Al-Qadisiya Journal For Science Vol.21 No. 1 Year 2016

Centralizing Higher Left Centralizers On Prime Rings

Receved :1/11/2015 Accepted :5/1/2016

Salah M. SalihMazen O. Karim

Department Of mathematics **Department Of mathematics**

College of Educations **College Of Educations**

Al-Mustansiriya University **Al-Qadisiyah University**

Dr.salahms2014@gmail.commazin792002@yahoo.com

Abstract :
In this paper we study the commutativity of prime rings satisfying certain identities involving higher left centralizer on it .

Math. Classification QAISO -27205

Key words: prime rings , higher left centralizer

1.Introduction :

Throughout this paper R is denote to an associative ring and it is center will denoted by $Z(R)$ which equal to the set of all elements $x \in R$ such that $xy = yx$ for all $y \in R$.

Now for any $x, y \in R$, the symbols $[x, y]$ and $\langle x, y\rangle$ will denoted to $xy - yx$ and $xy + yx$ respectively which are called commutator(Lie product) and anti- commutator (Jordan product) respectively .[1],[2] . A ring R is called commutative if $[x, y] = 0$ for all $x, y \in R$.

The above commutator and anti- commutator

satisfies the following[1] ,[2] :

1) $[xy, z] = [x, z]\, y + x[y, z]$
2) $[x, yz] = y[x, z] + [x, y]\, z$
3) $\langle x, yz\rangle = \langle x, y\rangle\, z - y[x, z]$

$= y\langle x, y\rangle + [x, y]\, z$

3) $\langle xy, z\rangle = x\langle y, z\rangle - [x, z]\, y$
$= \langle x, z\rangle\, y + x[y, z]$

A ring R is called prime if $xRy = \{0\}$ implies that $x = 0$ or $y = 0$ and it is called semi-prime if $xRx = \{0\}$ implies that $x = 0$[3] .

An additive mapping $F: R \to R$ is called centralizing on asubset S of ring R if $[F(x), x] \in Z(R)$ and it is called commuting if $[F(x), x] = 0$ for all $x \in S$[4], [5].

An additive mapping $T: R \to R$ is called left(right) centralizer on a ring R if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$[6] .

Many authors covers the concept of left centralizer and study the relation between the commutativity of ring and left centralizers .

K.K.Dey and A.C. Paul in [7] study the commutativety of Γ- ring in which satisfying certain identities involving left centralizers .

In this paper , we obtain the commutativity of a ring satisfying certain identities involving higher left centralizers on ring R , this work motivated from the work of K.K.Dey and A.C.Paul [7] .

We generalized the definition of higher k- left centralizer on a Γ- ring [8] into a higher left centralizer on a ring $R$by taking k as the identity automorphism as the following

Definition 1.1 :let R be a ring and let $T = (T_i)_{i\in N}$ be a family of left centralizers on R , then $T_n: R \to R$ is called higher left centralizer on R if

$$T_n(xy) = \sum_{i=1}^{n} T_i(x)y$$

holds for all $x, y \in R$.

2. Commutativity of prime gamma rings :: in this section we study the commutativity of the ring$R$ by using higher left centralizer on it .

Theorem 2.1 : let R be a prime ring and I be anon –zero ideal of R. suppose that R admits a family of non – zero higher left centeralizers $T = (T_i)_{i\in n}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and $i \in N$. if $T_n([x,y]) - [x,y] = 0$ for all x,y $\in I$ then R is commutative .

Proof : Given that $T = (T_i)_{i\in N}$ afamily of left centralizens of R such that.

$T_n([x,y]\ ) - [x,y] = 0$ …….. (1)

for all $x, y \in I$.

Then $T_n(xy - yx) - (xy - yx) = 0$

So that

$(\sum_{i=1}^{n} T_i(x)y - \sum_{i=1}^{n} T_i(y)x) - (xy - yx) = 0$

Which leads to

$(\sum_{i=i}^{n} T_i(x) - x)\,y - \sum_{i=1}^{n} T_i(y) - y)x = 0$ …………….(2)

Replace x by xr in (2) we get

$(\sum_{i=i}^{n} T_i(xr) - xr)\,y - \sum_{i=1}^{n} T_i(y) - y)xr = 0$

Hence

$(\sum_{i=1}^{n} T_i(x) - x)ry - (\sum_{i=1}^{n} T_i(y) - y)xr = 0$ ………..(3)

For all $x, y \in I, r \in R$

Using (2) in (3) to simplify , we obtain

$(\sum_{i=1}^{n} T_i(x) - x)\,[r,y] = 0$ …………….(4)

For all $x, y \in I, r \in R$.

Again replacing r by rs in (4)

$$(\sum_{i=1}^{n} T_i(x) - x)\,r\,[s,y] = 0$$

For all $x, y \in I$ and$r, s \in R$

$$(\sum_{i=1}^{n} T_i(x) - x)\,R\,[s,y] = 0$$

for all $x, y \in I, s \in R$

by primness' of R and since $\sum_{i=i}^{n}(T_i(x) - x) \neq 0$

hence $[s,y] = 0$ for all $y \in I$, $s \in R$

there fore $I \subset Z(R)$ and hence R is commutative . ■

Corollary 2.2 : In theorem 2.1 , if the family T of higher left centralizers is zero then R is commutative

proof : suppose that $T_n([x,y]) - [x,y] = 0$ for any $x, y \in I$

if $T_n = 0$ then $[x,y] = 0$ for all $x, y \in I$

There fore I is commutative hence R is commutative . ■

Theorem 2.3: let R be a prime ring and I be a non – zero ideal of R suppose that R admits a family T of non – zero higher left centralizer

$T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq -x$ for all $x \in I$ and $i \in N$, for ther if $T_n([x,y]) + [x,y] = 0$ for all $x, y \in I$, then R is commutative .

Proof : Given that $T = (T_i)_{i \in N}$ is a family of higher left centralizers of R such that .

$T_n([x,y]) + [x,y] = 0$ for all $x, y \in I$ ....(1)

Then

$$T_n(xy - yx) + (xy - yx) = 0$$

So that

$(\sum_{i=1}^{n} T_i(x)y - \sum_{i=1}^{n} T_i(y)x) + (xy - yx) = 0$

Which leads to

$(\sum_{i=1}^{n} T_i(x) + x)y - (\sum_{i=1}^{n} T_i(y) + y)x = 0$ ....(.2)

In (2) replace x by xr to get

$(\sum_{i=1}^{n} T_i(xr) + xr)y - (\sum_{i=1}^{n} T_i(y) + y)xr = 0$

Hence

$(\sum_{i=1}^{n} T_i(x) + x)ry - (\sum_{i=1}^{n} T_i(y) + y)xr = 0$ ......(.3)

For all $x, y \in I$, $r \in R$

Using (2) in (3) to simplify , we obtain

$(\sum_{i=1}^{n} T_i(x) + x)\ \beta\ [r,y]_\alpha = 0$ ....(4)

For all $x, y \in I$, $r \in R$

Replace r by rs in (4)

$$(\sum_{i=1}^{n} T_i(x) + x)\ r\ [s,y] = 0$$

For all $x, y \in I$, $r, s \in R$

in other words

$(\sum_{i=1}^{n} T_i(x) + x)\ R[s,y] = o$

For all $x, y \in I$, $s \in R$

By primness of R and since $\sum_{i=i}^{n}(T_i(x) + x) \neq 0$

we get $[s,y] = 0$ for all $y \in I$, $s \in R$

Therefore $I \subset Z(R)$ and hence R is commutative . ■

Theorem 2.4 : - let R be a prime ring and I be anon – zero ideal of R . suppose that R adimits afamily of non – zero higher left centralizers . $T = (T_i)_{i \in n}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and $i \in N$, further if

$$T_n(<x,y>) = \langle x,y \rangle$$

For all $x, y \in I$, then R is commutative .

Proof : - Given that

$T_n(<x,y>) - <x,y> = 0$ ....(1)

For all $x, y \in I$

This implies that

$(\sum_{i=1}^{n} T_i(x) - x)y + (\sum_{i=1}^{n} T_i(y) - y)x = 0$ ....(.2)

Replace x by xr in (2) we obtain .

$(\sum_{i=1}^{n} T_i(xr) - xr)y + (\sum_{i=1}^{n} T_i(y) - y)xr = 0$

Hence

$(\sum_{i=1}^{n} T_i(x) - x)ry - (\sum_{i=1}^{n} T_i(y) - y)xr = 0$ ....(3)

For all $x, y \in I, r \in R$

Using (2) in (3) we get

$(\sum_{i=1}^{n} T_i(x) - x)ry + (\sum_{i=1}^{n} T_i(x) - x)yr = 0$ ....(4)

For all $x, y \in I, r \in R$

That is

$$(\sum_{i=1}^{n} T_i(x) - x)[r, y] = 0$$

.............(5)

For all $x, y \in I, r \in R$

Replace r by rs in (5) we get .

$$(\sum_{i=1}^{n} T_i(x) - x)r[s, y] = 0$$

........(6)

For all $x, y \in I, r, s \in R$

i.e : $(\sum_{i=1}^{n} T_i(x) - x)R[s, y] = 0$

By primness of R and since $\sum_{i=1}^{n}(T_i(x) - x) \neq 0$

Then $[s, y] = 0$ for all $y \in I$

Hence $I \subset Z(R)$ there for R is commutative ■

Theorem 2.5 : - let R be a prime ring and I be anon – zero ideal of R . suppose that R adimits afamily of non – zero higher left centralizers . $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq -x$ for all $x \in I$ and $i \in N$, further if

$$T_n(< x, y >) + \langle x, y \rangle = 0$$

For all $x, y \in I$, then R is commutative

Proof : - Given that $T = (T_i)_{i \in N}$ be a family of non – zero higher left centralizers of R such that

$T_n(< x, y >) + < x, y > = 0$ ....(1)

For all $x, y \in I$.

Then

$(\sum_{i=1}^{n} T_i(xy + yx)) + (xy + yx) = 0$

H hence

$$\sum_{i=1}^{n} T_i(x)y + \sum_{i=1}^{n} T_i(y)x + (xy + yx) = 0$$

$$(\sum_{i=1}^{n} T_i(x) + x)y + (\sum_{i=1}^{n} T_i(y) + y)x) = 0$$

... .(2)

In the above relation eplace x by xr we obtain .

$(\sum_{i=1}^{n} T_i(xr) + xr)y + (\sum_{i=1}^{n} T_i(y) + y)xr = 0$

So we get

$(\sum_{i=1}^{n} T_i(x) + x)ry + (\sum_{i=1}^{n} T_i(y) + y)xr = 0$ ....(3)

For all $x, y \in I, r \in R$

Substitute (2) in (3) to get

$(\sum_{i=1}^{n} T_i(x) + x)[r, y] = 0$ ....(4)

For all $x, y \in I, r \in R$

Now again replace r by rs in (4) we have

$(\sum_{i=1}^{n} T_i(x) + x)r[s,y] = 0$ ....(5)

For all $x, y \in I$ and $r, s \in R$

i.e : $(\sum_{i=1}^{n} T_i(x) + x)R[s,y] = 0$

By primness of R and since $\sum_{i=1}^{n}(T_i(x) + x) \neq 0$

We have $[s, y] = 0$ for all $y \in I, s \in R$.

Hence $I \subset Z(R)$ there for R is commutative ■

Corollary 2 . 6 : In theorem 2.4 and 2.5 if a higher left centralizers T_n is zero . then R is commutative .

Proof : For any $x, y \in I$, we have

$$T_n(< x,y > = < x,y >$$

if $T_n = 0$ then $< x,y > = 0$ for all $x, y \in I$

replace x by xz and using the fact

$yx = -xy$ we conclude that

$x[z,y] = \{0\}$ for all $x, y, z \in I$

In other words we have

$IR[z,y] = 0$ for all $y, z \in I$.

Since R is prime and $I \neq \{0\}$

So that $[z,y] = 0$ for all $y, z \in I$

then I is commutative and hence R is commutative . ■

Theorem 2.7 :- let R be a prime ring and I be anon zero ideal of R . suppose that R admits a family of non –zero higher left centralizers $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, further if $T_n(xy) \mp (xy) = 0$ for all $x, y \in I$, then R is commutative .

proof :- for any $x, y \in I$ we have

$$T_n(xy) = (xy)$$

this implies that

$$T_n([x,y]) - ([x,y]) = 0$$

and hance by theorem 2.1 we have R is commutative

on the other hand if R is satisfy the condition $T_n(xy) + (xy) = 0$ for all $x, y \in I$.

then for any $x, y \in I$

$$\text{we have } T_n(xy + yx) = -(xy + yx)$$

So that $T_n(\langle x,y \rangle) + (\langle x,y \rangle) = 0$ for all $x, y \in I$.

Then by theorem 2.5 we have R is commutative ■

Corollary 2.8 : -let R be a prime ring and I be anon zero ideal of R . suppose that R admits a family of non –zero higher left centralizers $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq \mp x$ for all $x \in I$ and for all $i \in N$, further if $T_n(xy) \mp (yx) = 0$ for all $x, y \in I$ then R is commutative .

proof : For any $x, y \in I$ we have $T_n(xy) \mp (yx) = 0$

now if $T_n(xy) = (yx)$ this implies that $T_n([x,y]) - ([y,x]) = T_n([x,y]) + ([x,y]) = 0$

then by theorem 2.5 we have R is commutative

Now when $T_n(xy) + (yx) = 0$ then $T_n([x,y]) + ([y,x]) = 0$

this implies that $T_n([x,y]) + ([x,y]) = 0$ and hance by theorem 2.1 we have R is commutative ■

3.The main results: in this section we introduce the main results of this paper

Theorem 3.1: :let R be a prime ring and I be anon zero ideal of R . suppose that R admits a family of non –zero higher left centralizers $T=(T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x)\neq x$ for all $x\in I$ and for all $i\in N$, then the following conditions are equivalent :

(i) $T_n([x,y])-([x,y])=0$ for all $x,y\in I$

(ii) $T_n([x,y])+([x,y])=0$ for all $x,y\in I$

(iii) for all $x,y\in I$, either $T_n([x,y])-([x,y])=0$ or

$T_n([x,y])+([x,y])=0$

(iv) R is commutative

Proof :

(i)→(iv) suppose that $T_n([x,y])-([x,y])=0$

Then by theorem 2.1 we have R is commutative

(iv)→(i) suppose that R is commutative then $[x,y]=0$

and hence $T_n([x,y])-([x,y])=0$

(ii) →(iv) suppose that

$$T_n([x,y])+([x,y])=0$$

for all $x,y\in I$

Then by theorem 2.3 we have R is commutative

(iv)→(ii) suppose that R is commutative then $[x,y]=0$ for all $x,y\in I$

And hence $-[x,y]=0$ for all $x,y\in I$

Which implies that $T_n([x,y])-([x,y])=0$ for all $x,y\in I$

(iii) →(iv) suppose that for all $x,y\in I$ either $T_n([x,y])-([x,y])=0$ or

$$T_n([x,y])+([x,y])=0$$

Then by theorem2.1 or theorem 2.3 we have R is commutative

(iv)→(iii) suppose that R is commutative

For each fixed $y\in I$ we set

$$I_1=\{x\in I\,|\,T_n([x,y])-([x,y])=0\}$$

$$I_2=\{x\in I\,|\,T_n([x,y])+([x,y])=0\}$$

Then I_1 and I_2 are additive subgroups of I such that $I=I_1\cup I_2$.

But a group cannot be the set theoretic union of two proper subgroups , hance we have either

$I_1=I$ or $I_2=I$.

Further , using a similar argument , we obtain

$I=\{y\in I\,|\,I_1=I\}$ or $I=\{y\in I\,|\,I_2=I\}$

Thus we obtain that either $T_n([x,y])-([x,y])=0$ for all $x,y\in I$

or $T_n([x,y])+([x,y])=0$ for all $x,y\in I$

Hence R is commutative in both cases by theorem 2.1 (respectively theorem 2.3) ■

Theorem 3.2 ::let R be a prime ring and I be anon zero ideal of R . suppose that R admits a family of non –zero higher left centralizers $T=(T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x)\neq x$ for all $x\in I$ and for all $i\in N$, further if $T_n(xy)-(xy)\in Z(M)$ for all $x,y\in I$ then R is commutative .

Proof : for any $x,y\in I$ we have

$T_n(xy)-(xy)\in Z(R)$ ……………….(1)

This can be written as $\sum_{i=1}^{n} T_i(x)y-xy\in Z(R)$ for all $x,y\in I$ ……….(2)

That is $[(\sum_{i=1}^{n} T_i(x) - x)y , r] = 0$ for all $x, y \in I, r \in R$ ……….(3)

Which implies that

$(\sum_{i=1}^{n} T_i(x) - x)[y, r] + [\sum_{i=1}^{n} T_i(x) - x , r] \; y = 0$ …………..(4)

for all $x, y \in I, r \in R$

in (4) replace x by xz , we have

$(\sum_{i=1}^{n} T_i(x) - x)z[y, r] + [(\sum_{i=1}^{n} T_i(x) - x)z , r] \; y = 0$ …………(5)

for all $x, y, z \in I, r \in R$

from (3) we get that (5) becomes

$(\sum_{i=1}^{n} T_i(x) - x)z[y, r] = 0$ for all $x, y, z \in I, r \in R$.

This yields that

$(\sum_{i=1}^{n} T_i(x) - x)RI[y, r] = \{0\}$ for all $x, y \in I, r \in R$

By primness of R implies that

$I[y, r] = \{0\}$ or $\sum_{i=1}^{n} T_i(x) - x = 0$

and since $I \neq \{0\}$ and $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$

we get that I is central and hence R is commutative ■

Theorem 3.3:let R be a prime ring and I be anon zero ideal of . suppose that R admits a family of non –zero higher left centralizers $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq -x$ for all $x \in I$ and for all $i \in N$, further if $T_n(xy) - (xy) \in Z(M)$ for all $x, y \in I$, then R is commutative .

proof : suppose that $T = (T_i)_{i \in N}$ be a family of non –zero higher left centralizers satisfying the property $T_n(xy) - (xy) \in Z(R)$ for all $x, y \in I$

then the non- zero higher left centralizers $(-T)$ satisfies the condition

$(-T_n)(xy) - (xy) \in Z(R)$ for all $x, y \in I$

Hance by theorem 3.2 we have R is commutative . ■

Remark 3.4: in theorem 3.2 if the higher left centralizer is zero , then R is commutative .

Theorem 3.5 :let R be a prime ring and I be anon zero ideal of R . suppose that R admits a family of non –zero higher left centralizers $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, further if $T_n(xy) - (yx) \in Z(R) \; for \; all \; x, y \in I$ then R is commutative .

Proof : we are given that a higher left centralizer of R such that

$$T_n(xy) - (yx) \in Z(R)$$

for all $x, y \in I$

this implies that

$$[T_n(xy) - (yx) , \quad r] = 0 \quad \text{……………………(1)}$$

holds for all $x, y \in I , r \in R$

which implies that

$$\left[\sum_{i=1}^{n} T_i(x)y - yx , \; r\right] = 0 \quad \text{…………………..(2)}$$

for all $x, y \in I , r \in R$

replacing y by yx in the above relation and use it hence

$$\left[\sum_{i=1}^{n} T_i(x)yx - yx^2 , \; r\right] = 0$$

……………………(3)

for all $x, y \in I , r \in R$

we find that

$$(\sum_{i=1}^{n} T_i(x)y - yx)\,[x,r] = 0$$

……………………….(4)

for all $x, y \in I , r \in R$

again replace r by rs in (4) to get

$$(\sum_{i=1}^{n} T_i(x)y - yx)\,r[x,s]$$

$$+(\sum_{i=1}^{n} T_i(x)y - yx)[x,r] \quad s = 0$$

………..……..(5)

for all $x, y \in I , r, s \in R$

From (4) the relation (5) becomes

$$(\sum_{i=1}^{n} T_i(x)y - yx)\,r[x,s] = 0$$

……………….(6)

for all $x, y \in I , r, s \in R$

i.e.

$$(\sum_{i=1}^{n} T_i(x)y - yx)\,R\,[x,s] = 0$$

for all $x, y \in I , s \in R$

the primness of R implies that either $[x,s] = 0$ or $\sum_{i=1}^{n} T_i(x)y - yx = 0$

for all $x, y \in I , s \in R$

now put

$I_1 = \{x \in I | [x,s] = 0 \; for \; all \; s \in R \}$

$$I_2 = \left\{x \in I \left| \sum_{i=1}^{n} T_i(x)y - yx = 0 \; for \; all \; x, y \in I \right.\right\}$$

Then clearly that I_1 and I_2 are additive subgroups of$R$. moreover by the discussion given I is the set- theoretic union of I_1 and I_2 but can not be the set- theoretic of two proper subgroups .

Hence $I_1 = I$ or $I_2 = I$.

If $I_1 = I$, then $[x,s] = 0$ for all $x \in I , s \in R$ and hence R is commutative .

On the other hand if $I_2 = I$ then $\sum_{i=1}^{n} T_i(x)y = yx$ for all for all $x, y \in I$.

That is $\sum_{i=1}^{n} T_i(x)y - yx = 0$ for all for all $x, y \in I$

This implies that $T_n([x,y] \;) - ([x,y] \;) = 0$ for all for all $x, y \in I$.

Hence apply theorem 2.1 yields the required result . ■

References :

1) Ali S., Basudeb D. and Khan M. S. ,2014, " On Prime and Semiprime Rings with Additive Mappings and Derivations " ,Universal Journal of Computational Mathematics ,Vol.2, No.3 ,48 -55 .
2) Ur-Rehman N. , 2002,"On Commutativity of Rings with Generalized Derivations ", Math. J Okayama Univ. , Vol.44, 43-49.
3) Vukman J. ,1997,"Centralizers on Prime and Semiprime Rings " , Comment. Math. Univ. Caroline ,Vol.38 , No.2 , 231- 240.

4) Braser M. ,1993," Centralizing Mappings and Derivations in Prime Rings " ,Journal of Algebra ,Vol.156 ,385 -394 .
5) Vukman J.,1990,"Commuting and Centralazing Mappings in Prime Rings", Proceeding of the American Math. Society , Vol. 109 ,No.1 , 47 -52.
6) Ali S. and Dar N. A. ,2014," On Left Centralizers of Prime Rings with Involution " ,Palestine Journal of Mathematics,Vol.3,No.1,505-511.
7) Dey K.K. and Paul A.C ,2014,"commutativity of prime gamma rings with left centralizers " , J. Sci.Res. ,Vol.6 ,No.1 , 69-77.
8) Salih S.M.,kamal A.M. and hamad B. M. ,2013, “Jordan higher K-centralizer on Γ-rings ,ISOR Jornal of Mathematics ,Vol.7 No.1 ,6-14.

التمركزات العليا اليسرى على الحلقات الأولية

تاريخ الاستلام 2015/11/1 تاريخ القبول 2016/1/5

د. صلاح مهدي صالح **مازن عمران كريم**

قسم الرياضيات **قسم الرياضيات**

كلية التربية **كلية التربية**

الجامعة المستنصرية **جامعة القادسية**

Dr.salahms2014@gmail.commazin792002@yahoo.com

الملخص :

في هذا البحث ندرس ابدالية الحلقات الاولية التي تحقق شروط معينة تتضمن تمركزات يسرى من الدرجات العليامعرفة على تلك الحلقات الاولية.

الكلمات المفتاحية الحلقات الأولية , تمركزات يسرى