus M'$, M' is nonsingular strongly continuous. But M' strongly continuous implies M' is strongly extending by SC_1 condition of definition of strongly continuous. Thus M is strongly t-extending by [Ebrahimi ,2016, Theorem 3.2]. Let $A \leq M, A \supseteq Z_2(M)$ and $A \simeq$ direct summand, say $M = D\oplus E$. D and E are strongly t-extending by [Ebrahimi ,2016, theorem 3.5]. Hence $D = Z_2(D)\oplus D', E = Z_2(E)\oplus E'$ where D' is strongly extending of D, E' is strongly extending of E. Also $Z_2(M) = Z_2(D)\oplus Z_2(E)$ since $M = D\oplus E$. Hence $M = Z_2(D)\oplus D'\oplus Z_2(E)\oplus E' = Z_2(D)\oplus Z_2(E)\oplus D'\oplus E' = Z_2(M)\oplus(D'\oplus E')$. Thus $M' \simeq D'\oplus E'$. But $A \supseteq Z_2(M), M = Z_2(M) \oplus M'$,implies $A = Z_2(M)\oplus(M' \cap A)$ by modular law. Thus $M' \cap A \simeq A/_{Z_2(M)} \simeq D/_{Z_2(D)} \simeq D' \leq^{\oplus} M'$, so that $M' \cap A \simeq$a direct summand of M'. But M' is strongly continuous, so $M' \cap A$ is stable direct summand of M'. Hence $A \leq^{\oplus} M$. Also, $A = Z_2(M)\oplus(M' \cap A)$ and $Z_2(M)$ is stable in M, $(M' \cap A)$ is stable in M'. Now to prove A is stable in M, let $f \in End(M) \simeq \begin{pmatrix} End(Z_2(M)) & Hom(M', Z_2(M)) \\ 0 & End(M') \end{pmatrix}$.
Hence $f \simeq \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{pmatrix}$ for some $\alpha_1 \in End(Z_2(M)), \alpha_2 \in Hom(M', Z_2(M)), \alpha_3 \in End(M')$. Thus $f(A) = f(Z_2(M)\oplus(A \cap M')) \simeq \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{pmatrix}\begin{pmatrix} Z_2(M) \\ A \cap M' \end{pmatrix} \subseteq \begin{pmatrix} \alpha_1(Z_2(M))\alpha_2(A + Z_2(M)) \\ \alpha_3(A \cap M') \end{pmatrix} \subseteq \begin{pmatrix} Z_2(M) \\ A \cap M' \end{pmatrix} = A$, hence $f(A) \subseteq A$. Therefore A is a stable direct summand of M.

Remark s and Examples (2.3):

(1) Every Z_2-torsion module is strongly t-continuous.

Proof It is clear.

(2) Every strongly continuous module is strongly t-continuous, but the converse may be not true. For example :

Let $M = Z_{P^\infty}\oplus Z_{P^\infty}$ as Z-module. Then $Z_2(M) = Z_2(Z_{P^\infty})\oplus Z_2(Z_{P^\infty}) = M$, so that M is Z_2-torsion and hence M is strongly t-continuous by part (1). On other hand M is not strongly extending since $Z_{P^\infty} \leq^{\oplus} M$ (hence Z_{P^∞} is closed in M), but Z_{P^∞} is not stable direct summand because if it is so, then $Hom(Z_{P^\infty}, Z_{P^\infty}) = 0$ by [Saad, Lemma 2.6]. However, there exists $f: Z_{P^\infty} \mapsto Z_{P^\infty}$ defined by $f\left(\frac{a}{p^i} + Z\right) = \frac{a}{p^{i-1}} + Z, f \neq 0$. Thus M is not strongly continuous.

(3) It is clear that, every strongly t-continuous is t-continuous, but the converse may be not true as the following examples show.

(i) Let M be an R-module such that $E(M)$ is not strongly continuous. Let $M' = Z_2\oplus Z_8\oplus E(M)$. It is clear that $E(M)$ is a complement of $Z_2\oplus Z_8 = Z_2(Z_2\oplus Z_8)$. Hence $E(M)$ is nonsingular. But $E(M)$ is continuous and $Z_2(M') = Z_2\oplus Z_8$, that is $M' = Z_2(M')\oplus E(M)$. Hence M' is t-continuous by Theorem1.1(3). But M' is not strongly t-continuous, by Theorem 2.2

(ii) Let $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$ (Z_2 is not Z_2-torsion ring). Let M be an arbitrary R-module. Let $M' = Z_2(M)\oplus R$…………(1) . But M' is a t-extending R-module and M' not strongly t-extending since R_R is not strongly extending [6, Example 3.4]. Hence M' is not strongly t-continuous by Definition 2.1.By (1) R is a complement of $Z_2(M)$, so it is nonsingular. However, $Z_2(M') = Z_2(M)$. hence $M' = Z_2(M')\oplus R$. R is extending. Now let $N \leq R$, $N \simeq W \leq^{\oplus} R$, then $N \leq^{\oplus} R$. Hence R is continuous. Thus M' is t-continuous, by Theorem 1.1(3).

(4) Let $M = Z_n\oplus Z$ as a Z-module. M is strongly t-extending [Ebrahimi,2016]. $M/_{Z_2(M)} = Z_n\oplus Z/_{Z_n\oplus 0} \simeq Z$ and $End_Z(Z) \simeq Z$ is not Von Neumann regular. It follows that M is not t-continuous by Theorem 1.2(2).Thus M is not strongly t-continuous.

(5) Let M be a weak duo (every direct summand is fully invariant submodule)[Özcan ,2006]. Then M is t-continuous if and only if M is strongly t-continuous.

Proof (⇐) It is clear.

(⇒) Let N be a t-closed submodule of M. As M is t-continuous, is a direct summand. Then as M is weak duo, N is stable and so M is strongly t-extending . Let $N \leq M$ such that, $N \supseteq Z_2(M)$ and N is isomorphic to direct summand, then N is a direct summand by condition (2) of definition of t-continuous. Thus M is strongly t-continuous.

(6) Let M be a duo (hence if M is a fully stable or multiplication module). Then M is t-continuous if and only if M is strongly t-continuous. Where an R-module M is called multiplication if for each submodule N of M, there exists an ideal I of R such that $N = MI$. Equivalently M is a multiplication R-module if for each submodule N of M, $N = M(N:_R M)$, where $(N:M) = \{r \in R: Mr \leq N\}$[El-Bast,1988].

It is clear that every multiplication module is duo.

(7) Let M be a cyclic module over a commutative ring R. Then M is t-continuous if and only if M is strongly t-continuous, since every cyclic module over a commutative ring is a multiplication module.

In particular, every commutative ring R with unity, R is t-continuous if and only if R is strongly t-continuous.

An R-module M is called comultiplication (if ann_M $ann_R N = N$ for every submodule N of M [Ansari ,2007]).

(8) Let M be a comultiplication R-module. Then M is t-continuous if and only if M is strongly t-continuous.

Proof M is a comultiplication module implies M is a fully stable by [Inaam,2017, Lemma 2.11]. Thus the result follows by (6) above.

(9) Let M be an indecomposable module. Then M is t-continuous if and only if M is strongly t-continuous.

Proof: Since M is an indecomposable module, M is a weak duo. Hence the result is obtained by (5) above.

(10) Let M be an R-module with local endomorphism ring. Then M is t-continuous if and only if M is strongly t-continuous.

Proof As M has a local endomorphism ring, then M is indecomposable by [Lam ,1998, Theorem 3.5]. Hence the required result is obtained by (9) above.

(11) If M is strongly t-extending and satisfies C_2 then M is strongly t-continuous.

Proof Let $N \leq M$ and $N \supseteq Z_2(M)$ such that $N \simeq W \leq^{\oplus} M$. By (C_2), $N \leq^{\oplus} M$. Hence N is a closed submodule. But $N \supseteq Z_2(M)$, so that N is a t-closed submodule of M. Thus by strongly t-extending, N is stable direct summand.

(12) Let M be an indecomposable module. Then M is continuous if and only if M is strongly continues.

Proof(⇒) Let $N \leq_c M$. Since M is extending, $N \leq^{\oplus} M$. But M is indecomposable, so $N = (0)$ or M. Hence N is fully invariant direct summand. Thus M is strongly extending. Now let $N \leq M, N \supseteq Z_2(M)$ such that $N \simeq W \leq^{\oplus} M$. Hence $N \leq^{\oplus} M$, since M is continuous. But M is indecomposable $N = (0)$ or M. Thus $N$is stable direct summand and hence M is strongly continuous.

(⇐) It is clear.

(13) Let M be a nonsingular R-module. Then M is strongly t-continuous if and only if M is strongly continuous.

(⇒) Since M is nonsingular, every closed submodule N is t-closed, and hence N is stable direct summand. For any $N \leq M, N \supseteq Z_2(M) = (0)$ $N \simeq W \leq^{\oplus} M$. By condition (2) of strongly t-continuous, N is stable direct summand. Thus M is strongly continuous.

(⇐)It is obvious.

(14) Let M and M' be two R-modules such that $M \simeq M'$. Then M is strongly t-continuous if and only if M' is strongly t-continuous.

Inaam and Farhan in [Inamm,2017] introduced the notion of strongly t-semisimple module, where an R-module M is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $K \leq_{tes} N$. It is clear every strongly t-semisimple module is t-semisimple but not conversely by [Inaam, 2017].

The following Proposition shows the class of strongly t-continuous modules contains the class of strongly t-semisimple modules.

Proposition (2.4): Every strongly t-semisimple module is strongly t-continuous.

Proof Since M is strongly t-semisimple, $M = Z_2(M) \oplus M'$, M' is nonsingular semisimple , fully stable and M' is stable in M [11, Theorem2.3]. Hence M is strongly t-extending by [Ebrahimi ,2016, Theorem 3.2]. But $M' \simeq M/_{Z_2(M)}$ is semisimple, so $M/_{Z_2(M)}$ is endoregular. Thus M is t-endoregular and so that $M$is strongly t-continuous by Theorem 2.3.

Remark (2.5): The converse of previous Proposition is not true in general for example.

Let $M = Z_2 \oplus Q$ as Z-module. $Z_2(M) = Z_2$, so $M/_{Z_2(M)} \simeq Q$ which is not semisimple, hence M is not t-semisimple, by [11, Theorem 2.3]. Thus M is not strongly t-semisimple. However we can show that M is strongly t-continuous. $M = Z_2(M) \oplus Q$, where Q is nonsingular. Also, Q is strongly continuous. To show this: As Q and (0) are the only closed of Q and both of them are fully invariant direct summands, hence Q is strongly extending. If $N \leq Q$ and $N \simeq W \leq^{\oplus} Q$. Since Q has only two direct summands Q and (0), so that $W = (0)$ or Q and either $N \simeq (0)$ or $N \simeq Q$. If $N \simeq (0)$, then $N = (0)$ and N is a fully invariant direct summand. If $N \simeq Q$, then $N = Q$, since if we assume that $N < Q$, then there exists an isomorphism $0 \neq f \in Hom(Q, N)$ which is a contradiction. Thus Q is strongly continuous and then by Theorem 2.2(3), M is strongly t-continuous.

Proposition (2.6): If M is a fully stable t-extending module, then M is strongly t-continuous.

Proof: Let $N \leq_{tc} M$. As M is t-extending, $N \leq^{\oplus} M$. But M is a fully stable, so N is stable. Hence M is strongly t-extending. Now let $N \leq M$ and $N \supseteq Z_2(M)$ and $N \simeq W \leq^{\oplus} M$. Then, $N = W$ since distinct submodules of fully stable module cannot be isomorphic. Thus N is stable direct summand.

Corollary (2.7): Every comultiplication t-extending module M is strongly t-continuous.

Proof: Since M is a comultiplication module, then M is fully stable by [Inam,2017, Lemma 2.11]. Hence the result follows by Proposition 2.6.

Remark (2.8): Strongly t-continuous module need not be fully stable, for example:

Let $M = Z_2 \oplus Q$ as Z-module, M is strongly t-continuous Z-module. Let $N = Z_2 \oplus Z \leq M$. Let $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ where $f_1 =$ identity on Z_2, $f_2 \in End(Q)$ such that $f_2(x) = \frac{1}{2}x$. $f(N) \simeq \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}\begin{pmatrix} Z_2 \\ Z \end{pmatrix} = \begin{pmatrix} f_1 Z_2 \\ f_2(Z) \end{pmatrix} = \begin{pmatrix} Z_2 \\ 1/_2 Z \end{pmatrix} \nleq N$, hence N is not fully invariant in M, so it is not stable. Thus M is not fully stable.

Proposition (2.9): If M is t-extending module and satisfies condition (SC_2), then M is strongly t-continuous.

Proof Let $N \leq_{tc} M$. As M is t-extending, $N \leq^{\oplus} M$. But $N \simeq N \leq^{\oplus} M$, implies N is stable direct summand by condition (SC_2). Hence M is strongly t-extending. Also, by condition (SC_2) for any $N \leq M$, $N \supseteq Z_2(M), and\ N \simeq W \leq^{\oplus} M$, implies N is stable direct summand. Thus M is strongly t-continuous.

Remark (2.10): direct sum of strongly t-continuous module need not be strongly t-continuous, for example.

Let $M_1 = Z_2 \oplus Q$, $M_2 = Z_3 \oplus Q$ as Z-modules. Then M_1 and M_2 are strongly t-continuous, but $M_1 \oplus M_2$ is not strongly t-continuous since $M_1 \oplus M_2$ is not strongly t-extending [6, Example 3.10].

Proposition (2.11): A direct summand of strongly t-continuous is strongly t-continuous.

Proof Let M be a strongly t-continuous and let $K \leq^{\oplus} M$. Say $M = K \oplus K'$, for some $K' \leq M$. Assume $A \leq K$, $A \supseteq Z_2(K)$ and $A \simeq D \leq^{\oplus} K$. Since $A \simeq D$, then $A + Z_2(K') \simeq D + Z_2(K')$. But $A + Z_2(K') \supseteq Z_2(K) \oplus Z_2(K') = Z_2(M)$, hence $A + Z_2(K') \supseteq Z_2(M)$. On the other hand, $D \leq^{\oplus} K$ implies $K = D \oplus D'$ for some $D' \leq K$ and hence $M = (D \oplus D') \oplus K'$. As M is strongly t-continuous by hypothesis, then M is strongly t-extending and by [Ebrahimi ,2016, Theorem 3.5] K and K' are strongly t-extending. Hence $K' = Z_2(K') \oplus W$ where W is strongly extending module by [Ebrahimi ,2016, Theorem 3.2(4)]. Thus $M = D \oplus D' \oplus Z_2(K') \oplus W$ which implies that $D \oplus Z_2(K') \leq^{\oplus} M$. But $A \oplus Z_2(K') \supseteq Z_2(M)$ and $A \oplus Z_2(K') \simeq D \oplus Z_2(K') \leq^{\oplus} M$, so that $A \oplus Z_2(K')$ is a stable direct summand of M. Since $M = K \oplus K'$,then $A \oplus Z_2(K') = [(A \oplus Z_2(K') \cap K] \oplus [A \oplus Z_2(K') \cap K']$ where $[(A \oplus Z_2(K')) \cap K]$ is fully invariant in K and $[A \oplus Z_2(K') \cap K']$ is fully invariant in K'. But $(A \oplus Z_2(K')) \cap K = A$, hence A is fully invariant submodule of K. On other hand, $A \oplus Z_2(K') \leq^{\oplus} M$ implies that $A \oplus Z_2(K') \oplus B = M$ for some $B \leq M$. It follows that $K = (A \oplus Z_2(K') \oplus B) \cap K$ and so $K = A \oplus [(Z_2(K') \oplus B) \cap K]$. Thus $A \leq^{\oplus} K$ and therefore K is strongly t-continuous.

Corollary (2.12): For an R-module M. M is strongly t-continuous if and only if M is strongly t-extending and for every nonsingular submodule A of M which is isomorphic to a direct summand of M, $A \oplus Z_2(M)$ is a stable direct summand.

Proof (⇒) As M is strongly t-continuous, M is strongly t-extending and so $M = Z_2(M) \oplus M'$, M' is nonsingular strongly extending. Let A be a nonsingular submodule of M and $A \simeq B \leq^{\oplus} M$. As B is a direct summand of M, then $M = B \oplus D$, for some $D \leq M$. Hence $Z_2(M) = Z_2(B) \oplus Z_2(D)$. But B is nonsingular since $A \simeq B$, so that $Z_2(M) = Z_2(D)$. Moreover $D \leq^{\oplus} M$,implies that D is strongly t-extending , that is $D = Z_2(D) \oplus D'$ where D' is strongly extending and nonsingular. It follows $M = B \oplus Z_2(M) \oplus D'$ and hence $B \oplus Z_2(M) \leq^{\oplus} M$, hence $B \oplus Z_2(M)$ is a closed submodule of M containg $Z_2(M)$. Thus $B \oplus Z_2(M)$ is t-closed and so that $B \oplus Z_2(M)$ is a fully invariant direct summand of M, since M is strongly t-extending. Next as $A \simeq B$, then $A \oplus Z_2(M) \simeq B \oplus Z_2(M) \leq^{\oplus} M$. But M is strongly t-continuous, implies that $A \oplus Z_2(M)$ is a fully invariant direct summand (stable direct summand) of M.

(⇐) As M is strongly t-extending, so $M = Z_2(M) \oplus M'$where M' is a nonsingular strongly extending module. To prove M is strongly t-continuous, it is

enough to show that M' is strongly continuous by Theorem 2.2(3). Let $A \leq M'$ and $A \simeq B \leq^{\oplus} M'$. As M' is nonsingular, A is nonsingular submodule of M and $M' \leq^{\oplus} M$, so that $A \simeq B \leq^{\oplus} M$. Hence by hypothesis, $A \oplus Z_2(M)$ is stable direct summand of M and so that $(A \oplus Z_2(M)) \oplus W = M$ for some $W \leq M$. It follows that that
$M' = [(A \oplus Z_2(M)) \oplus W] \cap M' = A \oplus [(Z_2(M)) \oplus W) \cap M']$ and so $A \leq^{\oplus} M'$. Thus A is closed in M', but M' is strongly extending, hence A is a fully invariant direct summand of M'. Therefore M' is strongly continuous by definition of strongly continuous.

The following statements are characterizations for a right strongly t-continuous ring R.

Theorem (2.13): The following statements are equivalent for a ring R.

(1) Every nonsingular cyclic R-module is projective and strongly continuous.
(2) Every cyclic R-module is strongly t-continuous.
(3) $R = Z_2(R_R) \oplus R'$, R' is nonsingular strongly continuous.
(4) R is right strongly t-continuous.

Proof The equivalence (3) ⇔ (4) follows by Theorem 2.2(3), and the implication (2) ⇒ (4) is clear.

(4)⇒(1) Let M be a nonsingular cyclic R-module. By condition (4), R_R is strongly t-continuous and so it is t-extending, hence M is projective by [Asgari,2017, Proposition 3.18(2)]. But $M \simeq R/ann_R(M)$, which implies that $ann_R(M)$ is a t-closed and so $ann_R(M) \leq^{\oplus} R$ and stable (since R_R is strongly t-continuous). It follows that $M \simeq$ a direct summand of R. However a direct summand of R is strongly t-continuous by Proposition (2.11), hence M is a strongly t-continuous by Remarks and Examples 2.3(14). Then as M is nonsingular, M is strongly continuous by Remarks and Examples 2.3(12).

(1)⇒(2) Let M be a cyclic R-module. Clearly, $M/Z_2(M)$ is nonsingular cyclic R-module. By condition (1), $M/Z_2(M)$ is projective and strongly continuous. Thus $M = Z_2(M) \oplus M'$ and $M' \simeq M/Z_2(M)$ is strongly continuous. Therefore M is strongly t-continuous by Theorem 2.2(3).

Corollary (2.14): The following statements are equivalent for a ring R.

(1) Every nonsingular cyclic R-module is strongly continuous.
(2) $R/Z_2(R)$ is a right strongly continuous.

Proof: (1)⇒(2) Since $R/Z_2(R)$ is a nonsingular cyclic R-module. The result is clear by (1).

(2)⇒(1) Let M be a nonsingular cyclic R-module and set $\bar{R} = R/Z_2(R)$. Then M as $\bar{R}$-module is a nonsingular cyclic $\bar{R}$-module, hence $M_{\bar{R}}$ is strongly continuous (by Theorem 2.13). So that M_R is strongly continuous.

Theorem (2.15): The following statements are equivalent for a ring R.

(1) R is a right t-semisimple.
(2) Every nonsingular R-module is strongly t-continuous (strongly continuous).
(3) Every R-module is strongly t-continuous.
(4) Every R-module is t-continuous.
(5) Every free (projective) R-module is strongly t-continuous.
(6) $R^{(N)}$ is strongly t-continuous.

Proof (1)⇒(3) Since R is t-semisimple, then every R-module M is t-semisimple by [asgari,2013, Theorem 3.2], and $M = Z_2(M) \oplus M'$, for some injective submodule M' of M by [Asgari,2013, Proposition 3.2(5)]. Hence M' is t-semisimple by [Asgari,2013, Corollary 2.4(1)]. It follows that M' is strongly t-semisimple by [Inaam,2017, Corollary 2.13]. Hence M' is strongly t-continuous by Proposition 2.4. But M' is nonsingular, so that M' is strongly continuous. Thus M is strongly t-continuous by Theorem 2.2(3).

(3)⇒(2) It is obvious.

(2)⇒(1) Since every nonsingular R-module is strongly t-continuous (strongly continuous), then every nonsingular module is continuous R-module. Hence by Theorem 1.2(2⇔5), R_R is t-semisimple.

(3)⇒(4) , (3) ⇒ (5) and (5)⇒ (6) are clear.

(4)⇒(1) It follows by Theorem 1.2(3→5).

(6)⇒(1) Since R^N is strongly t-continuous, R^N is t-continuous by Remarks and Examples 2.3. Hence R is a right t-semisimple by, Theorem 1.3.

Recall that an R-module M is called F-regular (simply regular) if every submodule is pure, where a submodule N of M is pure if for every ideal I of R we have $MI \cap N = NI$[Asgari, 2011].

Theorem (2.16): The following statements are equivalent.

(1) Every nonsingular finitely generated is projective and strongly continuous.
(2) Every finitely generated is strongly t-continuous.

(3) $R^{(2)}$ is strongly t-continuous.

Proof (1)⟹(2) If M is a finitely generated R-module then $M/_{Z_2(M)}$ is finitely generated nonsingular. Hence $M/_{Z_2(M)}$ is projective and strongly continuous. It follows that $M = Z_2(M) \oplus M'$ ($M' \simeq M/_{Z_2(M)}$). Hence M is strongly t-continuous by Theorem 2.3(3).

(2)⟹(3) It is clear.

(3)⟹(1) $R^{(2)}$ is strongly t-continuous implies $R^{(2)}$ is t-continuous. Hence every finitely generated nonsingular is projective and continuous by Theorem 1.4(6→1). Hence every finitely generated nonsingular is projective and strongly continuous by Remarks and Examples 2.3(12).

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