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On Monotony and Comonotony Approximation by algebraic polynomial

By

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Abstract: In this paper we are proved that if the function $\mathbb{f} \in L_{\psi,p}(I) \cap \triangle^1 (I)$, change the monotone finitely many times in an interval $I = [-b, b]$, then there exist an algebraic polynomial $\wp_n \in \prod_n$, which comonotony of approximation with $f$at every point in an interval I, such that the best approximation can be estimate by $c(p,\delta)\omega_3^{\varphi}(\mathbb{f}, n^{-1})_{\psi,p}$.

Keyword: Monotony, Comonotony, Degree of best approximation, Modulus of smoothens , Algebraic polynomial **.**

حول التقريب الرتيب والمحافظ على الرتابة باستخدام متعددة الحدود الجبرية

بواسطة

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الكلمات المفتاحية :التقريب الرتيب , التقريب المحافظ على الرتابة , درجة افضل تقريب , مقياس النعومة , متعددة الحدود الجبرية .

المستخلص : في هذا البحث برهنا انه اذا كانت الدالة $\mathbb{f} \in L_{\psi,p}(I) \cap \triangle^1 (I)$ تتغير رتابتها بشكل منتهي في الفترة $I = [-b, b]$ فانه توجد متعددة حدود جبرية $\wp_n \in \prod_n$ والتي تكون محافظة على الرتابة مع الدالة $\mathbb{f}$ عند كل نقطة من نقاط الفترة I , بحيث انه افضل تقريب يمكن تخمينه بواسطة مقياس النعومة $c(p,\delta)\omega_3^{\varphi}(\mathbb{f}, n^{-1})_{\psi,p}$.

1. Introduction and definitions:

We are interested in this paper in how well can approximations of a function $\in L_{\psi,p}(I) \cap \triangle^1 (I)$,$I \subseteq R$, $0 < \mathrm{p} < 1$ which changes its monotony by a polynomial $\wp_n \in \prod_n$. In this case the polynomial $\wp_n$ are comonotony of approximation with the function $\mathbb{f}$ at every point in an interval I . Let $\mathbb{f}$ be a function which changes monotonicity finitely many time say $\mathcal{S} \geq 1$, times on I say $-b < \mathcal{I}_{\mathcal{S}} < \cdots < \mathcal{I}_1 < b = \mathcal{I}_0$ for $\mathcal{I}_{\mathcal{S}} \in J_{\mathcal{S}}$ and a function $\mathbb{f}$ in $\triangle^1 (J_{\mathcal{S}})$ we denote to the error of best approximation by : $E_n^{(1)}(\mathbb{f}, J_{\mathcal{S}})_{\psi,p} = \inf_{\wp_n \cap \Delta^1(J_{\mathcal{S}})} \left\| \triangle_h^k (\mathbb{f}, .) \right\|_{L_{\psi,p}(I)}$ and note that

$$E_n^{(1)}(\mathbb{f}, J_{\mathcal{S}})_{\psi,p} = E_n^{(1)}(\mathbb{f}, 0, J_{\mathcal{S}})_{\psi,p}$$

(where $\Delta^1(J_S)$ be the set of all functions $\mathbb{f}$ which change monotony at the points $\mathcal{I}_S \in J_S$. Recall that the order Ditizain-Totik modulus of smoothness is given by ([3]):

$$\omega_\varphi^k(\mathbb{f},\delta,I)_{\psi,p} = \sup_{0<h\le\delta} \left\| \triangle_h^k(\mathbb{f},.) \right\|_{L_{\psi,p}(I)}$$

Where $\|.\|_{L_{\psi,p}(I)}$ denotes the weighted quasi normed space [3] on an interval $[-b,b] \subseteq I \subseteq R$. The weighted quasi normed space $L_{\psi,p}(I), 0 < p < 1$ have form :

$$L_{\psi,p}(I) = \left\{ f \ni f: I \subset R \longrightarrow R: \left(\int_I \left| \frac{f(x)}{\psi(x)} \right|^p dx \right)^{\frac{1}{p}} < \infty , 0 < p < 1 \right\}$$

and the quasi normed $\|f\|_{L_{\psi,p}(I)} < \infty$, and

$$\Delta_h^k(\mathbb{f},x,I)_\psi = \Delta_h^k(\mathbb{f},x)_\psi = \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \frac{\mathbb{f}(x-\frac{kh}{2}-ih)}{\psi(x+\frac{kh}{2})} & , x \pm \frac{kh}{2} \in I \\ 0 & o.w \end{cases}$$

is the symmetric difference ([3]) and in this paper we used new chebyshev partition $X_j = a\cos\frac{j\pi}{n}$ ([3]), and proved the following theorem:

Theorem (1.1): There exists absolute constant $c = c(p,\delta)$ such that for every $\mathbb{f}$ which change monotony in $I, 0 < p < 1$ there is a polynomial $\wp_n \in \prod_n$ which is comonotony of approximation with $\mathbb{f}$ and satisfies:

$$\|\mathbb{f} - \wp_n\|_{L_{\psi,p}(I)} \le c(p,\delta)\omega_\varphi^3(\mathbb{f},\delta,I)_{\psi,p} .$$

2. Auxiliary Results: Now the following Lemmas are crucial for the proof of theorem (1.1).

Lemma (2.1): Let $\mathbb{f}$ be the same as in theorem (1.1) then there exist a function $\mathrm{g}_n \in L_{\psi,p}(I)$,comonotony of approximation with f in ℓ_i ,such that:

$$\|\mathbb{f} - \mathrm{g}_n\|_{L_{\psi,p}(I)} \le c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} \qquad \dots (2.1.1$$

$$\left\|\Delta_n \check{\acute{\mathrm{g}}}_i\right\|_{L_{\psi,p}(I)} \ge \omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} \qquad \dots (2.1.2$$

are satisfy.

Proof: To prove (2.1.1) we must to show that

$$\|\acute{\mathrm{g}}_i - \mathbb{f}\|_{L_{\psi,p}(I)} \le c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$$

Let$\mathcal{L}(x,\mathbb{f}) = \mathcal{L}\left(x,\mathbb{f},\acute{\mathcal{J}}_i,\mathcal{J}_i,\acute{\acute{\mathcal{J}}}_i\right) = \frac{x-\mathcal{J}_i}{\acute{\acute{\mathcal{J}}}_i-\acute{\mathcal{J}}_i}\left(\frac{x-\acute{\mathcal{J}}_i}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\mathbb{f}\left(\acute{\acute{\mathcal{J}}}_i\right) + \frac{x-\acute{\acute{\mathcal{J}}}_i}{\mathcal{J}_i-\acute{\mathcal{J}}_i}\mathbb{f}(\acute{\mathcal{J}}_i)\right)$,

is the **Lagrange polynomial** of degree ≤ 2 ([2]), which interpolates at $\acute{\mathcal{J}}_i,\mathcal{J}_i$ and $\acute{\acute{\mathcal{J}}}_i$ by using the inequality

$\|\mathbb{f} - \mathcal{L}(.,\mathbb{f})\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^k(\mathbb{f},n^{-1})_{\psi,p}$,

when $k = 3$, for $x \in \ell_i$ we set $\breve{\mathfrak{g}}_i$ be the polynomial of degree ≤ 2 which vanishes at $\mathcal{J}_i$ in the form

$$\breve{\mathfrak{g}}_i(x) = \frac{x-\mathcal{J}_i}{\acute{\acute{\mathcal{J}}}_i-\acute{\mathcal{J}}_i}\left(\frac{x-\acute{\mathcal{J}}_i}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\breve{\mathfrak{g}}_i\left(\acute{\acute{\mathcal{J}}}_i\right) + \frac{x-\acute{\acute{\mathcal{J}}}_i}{\mathcal{J}_i-\acute{\mathcal{J}}_i}\breve{\mathfrak{g}}_i(\acute{\mathcal{J}}_i)\right) ,$$

and by using the above presentation of $\breve{\mathfrak{g}}_i$ and $\mathcal{L}(x,\mathbb{f}), x \in \ell_i$ we get

$$\|\breve{\mathfrak{g}}_i - \mathbb{f}\|_{L_{\psi,p}(I)} \leq c(p)\|\acute{\mathfrak{g}}_i - \mathcal{L}(.,\mathbb{f})\|_{L_{\psi,p}(I)} + \|\mathcal{L}(.,\mathbb{f}) - \mathbb{f}\|_{L_{\psi,p}(I)}$$

$$\leq \left|\frac{(x-\mathcal{J}_i)(x-\acute{\mathcal{J}}_i)}{\left(\acute{\acute{\mathcal{J}}}_i-\acute{\mathcal{J}}_i\right)\left(\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i\right)}\right| \left\|\breve{\mathfrak{g}}_i\left(\acute{\acute{\mathcal{J}}}_i\right) - \mathbb{f}\left(\acute{\acute{\mathcal{J}}}_i\right)\right\|_{L_{\psi,p}(I)} + \left|\frac{(x-\mathcal{J}_i)(x-\acute{\acute{\mathcal{J}}}_i)}{(\acute{\acute{\mathcal{J}}}_i-\acute{\mathcal{J}}_i)(\mathcal{J}_i-\acute{\mathcal{J}}_i)}\right| \|\breve{\mathfrak{g}}_i(\acute{\mathcal{J}}_i) - \mathbb{f}(\acute{\mathcal{J}}_i)\|_{L_{\psi,p}(I)}$$

$$\leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} \qquad \ldots(2.1.3$$

Hence (2.1.3) is proved. It is well known ([5],[6]) that there exists a polynomial $Q(x)$, of degree$\leq n$ satisfying

$$\|\mathbb{f} - Q\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} \qquad \ldots. (2.1.4$$

And by using definition the piecewise polynomial function $\mathfrak{F}(x)$, and the function $\mathfrak{g}_n(x)$,$x \in \ell_i$([3]), not that $\mathfrak{g}_n$ is comonotony of approximation with $\mathbb{f}$ in $\mathcal{J}_i$, hence it is comonotony of approximation with $\mathbb{f}$ in $\bigcup \mathcal{J}_i$,hence

$$\|\mathbb{f} - \mathfrak{g}_n\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$$

Satisfy from the inequality (2.1.3) and (2.1.4) .

Know to prove (2.1.2) for $x \in \ell_i$ we use the above presentation $\breve{\mathfrak{g}}_i$

$$\breve{\mathfrak{g}}_i(x) = \frac{x-\mathcal{J}_i}{\acute{\acute{\mathcal{J}}}_i-\acute{\mathcal{J}}_i}\left(\frac{x-\acute{\mathcal{J}}_i}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\breve{\mathfrak{g}}_i\left(\acute{\acute{\mathcal{J}}}_i\right) + \frac{x-\acute{\acute{\mathcal{J}}}}{\mathcal{J}_i-\acute{\mathcal{J}}_i}\breve{\mathfrak{g}}_i(\acute{\mathcal{J}}_i)\right)$$

Where ([1]),

$$\breve{\mathfrak{g}}_i(\acute{\mathcal{J}}_i) = \begin{cases} 2\Delta_{h\varphi(x)}^3(\mathbb{f},x)sign(\mathbb{f}(\acute{\mathcal{J}}_i)) & if\ |\mathbb{f}(\acute{\mathcal{J}}_i)| \leq c\Delta_{h\varphi(x)}^3(\mathbb{f},x) \\ \mathbb{f}(\acute{\mathcal{J}}_i) & o.w \end{cases}$$

and

$$\breve{g}_i\left(\acute{\mathcal{J}}_i\right)=\begin{cases}2\Delta^3_{h\varphi(x)}(\mathbb{f},x)sign(\mathbb{f}\left(\acute{\mathcal{J}}_i\right)) & if\ \left|\mathbb{f}(\acute{\mathcal{J}}_i)\right|\le c\Delta^3_{h\varphi(x)}(\mathbb{f},x)\\ \mathbb{f}(\acute{\mathcal{J}}_i) & o.w\end{cases}$$

Since $\breve{g}_i\in\prod_2$, $\breve{g}_i(\acute{\mathcal{J}}_i)$ and $\breve{g}_i(\acute{\acute{\mathcal{J}}}_i)$are monotony then the only zero of $\breve{g}_i$ in ℓ_i is $\mathcal{J}_i$. hence $\breve{g}_i$ is comonotony of approximation with $\mathbb{f}$ in ℓ_i .Also

$$\acute{\breve{g}}_i(x)=\frac{2x-\mathcal{J}_i-\acute{\mathcal{J}}_i}{\left(\acute{\mathcal{J}}_i-\mathcal{J}_i\right)\left(\acute{\mathcal{J}}_i-\mathcal{J}_i\right)}\breve{g}_i\left(\acute{\mathcal{J}}_j\right)+\frac{2x-\mathcal{J}_i-\acute{\acute{\mathcal{J}}}_i}{\left(\acute{\mathcal{J}}_i-\mathcal{J}_i\right)\left(\mathcal{J}_i-\acute{\mathcal{J}}_i\right)}\breve{g}_i(\acute{\mathcal{J}}_i)$$

is a linear function and

$\acute{\breve{g}}_i\left(\frac{\mathcal{J}_i+\acute{\mathcal{J}}_i}{2}\right)=\frac{-\breve{g}_i(\acute{\mathcal{J}}_i)}{\mathcal{J}_i-\acute{\mathcal{J}}_i}$,and $\acute{\breve{g}}_i\left(\frac{\mathcal{J}_i+\acute{\acute{\mathcal{J}}}_i}{2}\right)=\frac{\breve{g}_i(\acute{\acute{\mathcal{J}}}_i)}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}$, are of the same monotony which implies that $\acute{\breve{g}}_i$ does not change monotony at J_i for $x\in J_i$ we get

$$\left|\acute{\breve{g}}_i(x)\right|\ge inf\left\{\left|\acute{\breve{g}}_i\left(\frac{\mathcal{J}_i+\acute{\mathcal{J}}_i}{2}\right)\right|,\left|\acute{\breve{g}}_i\left(\frac{\mathcal{J}_i+\acute{\acute{\mathcal{J}}}_i}{2}\right)\right|\right\}=inf\left\{\left|\frac{\breve{g}_i(\acute{\mathcal{J}}_i)}{\mathcal{J}_i-\acute{\mathcal{J}}_i},\left|\frac{\breve{g}_i(\acute{\acute{\mathcal{J}}}_i)}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\right|\right|\right\}$$

$$\ge 2inf\left\{\frac{\left|\Delta^3_{h\varphi(x)}(\mathbb{f},x)\right|}{\mathcal{J}_i-\acute{\mathcal{J}}_i},\frac{\left|\Delta^3_{h\varphi(x)}(\mathbb{f},x)\right|}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\right\}$$

$$\ge\frac{2}{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}\left\{\left|\Delta^3_{h\varphi(x)}(\mathbb{f},x)\right|\right\}$$

Since $|J_i|=\frac{\acute{\acute{\mathcal{J}}}_i-\mathcal{J}_i}{2}$ when $k=3$ and $2|J_i|=|\ell_i|$,then

$$\left\|\acute{\breve{g}}_i\right\|_{L_{\psi,p}(J^*)}\ge\frac{1}{|\ell_i|}\left\{\left\|\Delta^3_{h\varphi(x)}(\mathbb{f},x)\right\|_{L_{\psi,p}(J^*)}\right\}$$

From the relationship $|\ell_i|\approx\Delta_n(x)$ and $x\in J^*=\cup_{i=1}^{s}J_i$ ([3]),we get

$|J_i|\le|\ell_i|\approx\Delta_n(x)$, $x\in J^*$ that is $\frac{1}{\Delta_n(x)}\approx\frac{1}{|\ell_i|}\le\frac{1}{|J_i|}$ hence

$$\left\|\acute{\breve{g}}_i\right\|_{L_{\psi,p}(J^*)}\ge\frac{1}{\Delta_n(x)}\left\{\left\|\Delta^3_{h\varphi(x)}(\mathbb{f},.)\right\|_{L_{\psi,p}(J^*)}\right\}$$

$$\left\|\Delta_n\acute{\breve{g}}_i\right\|_{L_{\psi,p}(J^*)}\ge\omega^3_\varphi(\mathbb{f},n^{-1})_{\psi,p}\ .$$

Lemma (2.2): Let $\mathbb{f}$ be the same as in theorem (1.1) then there exist a polynomial $\mathcal{Q}_n(x)\in\prod_n$, $\in N$, such that

$$\left\|\varphi^3\mathcal{Q}_n^{(3)}\right\|_{L_{\psi,p}(I)}\le c(p,\delta)\omega^3_\varphi(\mathbb{f},n^{-1})_{\psi,p}$$

Proof: by (2.1.4) in Lemma (2.1) there exist a polynomial $\mathcal{Q}_n(x)$ comonotony of approximation with $\mathbb{f}$ by a well-known on difference $\left(\frac{\varphi}{n}\right)^3\mathcal{Q}_n^{(3)}(x)=\left(\frac{\varphi}{n}\right)^3D^3\mathcal{Q}_n(x)=\Delta^3_{\frac{\varphi}{n}}\mathcal{Q}_n(x)$

$$\left\|\left(\frac{\varphi}{n}\right)^3 \mathcal{Q}_n^3\right\|_{L_{\psi,p}(I)} \leq c(p)\left\|\Delta^3_{\frac{\varphi}{n}}(\mathbb{f}-\mathcal{Q}_n,.)\right\|_{L_{\psi,p}(I)} + c(p)\left\|\Delta^3_{\frac{\varphi}{n}}(\mathbb{f},.)\right\|_{L_{\psi,p}(I)}$$

By definition of $\omega_\varphi^k(\mathbb{f},n^{-1})_{\psi,p}$, when $k=3$ we get

$$\left\|\left(\frac{\varphi}{n}\right)^3 \mathcal{Q}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$$

$$\left\|\varphi^3\mathcal{Q}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq c(p,\delta)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} .$$

Lemma (2.3): Let $\mathbb{f}$ be the same as in theorem (1.1), then there exist a function $\mathbb{f}_n \epsilon L_{\psi,p}(I)$, comonotony of approximation with $\mathbb{f}$ in J_i, such that

$$\left\|\varphi^3\mathbb{f}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq c(p,\delta)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$$

Proof: Now by using Kolmogorov type inequality ([4]),

$\left\|h^{(v)}\right\|_{L_{\psi,p}(I)} \leq (2b)^{r-v}\left\|h^{(r)}\right\|_{L_{\psi,p}(I)} + (2b)^{-v}\|\mathbb{g}\|_{L_{\psi,p}(I)}$. … (2.3.1

Where $h \epsilon L_{\psi,p}(I)$, $0 \leq v \leq r$, using fact that $\varphi(x) \sim n\Delta_n(x) \sim n|\ell_i|$, $x \in \ell_i$, ([3]) we get $\left\|\varphi^3\mathbb{f}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq cn^3|\ell_i|^3 \sum_{v=0}^3 \left\|\mathcal{Q}^{(v)} - \breve{\mathbb{g}}_i^{(v)}\right\|_{L_{\psi,p}(I)} \left\|\delta^{(3-v)}\right\|_{L_{\psi,p}(I)}$

Applying (2.3.1) for $\left\|\mathcal{Q}^{(v)} - \breve{\mathbb{g}}_i^{(v)}\right\|_{L_{\psi,p}(I)}$, and Markova's inequality for $\left\|\mathfrak{F}^{(3-v)}\right\|_{L_{\psi,p}(I)}$, together with (2.1.3) and the fact $\|\mathbb{f}-\mathcal{Q}\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$, and lemma (2.2) we get

$\left\|\varphi^3\mathbb{f}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq$
$cn^3|\ell_i|^3 \sum_{v=0}^3 (|\ell_i|^{3-v}\left\|\mathcal{Q}^{(3)}\right\|_{L_{\psi,p}(I)} + |\ell_i|^{-v}\left\|\mathcal{Q}-\breve{\mathbb{f}}_i\right\|_{L_{\psi,p}(I)})\,|\ell_i|^{v-3}\|\delta\|_{L_{\psi,p}(I)}$.

Hence

$$\left\|\varphi^3\mathbb{f}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq c(p,\delta)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p} .$$

Proof (1.1): Let $n \geq 4\delta^{-1}$ be fixed and let $n \leq N(n) = N$, be an integer, also let $\mathbb{g}_n \in L_{\psi,p}(I)$, be a function which was described in lemma(2.1). Lemma (2.3) can be written as: $\left\|(1-x^2)^{3/2}\mathbb{f}_n^{(3)}\right\|_{L_{\psi,p}(I)} \leq c(p)\omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$

It follows from lemma (2.2) that there exist a polynomial $\wp_N(\mathbb{f}_n,x) \in \prod_N$, which $\wp_N(\mathbb{f}_n,\mathcal{J}_i) = 0$, $i = 1,\dots,s$, and such that : $\|\mathbb{f}_n - \wp_N(f_n)\|_{L_{\psi,p}(I)} \leq \mathcal{K}_1\mathcal{K}_2\left(\frac{n}{N}\right)^3 \omega_\varphi^3(\mathbb{f},n^{-1})_{\psi,p}$, and $\left\| \Delta_N (\acute{\mathbb{f}}_n - \acute{\wp}_N(f_n)) \right\|_{L_{\psi,p}(I)} \le \mathcal{K}_1\mathcal{K}_3 \left(\frac{n}{N}\right)^3 \omega_\varphi^3(\mathbb{f}, n^{-1})_{\psi,p}$, … (1.1.1

we prescribe N , to be such that $\mathcal{K}_1\mathcal{K}_2 \left(\frac{n}{N}\right)^3 \le 1$, and $\mathcal{K}_1\mathcal{K}_3 \left(\frac{n}{N}\right)^3 \le \frac{1}{4}$

for instance $N = n\mathcal{K}_7 = \left(\left[(\mathcal{K}_1\mathcal{K}_2)^{\frac{1}{3}}\right] + \left[2\sqrt{\mathcal{K}_1\mathcal{K}_3}\right] + 2\right) n$.

It follows from (1.1.1) that for $x \in J_i$, $i = 1, \dots, s$ the following estimate is valid

$$\left\| \acute{\mathbb{f}}_n - \acute{\wp}_N(f_n) \right\|_{L_{\psi,p}(I)} \le \mathcal{K}_1\mathcal{K}_3 \frac{n}{N\Delta_N} \left(\frac{n}{N}\right)^2 \omega_\varphi^3(\mathbb{f}, n^{-1})_{\psi,p}$$

$$\le \mathcal{K}_1\mathcal{K}_3 \frac{n}{\sqrt{1-x^2}} \left(\frac{n}{N}\right)^2 \omega_\varphi^3(\mathbb{f}, n^{-1})_{\psi,p}$$

$\le \frac{1}{2} \Delta_n^{-1}(x) \omega_\varphi^3(\mathbb{f}, n^{-1})_{\psi,p}$,

since $N\Delta_N(x) \sim \varphi(x)$.

Together with lemma (2.1.2) this implies that monotony of $\wp_N(\mathbb{f}_n, x)$ the same as of monotony of $\mathbb{f}_n(x)$, $x \in \bigcup_{i=1}^{s} J_i$.In turn, it follows that $\wp_N(\mathbb{f}_n)$, is comonotony of approximation with $\mathbb{f}$ in $\bigcup_{i=1}^{s} J_i$, and also by lemma (2.1.1) and (1.1.1), we get

$$\|\mathbb{f} - \wp_N(\mathbb{f}_n)\|_{L_{\psi,p}(I)} \le c(p)\|\mathbb{f} - \mathbb{f}_n\|_{L_{\psi,p}(I)} + c(p)\|\mathbb{f}_n - \wp_N(\mathbb{f}_n)\|_{L_{\psi,p}(I)}$$

$$\le c(p,\delta)\omega_\varphi^3(\mathbb{f}, n^{-1})_{\psi,p}$$

Together with lemma (2.1.1) this yields the assertion of theorem (1.1) for $n > \mathcal{K}_8 = 4\delta^{-1}\mathcal{K}_6\mathcal{K}_7$, $\mathcal{K}_8 = \mathcal{K}_8(p,\delta)$.

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