ON THE EXPECTED VALUE OF UNCERTAIN VARIABLES

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Abstract

In this work , some mathematical properties and several inequalities for uncertain variables that based on uncertain measure and expected value are proved. Finally the relation between convergence in p th moment and convergence in measure for uncertain sequence were investigated.

Keyword: Uncertain measure, uncertain variable, expected value.

1-Introduction

Liu [2] in 2007 founded uncertainty theory, and Liu [8] refined it in 2010 . The first basic concepts of uncertainty theory is an uncertain measure which defined as a set function from $M : F \to R$, satisfies the following axioms:

(A1) (Normality) : $M(\Omega) = 1$.

(A2)(Monotonicity) : $M(A) \leq M(B)$ whenever $A \subset B$.

(A3)(Self-duality): $M(A) + M(A^c) = 1$ for any event A.

(A4)(Countable Subadditivity): For every countable sequence of events $\{A_i\}$, we have

$$M(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} M(A_i)$$

(A5) (Product Measure Axiom): Let (Ω_k, F_k, M_k) be uncertainty spaces for $k = 1,2,\ldots,n$. Then the product uncertain measure M is an uncertain measure on the product σ - filed $F_1 \times F_2 \times \ldots \times F_n$ satisfying :

$$M(\prod_{k=1}^{n} A_k) = \min\{M_k(A_k) : A_k \in F_k, k = 1,2,\ldots,n\}$$

That is, for each event $A \in F = F_1 \times F_2 \times \ldots \times F_n$, we have

$$M(A) = \begin{cases} \alpha, & \alpha > 0.5 \\ 1-\beta, & \beta > 0.5 \\ 0.5, & o.w \end{cases}$$

where

$$\alpha = \sup\left\{\begin{matrix} \min\{M_k(A_k)\} : k = 1,2,\ldots,n\} : \\ A_1 \times A_2 \times \ldots \times A_n \subseteq A \end{matrix}\right\}$$

$$\beta = \sup\left\{\begin{matrix} \min\{M_k(A_k)\} : k = 1,2,\ldots,n\} : \\ A_1 \times A_2 \times \ldots \times A_n \subseteq A^c \end{matrix}\right\}$$

Definition (1-1) [3]

The triple (Ω, F, M) is called an uncertainty space which Ω be a nonempty set, F be a σ - field over Ω, and M be an uncertain measure.

Definition (1-2) [1]

An uncertain variable X defined as measurable function from an uncertainty space to the set of real numbers such that $\{X \in B\}$ is an event for any set B of $\Re$

Definition (1-3) [7]

Assume that $X_1, X_2, ..., X_n$ be an uncertain variables. We say that $X_i, i=1,2,...,n$ be an independent uncertain variables if $M(\prod_{i=1}^{n}(X_n \in B_i)) = \min_{1\le i\le n} M_i(X_i \in B_i)$, for any Borel $B_1, B_2, ..., B_n$ of $\Re$.

Definition(1-4)[10]

Assume that $\{x_n\}$ be an uncertain variables sequence .we say that $\{x_n\}$ convergence in p th moment to uncertain variable x if $\lim_{n\to\infty} E\{|x_n - x|^p\} = 0$ for any $p \in [1,\infty)$. we say that $\{x_n\}$ convergence in mean to uncertain variable x whenever $p=1$ and we say that $\{x_n\}$ convergence in mean square to uncertain variable x whenever $p=2$.

Definition(1-5)[9]

Assume that $\{x_n\}$ be an uncertain variables sequence . we say that $\{x_n\}$ is convergence in measure to uncertain variable x if $\lim_{n\to\infty} M\{|x_n - x| \ge \varepsilon\} = 0$, for every $\varepsilon > 0$.

Theorem (1-6) [9]

Assume that X be an uncertain variable. Then we get $M(|X| \ge \varepsilon) \le \dfrac{E(|X|^p)}{\varepsilon^p}$, for any given numbers $\varepsilon > 0$ and $p > 0$.

Theorem (1-7)[5]

Assume that X and Y be two independent uncertain variables such that $E(|X|^p)$ and $E(|Y|^p)$ are finite. Then we get $E(XY) \le \sqrt[p]{E(|X|^p)} + \sqrt[q]{E(|Y|^q)}$, for any positive numbers p and q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem (1-8)[5]

Assume that X and Y be two independent uncertain variables such that $E(|X|^p)$ and $E(|Y|^p)$ are finite. Then we get $\sqrt[p]{E(|X+Y|^p)} \le \sqrt[p]{E(|X|^p)} + \sqrt[p]{E(|Y|^p)}$ for any real number p such that $p \ge 1$.

2- Expected value of uncertain variables

Definition (2-1) [6,8]

The expected value operator of uncertain variable X is defined by

$$E(X) = \int_0^{+\infty} M(X \ge r)dr - \int_{-\infty}^{0} M(X \le r)dr,$$

provided that at least one of the two integrals is finite. The variance of X is defined by $E(X) = E(X-e)^2$ where X is the finite expected value of X. Generally, the expected value $E((X)^k)$ is called the k th absolute moment of the uncertain variable X for any positive integer k.

Theorem(2-2) [5]

Assume that X and Y be two independent uncertain variables such that $E(X) < \infty$, and $E(Y) < \infty$. Then we get $E(aX + bY) = aE(X) + bE(Y)$ for any $a, b \in \Re$.

Theorem(2-3)

Assume that X be an uncertain variable. Then $E(-X) = -E(X)$.

Proof:

By using definition (2-1), then we get

$$E(-X) = \int_0^{+\infty} M(-X \ge r)dr - \int_{-\infty}^{0} M(-X \le r)dr$$

$$= \int_0^{+\infty} M(X \le -r)dr - \int_{-\infty}^{0} M(X \ge -r)dr$$

$$= \int_{-\infty}^{0} M(X \le r)dr - \int_0^{+\infty} M(X \ge r)dr$$

$$= -\left[\int_0^{+\infty} M(X \ge r)dr - \int_{-\infty}^{0} M(X \le r)dr\right]$$

$$= -E(X)$$

Theorem(2-4)

Assume that X and Y be two an independent uncertain variables such that $E(|X|) < \infty$, and $E(|Y|) < \infty$. Then we get $E(|X + Y|) \le E(|X|) + E(|Y|)$

Proof:

By using theorem (2-2), then we get $E(|X + Y|) \le E(|X| + |Y|) = E(|X|) + E(|Y|)$

3- Main Results

Theorem(3-1)

Assume that X and Y be two independent uncertain variables, and $f : R^2 \to R$ a convex function such that $E(X) < \infty$, and $E(Y) < \infty$. Then $f(E(X), E(Y)) \le E(f(X,Y))$

Proof:

Since $f : R^2 \to R$ is a convex function, then for any given $(x_1, y_1) \in R^2$, then there exist $a_1, a_2 \in R$ such that

$f(x, y) - f(x_1, y_1) \ge a_1(x - x_1) + a_2(y - y_1)$, for all $x, y \in R$

Put $x_1 = E(X), y_1 = E(Y)$ and $x = X, y = Y$

Then, $f(X,Y) - f(E(X), E(Y))$
$\ge a_1(X - E(X)) + a_2(Y - E(Y))$

$E[f(X,Y) - f(E(X), E(Y))]$
$\ge E[a_1(X - E(X)) + a_2(Y - E(Y))]$

From theorem (2-2), we get

$E(f(X,Y)) - f(E(X), E(Y)) \ge$
$a_1(E(X) - E(X)) + a_2(E(Y) - E(Y)) = 0$

$f(E(X), E(Y)) \le E(f(X,Y))$

Theorem(3-2)

Assume that X and Y be two an independent uncertain variables such that $E(|X|^p) < \infty$ and $E(|Y|^p) < \infty$, where $p > 0$. Then $E(|X+Y|^p) \le CE(|X|^p) + CE(|Y|^p)$.

Holds, where $C = \begin{cases} 1 & , if\, 0 < p \le 1 \\ 2^{p-1} & , if p > 1 \end{cases}$

Proof:

for any $a_1, a_2 \in \Re$, we get

$|a_1 + a_2|^p \le C|a_1|^p + C|a_2|^p$

Put $a_1 = X$ and $a_2 = Y$, we get

$|X+Y|^p \le C|X|^p + C|Y|^p$

$E[|X+Y|^p] \le E[C|X|^p + C|Y|^p]$, by theorem (2-2), then we get

$E(|X+Y|^p) \le CE(|X|^p) + CE(|Y|^p)$

Theorem (3-3)

Assume that $\{x_n\}$ be an uncertain variables sequence and x, y be two uncertain variables . If $x_n \to x$ as $n \to \infty$ (in p th moment) , and $x_n \to y$ as $n \to \infty$ (in p th moment) , Then $x = y$.

Proof:

Since $\lim\limits_{n\to\infty} E\{|x_n - x|^p\} = 0$ and $\lim\limits_{n\to\infty} E\{|x_n - y|^p\} = 0$

By using Theorem (3-2) that

$$E\{|x-y|^p\} = E\{|x - x_n + x_n - y|^p\}$$
$$\le CE\{|x_n - x|^p\} + CE[|y_n - y|^p] \to 0 ,$$

as $n \to \infty$. we get $E\{|x-y|^p\} = 0$.

Theorem (3-4)

Assume that $\{x_n\}$, $\{y_n\}$ be two uncertain variables sequence and x, y be two uncertain variables. If $x_n \to x$ as $n \to \infty$ (in p th moment), and $y_n \to y$ as $n \to \infty$ (in p th moment), if $E\{|x_n - x|^p\}$ and $E\{|y_n - y|^p\}$ are independent for all n. Then $x_n - y_n \to x - y$ as $n \to \infty$.

Proof:

Since $\lim\limits_{n\to\infty} E\{|x_n - x|^p\} = 0$ and $\lim\limits_{n\to\infty} E\{|y_n - y|^p\} = 0$

By using Theorem (3-2) that

$$E\{|(x_n - y_n) - (x-y)|^p\}$$
$$= E\{|(x_n - x) + (y - y_n)|^p\}$$
$$\le CE\{|x_n - x|^p\} + CE\{|y_n - y|^p\} \to 0,$$

As $n \to \infty$, then we get $\lim\limits_{n\to\infty} E\{|(x_n - y_n) - (x-y)|^p\} = 0$.

Theorem (3-5)

Assume that $\{x_n\}$ be an uncertain variables sequence and x be an uncertain variable . If $x_n \to x$ (in p th moment), then $\lim_{n\to\infty} E\{|x_n|^p\} = E\{|x|^p\}$.

Proof:

There is two cases:

Case(1) : If $0 < p \le 1$. Put $x_n = x + (x_n - x)$ and $x = x_n + (x - x_n)$

By using theorem (3-2), we get $E\{|x_n|^p\} \le E\{|x|^p\} + E\{|x_n - x|^p\}$, and $E\{|x|^p\} \le E\{|x_n|^p\} + E\{|x_n - x|^p\}$, furthermore

$$\lim_{n\to\infty}\left(E\{|x_n|^p\} - E\{|x|^p\}\right) \le \lim_{n\to\infty}\left(E\{|x_n - x|^p\}\right) = 0,$$

Which implies $\lim_{n\to\infty}\left(E\{|x_n|^p\}\right) = \left(E\{|x|^p\}\right)$

Case(2) : If $p > 1$. Put $x_n = x + (x_n - x)$ and $x = x_n + (x - x_n)$

By using theorem (1-8), that

$$\left(E\{|x_n|^p\}\right)^{\frac{1}{p}} \le \left(E\{|x|^p\}\right)^{\frac{1}{p}} + \left(E\{|x_n - x|^p\}\right)^{\frac{1}{p}}$$

$$\left(E\{|x|^p\}\right)^{\frac{1}{p}} \le \left(E\{|x_n|^p\}\right)^{\frac{1}{p}} + \left(E\{|x_n - x|^p\}\right)^{\frac{1}{p}}$$

Then we get

$$\lim_{n\to\infty}\left(\left(E\{|x_n|^p\}\right)^{\frac{1}{p}} - \left(E\{|x|^p\}\right)^{\frac{1}{p}}\right) \le \lim_{n\to\infty}\left(E\{|x_n - x|^p\}\right)^{\frac{1}{p}} = 0$$

Which implies $\lim_{n\to\infty}\left(E\{|x_n|^p\}\right)^{\frac{1}{p}} = \left(E\{|x|^p\}\right)^{\frac{1}{p}}$.

Since $f(x) = x^{\frac{1}{p}}$ is continuous function , thus $\lim_{n\to\infty} E\{|x_n|^p\} = E\{|x|^p\}$

Theorem (3-6)

Assume that $\{x_n\}$ and $\{y_n\}$ be two uncertain variables sequence and x, y be two uncertain variables . If $x_n \to x$ as $n \to \infty$ (in p th moment) , and $y_n \to y$ as $n \to \infty$ (in q th moment) , for any positive numbers p and q such that $\frac{1}{p} + \frac{1}{q} = 1$,and $p > 1$. Then $x_n y_n \to xy$ as $n \to \infty$ (in mean).

Proof:

Since $\lim_{n\to\infty} E\{|x_n - x|^p\} = 0$

, $\lim_{n\to\infty} E\{|y_n - y|^q\} = 0$ and $\lim_{n\to\infty} E\{|x_n|^p\} = E\{|x|^p\}$

by using theorem (1-7)

$$E\{|x_n y_n - xy|\} = E\{|x_n y_n - x_n y + x_n y - xy|\}$$
$$\le E\{|x_n(y_n - y)|\} + E\{|y(x_n - x)|\}$$
$$\le \sqrt[p]{E\{|x_n|^p\}}\sqrt[q]{E\{|y_n - y|^q\}}$$
$$+ \sqrt[q]{E\{|y|^q\}}\sqrt[p]{E\{|x_n - x|^p\}} \to 0,$$

As $n \to \infty$, then we get $\lim_{n\to\infty} E\{|x_n y_n - xy|\} = 0$

Theorem (3-5)

Assume that $\{x_n\}$ be an uncertain variables sequence and x be an uncertain variable . If $x_n \to x$ as $n \to \infty$ (in p th moment), then $x_n \to x$ as $n \to \infty$ (in measure) .

Proof:

By using theorem (1-6) then for any given numbers $\varepsilon > 0$ we get

$$M[|x_n - x| \ge \varepsilon] \le \frac{E[|x_n - x|^p]}{\varepsilon^p}$$

then

$$\lim_{n\to\infty} M[|x_n - x| \ge \varepsilon] \le \lim_{n\to\infty}\left(\frac{E[|x_n - x|^p]}{\varepsilon^p}\right)$$

Which means that $\{x_n\}$ is converges to uncertain variable x in measure.

Example(3-9)

The convergence uncertain sequence in measure does not be convergence uncertain sequence in p th moment. for example , Let (Ω, F, M) to be $\{y_1, y_2, ...\}$ such that $M(y_k) = \frac{1}{k}$ for $k = 1,2,3,....$,and define the uncertain variables by

$$X_n(y_k) = \begin{cases} n & , if n = k \\ 0 & , o.w \end{cases}$$

for $n = 1,2,3,....$ and $X = 0$, then for any small number $\varepsilon > 0$ we get that $\lim_{n\to\infty} M[|X_n - X| \ge \varepsilon] = \lim_{n\to\infty}(\frac{1}{n}) = 0$, but $E[|X_n - X|] = 1$

Since

$$E[|X_n - X|^p] = \int_0^\infty M(|X_n - X|^p \ge r)dr$$
$$\ge \int_0^\infty M(|X_n - X| \ge r)dr = E(|X_n - X|) = 1$$

Thus, $\lim_{n\to\infty} E[|X_n - X|^p] = 1 \ne 0.$

4-Conclusion

We obtained in this present paper some properties of expected value and several inequalities that based on expected value and uncertain measure, which is analogous inequality. Then by using this inequality, we discussed some mathematical properties of convergence in p th moment for uncertain sequence.

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حول القيمة المتوقعة للمتغيرات اللايقينية

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الخلاصة

في هذا العمل، برهنا بعض الخواص الرياضية و بعض المتراجحات القائمة على القياس اللايقيني و القيمة المتوقعة ، أخيرا حققنا العلاقة بين التقارب في (p th moment) والتقارب في القياس للمتتابعات اللايقينية.

الكلمات الدليلية

القياس اللايقيني (غير المؤكد) ، المتغيرات اللايقينية، القيمة المتوقعة .