On Pure Multiplication Modules

Received :11/1/2016 Accepted : 18/4/2016

Tha'ar Younis Ghawi

Department of Mathematics , College of Education

University of AL-Qadisiyah \ IRAQ

E-mail : thaar_math83@yahoo.com

Abstract. Let R be a commutative ring with identity and M be a unitary left R-module. A submodule N of an R-module M is called pure if, $IN = N \cap IM$ for all ideal I of R. In this paper, we shall investigate and study some various properties of the concept of pure multiplication modules, which is appeared in [3] as a proper generalization of multiplication modules, where an R-module M is called pure multiplication if for every pure submodule N of M there exists an ideal I of R such that $N = IM$. We give a number of results concerning pure multiplication modules. Also, we discussed some properties of pure submodules of pure multiplication modules.

Key Words: Pure Multiplication Modules; Pure Submodules; Cancellation Modules.

Mathematics Classification QA 150 – 272.5

Introduction

Let M be an R-module. For submodule K of M, $[K :_R M] = \{r \in R : rM \subseteq K\}$ is the residual ideal of K by M in R. The annihilator of M in R is $ann_R M = [0 :_R M]$. Recall that an R-module M is called faithful if, $ann_R M = 0$. An R-module M is called multiplication, provided for each submodule N of M there exists an ideal I of R such that $N = IM$ [4]. An R-submodule N of a module M is called pure if, $IN = N \cap IM$ for each ideal I of R, and an R-module M is called regular if all its submodules are pure [7]. Also, we say that an R-module M is pure simple if, the trivial submodules of M are the only pure submodules [6]. Atani in [3], introduced and study the concept of pure multiplication modules as a proper generalization of multiplication modules. An R-module M is said to be pure multiplication if for each pure submodule N of M, there exists an ideal I of R such that $N = IM$. Our objective is to investigate the concept of pure multiplication modules. This work consists of two sections. In section one, we give several properties of pure multiplication modules. We prove that, if M is an anti-hopfian module then $End_R(M)$ is a pure multiplication ring, where an R-module M is said to be anti-hopfian if M is nonsimple and all nonzero factor modules of M are isomorphic to M [10]. Also, we presented some relations between pure multiplication modules and other modules. In section two, we investigate pure submodules of pure multiplication modules and presented some properties of such submodules.

Finally, it is remarked that all rings R considered in this work are commutative with identity and all R-modules are left unitary. We will refer to the submodule (direct summand) N of a module M by $N \leq M$ ($N \leq^{\oplus} M$), the endomorphism ring of a left R-module M by $End_R(M)$.

1. Pure Multiplication Modules And Related Modules

We begin this section with the following Remarks and Examples :

Remarks and Examples 1.1

(i) The Z-module Z_{p^∞} is pure multiplication, where p is prime, since it is pure simple. But it is well-known that not multiplication.

(ii) Every simple module is pure multiplication. In particular Z-module Z_P is pure multiplication, where p is a prime number.

(iii) Each of Z, Z_4 and Z_8 as Z-modules are pure multiplication .

(iv) Z_6 as Z-module is pure multiplication, but it is not pure simple .

(v) Z-module $Z \oplus Z$ is not pure multiplication, since $N = Z \oplus (0)$ is a pure submodule of $Z \oplus Z$,but $N = Z \oplus (0) \neq I.(Z \oplus Z)$ for all ideal I of Z . Notice, this example shows the direct sum of pure multiplication modules need not be pure multiplication.

(vi) If M is a regular module. Then the concepts multiplication module and pure multiplication module are coincide for M.

Recall that an R-module M is said to be semisimple if every submodule of M is a direct summand [11].

(vii) Since every semisimple module is regular, thus if M is a semisimple module, then M is a pure multiplication module if and only if M is a multiplication module.

($viii$) Every commutative ring with identity is multiplication, and so it is pure multiplication.

Recall that an R-module M is cancellation if for any two ideals A, B of R with $AM = BM$ implies that $A = B$ [13] .

In the next result, we state Proposition for pure multiplication module under multiplication finitely generated faithful modules.

Proposition 1.2 Let M be a multiplication finitely generated faithful R-module and A be an ideal of R. Then AM is a pure multiplication module if and only if A is a pure multiplication R-module.

Proof. Suppose that AM is a pure multiplication R-module. Let I be a pure ideal of A, then $JI = I \cap JA$ for all ideal J of R. Since M is a multiplication finitely generated faithful module, so $J(IM) = (JI)M = (I \cap JA)M = (IM) \cap J(AM)$ this mean IM is a pure in AM. Since AM is pure multiplication, then there exists an ideal K of R such that $IM = K(AM) = (KA)M$, hence $I = KA$, since M is a cancellation module, by [1,Th. 3.1]. Thus A is pure multiplication.

Conversely, let N be a pure submodule of AM. Since M is a finitely generated faithful multiplication R-module, so $N = BM$, where B is an ideal of A. Thus, for all ideal I of R, $IN = N \cap I(AM)$,then $(IB)M = BM \cap (IA)M = (B \cap IA)M$, but M is a cancellation module, so $IB = B \cap IA$, this mean that B is a pure ideal of A, but A is pure multiplication, so $B = JA$ for some ideal J of R, hence $N = BM = (JA).M = J(AM)$. Thus AM is pure multiplication. □

Remarks 1.3

(i) Let M be an R-module, $N \leq M$. If M/N is a pure multiplication R-module then it is not necessarily that M is so, as the following example: consider $M = Z \oplus Z$ as Z-module and let $N = 2Z \oplus 3Z$ be a submodule of M, so we have $\frac{M}{N} = \frac{Z \oplus Z}{2Z \oplus 3Z} \cong Z_2 \oplus Z_3 \cong Z_6$ which is a pure multiplication as Z-module, but $M = Z \oplus Z$ is not a pure multiplication as Z-module [see (Rem.and.Ex. 1.1) (iv), (v)] .

(ii) Consider the following short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$. If each of A and C is a pure multiplication R-module, then B may not be Pure multiplication, as the following example shows: consider $0 \longrightarrow Z \xrightarrow{i} Z \oplus Z \xrightarrow{\rho} Z \longrightarrow 0$, where i is the inclusion map, and ρ is the projection map. By [(Rem.and.Ex 1.1)(iii),(v)], Z as Z-module is pure multiplication, but $Z \oplus Z$ is not pure multiplication

Proposition 1.4 Let M be a divisible module over an integral domain R. Then M is a pure simple R-module if and only if M is a pure multiplication R-module.

Proof. Let M is a pure simple R-module, so the trivial submodules are the only pure submodules of M, hence clear that M is a pure multiplication R-module.

Conversely, assume M is pure multiplication and let N be a non-zero pure submodule of M, then there exists an ideal I of R such that $N = IM$, but M is a divisible module over an integral domain R, then $IM = M$ and hence $N = M$. Thus M is a pure simple R-module. □

Proposition 1.5 If M is an anti-hopfian R-module, $S = End_R(M)$ is a pure multiplication ring.

Proof. Since M is an anti-hopfian module, thus $S = End_R(M)$ is an integral domain, by [10]; this mean $S = End_R(M)$ is a commutative ring with identity, thus by [(Rem.and.Ex.1.1)($viii$)] $S = End_R(M)$ is a pure multiplication ring . □

Recall that an R-module M is said to have the pure intersection property (briefly PIP) if the intersection of any two pure submodules of M is again pure [2] .

Now, we shall introduce the next result .

Proposition 1.6 Every pure multiplication module has the PIP.

Proof. Let M be a pure multiplication R-module and let A, B be two pure submodules of M, thus $A = IM$ and $B = JM$ for some ideals I and J in R. Since A and B are pure submodules of M, then $KA = A \cap KM$ and $KB = B \cap KM$ for every ideal K in R. Thus, $(A \cap B) \cap KM =$ $(IM \cap JM) \cap KM$, also $KIM = IM \cap KM$, $KJM = JM \cap KM$. Therefore $(IM \cap JM) \cap KM = IM \cap (JM \cap KM) = IM \cap KJM = KIJM \subseteq K(A \cap B)$, this mean that $(A \cap B) \cap KM \subseteq K(A \cap B)$. But, it is clear that $K(A \cap B) \subseteq (A \cap B) \cap KM$, hence we get $K(A \cap B) = (A \cap B) \cap KM$ for every ideal K in R. Thus $A \cap B$ is pure in M . □

Let M be a module over an integral domain R. The set $T(M) = \{m \in M : rm = 0$ for some nonzero $r \in R\}$ is called a torsion submodule of

M. If $T(M)=0$, then M is called a torsion free module [16].

The converse of Proposition 1.6 is not true in general, as the following example shows.

Example 1.7 Let $M=Z\oplus Z$ as Z-module. Since M is a torsion free module over Z, thus by [2, Cor. 2.3.2] M has the PIP. But M is not a pure multiplication as Z-module [(see Rem.and. Ex. 1.1) (v)] .

Recall that an R-module M is called scalar if, for each $f\in End_R(M)$,there exists $a\in R$ such that $f(m)=am$ for all $m\in M$ [17] .

Proposition 1.8 Let M be a scalar R-module with $ann_R M$ is prime in R, then $S=End_R(M)$ is a pure multiplication ring .

Proof. By [14, Lemma 6.2, p.80], we have that $S=End_R(M)\cong R/ann_R M$, since M is a scalar R-module. On the other hand, $ann_R M$ is prime, implies $R/ann_R M$ is an integral domain, then $R/ann_R M$ is a commutative ring with identity; that is $S=End_R(M)$ a commutative ring with identity, and hence by [(Rem.and.Ex. 1.1)($viii$)] $S=End_R(M)$ is a pure multiplication ring . □

Corollary 1.9 If M is a prime scalar R-module, then $S=End_R(M)$ is a pure multiplication ring.

Proof. It is easy to check . □

Corollary 1.10 Let M be a faithful scalar R-module. Then R is a pure multiplication ring if and only if $S=End_R(M)$ is pure multiplication.

Proof. Since M is a scalar R-module, then $S=End_R(M)\cong R/ann_R M$ by [14, Lemma 6.2, p.80], but M is a faithful R-module, therefore $S=End_R(M)\cong R/(0)\cong R$. Hence the result is obtained . □

A submodule N of a module M is said to be essential if, every nonzero submodule of M has nonzero intersection with N. Recall that an R-module M is called uniform if, $M\neq 0$ and every nonzero submodule of M is essential in M [9]. An R-module M is said to be essentially quasi-Dedekind if, $Hom_R(M/N,M)=0$ for all essential submodule N of M , that is; every essential submodule of M is quasi-invertible [8].

Proposition 1.11 Let M be a pure multiplication uniform module over a regular ring R, then every finitely generated submodule of M is equal to zero .

Proof. Since R is a regular ring, thus M is a regular R-module. If M is a pure multiplication module over a regular ring R, then by [Rem. and.Ex 1.1(vi)] M is a multiplication module over regular ring R, thus by [15, Cor. 3.8] $End_R(M)$ is a regular ring, so by [8, Prop 2.3.8] M is essentially quasi-Dedekind, and so every essential submodule of M is quasi-invertible. Thus every nonzero submodule of M is quasi-invertible, since M is uniform, and hence every nonzero submodule of M is not direct summand. On the other hand, since $End_R(M)$ is a regular ring, thus by [15, Th. 3.3] every finitely generated submodule of M is a direct summand, so every finitely generated submodule of M is equal to zero . □

Corollary 1.12 Let M be a pure multiplication uniform module over a regular ring R. If M is a finitely generated R-module, then $M=0$.

In the next, we will investigate the behavior of pure submodules and pure multiplication modules under localization. Before that, we need the following Lemma.

Lemma 1.13 Let M be an R-module, $N \leq M$ and let S be a multiplicative closed subset of R. If N is pure in M as R-module, then $S^{-1}N$ is pure in $S^{-1}M$ as $S^{-1}R$-module. The converse hold whenever $S^{-1}A = S^{-1}B$ *iff* $A = B$ for any R-modules A, B.

Proof. If N is a pure submodule of M as R-module, so $IN = N \cap IM$ for each ideal I of R, then $S^{-1}(IN) = S^{-1}(N \cap IM) = S^{-1}N \cap S^{-1}IM$,so $(S^{-1}I)(S^{-1}N) = S^{-1}(N) \cap (S^{-1}I)(S^{-1}M)$ for all ideal $S^{-1}I$ of $S^{-1}R$, therefore $S^{-1}N$ is a pure submodule of $S^{-1}M$ as $S^{-1}R$-module.

Conversely, assume that $S^{-1}N$ is pure in $S^{-1}M$ as $S^{-1}R$-module, so for every ideal $S^{-1}J$ of $S^{-1}R$, $(S^{-1}J)(S^{-1}N) = S^{-1}N \cap (S^{-1}J)(S^{-1}M)$, thus $S^{-1}JN = S^{-1}N \cap S^{-1}JM = S^{-1}(N \cap JM)$ so by assumption, $JN = N \cap JM$ for every ideal J of R. Thus N is pure in M as R-module . □

Theorem 1.14 Let M be an R-module and let S be a multiplicative closed subset of R such that $S^{-1}A = S^{-1}B$ *iff* $A = B$ for any R-modules A, B. Then M is a pure multiplication as R-module if and only if $S^{-1}M$ is a pure multiplication as $S^{-1}R$-module.

Proof. Assume that M is a pure multiplication as R-module. Let $S^{-1}N$ be a pure submodule of $S^{-1}M$ as $S^{-1}R$-module, then by Lemma 1.13, N is a pure submodule of M as R-module, $N = IM$ for some ideal I of R, since M is a pure multiplication module. Hence $S^{-1}N = S^{-1}(IM) = (S^{-1}I)(S^{-1}M)$ for some ideal $S^{-1}I$ of $S^{-1}R$. Thus $S^{-1}M$ is a pure multiplication as $S^{-1}R$-module.

Conversely, let N be a pure submodule of M as R-module, so by Lemma 1.13, $S^{-1}N$ is a pure submodule of $S^{-1}M$ as $S^{-1}R$-module, but $S^{-1}M$ is a pure multiplication module, then there exists an ideal $S^{-1}I$ of $S^{-1}R$ such that $S^{-1}N = (S^{-1}I)(S^{-1}M) = S^{-1}(IM)$, so $N = IM$, by assumption, for some ideal I of R. Thus the result is obtained. □

Corollary 1.15 Let M be an R-module. Then M is a pure multiplication as R-module if and only if M_P is a pure multiplication as R_P-module, for every maximal ideal P of R.

2. Pure Submodules Of Pure Multiplication Modules

In this section, our objective is to investigate pure submodules of pure multiplication modules. We give some properties of this concept of submodules, also we study the direct summand of pure multiplication module.

However, we begin this section with another proof of following Proposition which appeared in [3, Prop. 2.6(*ii*)].

Proposition 2.1 Every pure submodule of a pure multiplication module is pure multiplication.

Proof. Suppose that N is a pure submodule of a pure multiplication R-module M. Let K be a pure submodule of N, then by [19, notes 2.7(1), p.15] K is a pure submodule of M, but M is a pure multiplication module, then there exists an ideal I of R such that $K = IM$. Also, since N is pure in M, then $IN = N \cap IM = N \cap K = K$. Hence N is a pure multiplication R-module . □

The converse of Proposition 2.1 need not be true in general, as the following example shows: in $M = Z \oplus Z$ as Z-module, we have $Z \oplus (0)$ is a proper non-trivial pure submodule. On the other hand, we have $Z \oplus (0) \cong Z$ which is a pure multiplication as Z-module, but $Z \oplus Z$ as Z-module is not pure multiplication [see (Rem. and.Ex.1.1) (*iii*), (*v*)] .

Corollary 2.2 Let M be an R-module. Then M is pure multiplication if and only if every direct summand of M is pure multiplication.

Proof. Let M be a pure multiplication R-module. Suppose that N is a direct summand of M, so by [19, notes 2.7(3), p.15] N is pure in M. Hence, the result is obtained by Proposition 2.1.

Conversely, follows by taking $N \leq^{\oplus} M$ and $N = M$. □

Corollary 2.3 Every pure ideal of a ring R is a pure multiplication R-module .

Proof. It follows by [(Rem.and.Ex. 1.1)(*viii*)] and Proposition 2.1 . □

Let M be an R-module and N, L submodules of M. Consider the set $N * L := Hom(M, L)N = \sum\{\varphi(N) | \ \varphi : M \to L\}$. A submodule N of M is called idempotent if, $N * N = N$ [12]. Equivalently, a submodule N of M is called idempotent if and only if $N = (N :_R M)N$. An ideal I of a ring R is called idempotent if, $I^2 = I$. Then I is an idempotent submodule of $_R R$ if and only if I is an idempotent ideal of R.

Proposition 2.4 Let M be a pure multiplication R-module. Then every pure submodule of M is an idempotent submodule .

Proof. Suppose that N is a pure submodule of M. Since M is a pure multiplication module, thus by [3, p.35] we have that $N = (N :_R M)M$. On the other hand, N is pure in M and $(N :_R M)$ is an ideal of R, $(N :_R M)N = N \cap (N :_R M)M$; this means $(N :_R M)N = N$, and hence N is an idempotent submodule. □

We need the following Lemma to prove the next result .

Lemma 2.5 Let M be a pure multiplication cancellation R-module such that N is a pure submodule of M, then $I(N :_R M) = I \cap (N :_R M)$ for each ideal I of R .

Proof. Let N be a pure submodule of M, then $IN = N \cap IM$ for every ideal I of R. It is clear that $I(N :_R M) \subseteq I \cap (N :_R M)$ .Conversely, $I \cap (N :_R M) \subseteq (IM :_R M) \cap (N :_R M) = (IM \cap N :_R M) = (IN :_R M)$, but $(IN :_R M) \subseteq I(N :_R M)$; to see this: let $a \in (IN :_R M)$ then $a.M \subseteq IN$, but $N = (N :_R M)M$, thus $a.M \subseteq I(N :_R M)M$, hence $a \in I(N :_R M)$, since M is cancellation. Thus $I \cap (N :_R M) \subseteq I(N :_R M)$ and hence the result is obtained . □

Proposition 2.6 Let M be a pure multiplication cancellation R-module such that N is a pure submodule of M, then $(N :_R M)$ is an idempotent ideal of R .

Proof. By Lemma 2.5, we have $(N :_R M)^2 = (N :_R M).(N :_R M) = (N :_R M) \cap (N :_R M) = (N :_R M)$. Hence $(N :_R M)$ is an idempotent ideal of R. □

A submodule N of an R-module M is called characteristic if, for all automorphisms φ of M, $\varphi(N) = N$ [5]. And, we say that a submodule N of a module M is fully invariant if $\varphi(N) \subseteq N$ for each endomorphisms φ of M [18].

Proposition 2.7 Let M be a pure multiplication R-module. Then every pure submodule of M is a characteristic submodule.

Proof. Suppose that N is a pure submodule of M, so there exists an ideal I of R such that $N = IM$. Let $\varphi \in Aut(M)$, φ is an epimorphism, and so $\varphi(N) = \varphi(IM) = I\varphi(M) = IM = N$. Thus N is characteristic. □

The converse of above Proposition need not be true in general, as the following example: consider Z as Z-module and $N = 2Z$. It is well known that Z is a pure multiplication Z-module, also N is characteristic in Z, but N is not pure in Z. In fact, if $I = 2Z$ is an ideal of Z, then $2 \in N \cap IM = 2Z \cap (2Z)$, but $2 \notin IN = 2.(2Z)$.

Corollary 2.8 Let M be a pure multiplication R-module and N be a pure submodule of M. Then every pure submodule of M/N is a characteristic submodule.

Proof. By [3, Prop. 2.6(*ii*)], we have that M/N is a pure multiplication R-module. Thus by Proposition 2.7, the result follow. □

Proposition 2.9 Let M be a pure multiplication R-module. Then every pure submodule of M is a fully invariant submodule.

Proof. Analogous proof of Proposition 2.7. □

REFERENCES

[1] Abd EL-Bast Z., Smith P.F. ,1988, Multiplication Modules , Comm. In Algebra, 16, P. 755-779.

[2] Al-Bahrany B., 2000, Modules With The Pure Intersection Property, Ph.D. Thesis, Univ. of Baghdad.

[3] Atani S.E., 2008, Indecomposable Weak Multiplication Modules Over Dedekind Domains, Demonstratio math., Vol. XLI, No.1, P. 33-43.

[4] Branard A., 1981, Multiplication Modules, J. of Algebra, 71, P. 174-178 .

[5] Brandt J., 1982, Characteristic Submodules, J. of London. Math. Soc (2), 25, P. 35-38.

[6] Fieldhouse D.J., 1970, Pure Simple And Indecomposable Rings, Can.Math.Bull, 13, P. 71-78.

[7] Fieldhouse D.J., 1969, Pure Theories, Math. Ann, 184, P.1-18.

[8] Ghawi, Th.Y., 2010, Some Generalizations Of Quasi-Dedekind Modules, M.Sc. Thesis, College of Education Ibn Al-Haitham, Univ. of Baghdad.

[9] Goodearl K.R. ,1976, Ring Theory, Maracel Dekker, Nework.

[10] Hiraro Y., Mogami I., 1986, On Restricted Anti-hopfian Modules, Math.J.Okayama Univ., 28, P. 119-131.

[11] Kasch F. , 1982, Modules And Rings, Academic press, New York.

[12] Keskin D., Orhan N., Tribak R. and Smith P.F., 2011, On Fully Idempotent Modules, Comm. Algebra, 39, P. 2707-2722.

[13] Mijbass A.S. ,1992, Cancellation Modules, M.Sc. Thesis, Univ. of Baghdad.

[14] Mohamed-Ali E.A. ,2006, On Ikeda-Nakayama Modules, Ph.D. Thesis, Univ. of Baghdad.

[15] Naoum A.G., 1995, Regular Multiplication Modules, Periodica Math, Hungarica, Vol.31(2), P. 155-162.

[16] Sharpe D.W., Vamos P. ,1972, Injective Modules, Lectures in pure math., Combridge Unvi. Press.

[17] Shihab B.N. , 2004, Scalar Reflexive Modules, Ph.D. Thesis, Univ. of Baghdad.

[18] Wisbauer R., 1991, Foundations Of Modules and Ring Theory, reading, Gordon and Breach Science Publishers.

[19] Yaseen S.H. ,1985, F-Regular Modules, M.Sc. Thesis, Univ. of Baghdad.

حول المقاسات الجدائية النقية

تاريخ الاستلام 2016/1/11 **تاريخ القبول 2016/4/18**

ثائر يونس غاوي

كلية التربية– قسم الريّاضيات

جامعة القادسّـــــية – العراق

البريد الالكتروني : thaar_math83@yahoo.com

الكلمات المفتاحية :المقاسات الجدائية النقية، المقاسات الجزئية النقية، مقاسات الحذف.

المستخلص:

لتكن R حلقة أبدالية ذات عنصر محايد وليكن M مقاساً احادي ايسر على الحلقة R. يقال للمقاس الجزئي N من المقاس M على الحلقة R بأنه نقي إذا كان $IN = N \cap IM$ لكل مثالي I من الحلقة R. في بحثنا هذا تحققنا ودرسنا بعض الخصائص المختلفة لمفهوم المقاسات الجدائية النقية، و الذي ظهر في [3] ، كأعمام فعلي لمفهوم المقاسات الجدائية ، حيث يقال للمقاس M على الحلقة R بأنه مقاس جدائي نقي إذا كان لكل مقاس جزئي نقي N من M على الحلقة R يوجد مثالي I من R بحيث $N = IM$. أعطينا عدد من النتائج المعتبرة للمقاسات الجدائية النقية . كذلك ناقشنا البعض من خصائص المقاسات الجزئية النقية في المقاسات الجدائية النقية.