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ON THE PRIMARY SPECTRUM OF A MODULE AND A PRIMARY FUL MODULE

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Abstract

For any module M over a commutative ring R, Prim (M) is the collection of all primary submodules.

In this research we investigate the interplay between the topological properties of Prim (M) and module theoretic properties of M .

Also, for various types of modules M, we obtain some conditions under which Prim (M) is homomorphic with the Primary ideal space of some ring.

Key word : (primary submodule – Zariski topology – primary surjective module)

Mathematics Classification QA 440 - 699

1. Introduction

Throughout this research, R is a commutative ring with non zero identity and M is a unitary R-module. For any ideal I of R containing AnnR (M), $\overline{R}$ and $\overline{I}$ denote R/Ann(M) and I/Ann(M) respectively. Moreover the notation ”⊂” will denote the strict inclusion.

For M as an R-module and N a submodule, we recall the colon ideal of M into N ,

$$(N : M) = \{r \in R|rM \subseteq N \}.$$

A submodule P of M is said to be a primary submodule if $xm \in P$ for $x \in R$ and $m \in M$ imply that either $m \in P$ or $x^n \in (P:M)$ for some positive integer n[1]. If P is a primary submodule, then (P : M) is a primary ideal of R.

Let Prim (M) is the collection of all primary submodules of M.

If Prim (M) ≠ ∅,the mapping ψ: Prim(M)→ Prim($\overline{R}$) such that $\psi(P) = (\overline{P : M})$ for every P ∈ Prim (M) , is called the natural map of Prim (M).

M is said to be primary ful if either M=(0) or M ≠(0) and the natural map of Prim (M) is surjective.

M is said to be X -injective if either Prim (M) = ∅ or Prim (M) ≠ ∅ and the natural map of Prim(M) is injective .

The Zariski topology on X = Prim (M) is the topology τM described by taking the set

Z (M) = {VM (N)|N is a submodule of M }

as the set of closed sets of X , where

VM (N) = {P ∈ X |(P : M) ⊇ (N : M)}.

When M = R, τM = τR is the well known Zariski topology on Prim (R) [2].

In the rest of this research Prim (M) is always equipped with the zariski topology τM .

The present authors introduced the concept of Primary-injective modules and investigated some important properties of this family of modules. An R-modules M is called Primary-injective if the natural map of Prim (M) is injective [3].

A topological space W is said to be Primary-spectral if it is homeomorphic with the Primary ideal space of some ring (see Definition 3.17). Primary-spectral spaces have been characterized by Hochster in [4, Proposition 11].

In this research, we investigate the interplay between the topological properties of Prim (M) and module theoretic properties of M (see Proposition 3.2, Theorem 3.6, Theorem 3.13, Corollary 3.15, Proposition 3.19, and Theorem 3.22). Theorem 3.14 provides useful information about the relationship between topological properties of Prim (M) and Prim ($\overline{R}$). Also we consider the conditions under which Prim (M) is a Noetherian topological space (see Proposition 3.2, Theorem 3.6, Theorem 3.14, and Corollary 3.15). Moreover, we study the topological space Prim (M) from the point of view of Primary-spectral spaces (see Theorem 3.22). It is shown that if M is a Primary-injective module over a PID, then Prim (M) is a Primary-spectral topological space (see Theorem 3.22 (g)). These results enable us to provide a large family of modules such that their Primary submodules are Primary-spectral.

2. Preliminaries

In this section we review some preliminary results which will be needed in next section.

Definition 2.1. For a topological space X , we recall

(a) X is quasi compact if it satisfies one of the following two equivalent conditions.

(1) Every collection of open subsets whose union is X contains a finite subcollection whose union is X .

(2) Every collection of closed subsets whose intersection is empty set contains a finite subcollection whose intersection is empty set (see [5,Definition 2.135]).

(b) X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (or maximal condition). (see [6, Chap. 6, Example 5]).

(c) X is said to be connected if it is not the union

$X = X_0 \cup X_1$ of two disjoint closed non-empty subsets X_0 and X_1 (see [5, Definition 2.105]).

(d) X is said to be irreducible if X is not the union of two proper closed subsets.

For $X' \subseteq X$, X' is irreducible if it is irreducible as a space with the relative topology. This is equivalent to say that, if F , G are closed subsets of X such that $X' \subseteq F \cup G$, then $X' \subseteq F$ or $X' \subseteq G$ (see [7, Ch. II]).

(e) A maximal irreducible subset of X is called an irreducible component of X . It is well known that every irreducible component of X is closed in X (see [7, Ch. II]).

Remark 2.2. Let X and Y be two topological spaces.

(a) Let f be a continuous mapping from X to Y .

(1) If X is a connected (resp. quasi compact) topological space, then f (X) is a connected (resp. quasi compact) topological space (see [5,Theorem 2.107 and Theorem 2.138]).

(2) For every irreducible subset E of X , f (E) is an irreducible subset of Y (see [7, Ch. II]).

(b) If X is a Noetherian topological space, then every subspace of X is a Noetherian topological space, and X is a quasi compact topological space (see [6, Chap. 6, Exc. 5]).

(c) Every Noetherian topological space has only finitely many irreducible com-ponents (see [7, Proposition 10]).

(d) Closed subspaces of quasi compact topological spaces are quasi compact (see [5, Theorem 2.137]).

(e) Every finite topological space is quasi compact (see [5]).

(f) Closure of any connected (resp. irreducible) subspace is connected (resp. irreducible) (see [5, Corollary 2.112] and [7, Ch. II]).

(g) Let A and B be subsets of X such that A ⊆ B ⊆ X , where B is closed in X and equipped with the relative topology. Then A is an irreducible closed subset of B if and only if A is an irreducible closed subset of X (see Definition 2.1 (d)).

3. Main results

As it was mentioned before, Prim (M) is always equipped with Zariski topology τM .

Lemma 3.1. Let M be an R-module and let φ : Prim (M) → Prim ($\overline{R}$) be the natural map of Prim (M). Then the following hold.
(a) φ is a continuous map.
(b) If M is Primary-surjective, then φ is closed and open mapping.

Proof. (a) This follows from the fact that $\varphi^{-1}(V\overline{R}(\overline{I}))$ = VM (IM) for every ideal I of R containing Ann(M).
(b) Let N be a submodule of M and let VM(N) be a closed subset of Prim (M).
Then as in the proof part (a), we have $\varphi^{-1}(V\overline{R}((\overline{N : M})))$ = VM((N : M)M) = VM(N).
Hence φ(VM(N)) = $V\overline{R}((\overline{N : M}))$ because φ is surjective. Also φ is open by similar arguments and the proof is completed.
A topological space W is a cofinite topological space when its open sets are empty and W and all subsets with a finite complement. This topology is denoted by τ f c.

Proposition 32. Let R be a ring such that the intersection of every infinite collection of Primary ideals of R is zero (for example, when R is PID or one dimensional Noetherian domain) and let M be an R-module. Then Prim (M) is a Noetherian topological space.

Proof. Let VM(N) be a closed subset of Prim (M) for some submodule N of M . If VM(N) is infinite, then (N : M) is contained in an infinite number of Primary ideals of R. Since the intersection of every infinite collection of Primary ideals of R is zero, (N : M) = (0) so that VM(N) = Prim (M). It follows that $\tau M \subseteq \tau$ f c and hence Prim (M) is a Noetherian topological space because every cofinite topological space is Noetherian.

Notation 3.3. Let M be an R-module and W be a subset of Prim (M). We will denote the intersection of all elements in W by $\Im$(W) and the closure of W in Prim (M)by Cl(W).

Lemma 3.4. Let M be an R-module and W be a subset of Prim (M).
Then Cl (W) = VM ($\Im$ (W)). Hence, W is closed if and only if VM ($\Im$ (W)) = W .

Proof. Let W be a subset of Prim (M). It is well known that Cl(W) = Cl(W) ∩ Prim (M).
But Cl(W) = V ($\Im$ (W)) by [11, Proposition 5.1]. It follows that Cl(W) = VM($\Im$ (W)).

For a proper ideal I of R, we recall that the Primary-radical I , denoted by PJ(I), is the intersection of all Primary ideals containing I . An ideal I of R is a Primary-radical ideal if I = PJ(I).

Definition 3.5. Let M be an R-module. The Primary -radical of a submodule N of M , denoted by PJ (N), is the intersection of all members of VM(N). In case that VM (N) = ∅, we define PJ(N) = M. A submodule N of M is said to be a PJ-radical submodule if N =PJ(N).

Theorem 3.6. Let M be an R-module. Then the following are equivalent.
(a) Prim (M) is a Noetherian topological space.
(b) The ascending chain condition for PJ-radical submodules of M holds.

Proof. (a)⇒(b) Straightforward.
(b)⇒(a) Let
$VM(N1) \supseteq VM(N2) \supseteq \cdots \supseteq VM(Ni) \supseteq \cdots$ be a descending chain of closed sets VM(Ni) of Prim (M), where Ni is a submodule of M . Hence PJ (N1) ⊆ PJ (N2) ⊆ · · ⊆ PJ (Ni) ⊆ · · is an ascending chain of PJ-radical

submodules of M. So by hypothesis, there exists a $k \in N$ such that for all $n > k$, we have PJ (Nk+n) = PJ (Nk). Now by using Lemma 3.4, for all $n > k$,
VM (Nk+n) = VM (Nk) and the proof is completed.

Corollary 3.7. Let M be a Noetherian R-module. Then Prim (M) is a Noetherian topological space.

We recall that if I is an ideal of R, then the PJ-components of I are the minimal members of the family of PJ-radical primary ideals containing I .

Definition 3.8. Let M be an R-module and L a submodule of M . A submodule P of M is a PJ-component of L, if (P : M) is a PJ-component of (L : M). Clearly, this definition is the generalization of PJ-component of an ideal in rings.

Definition 3.9. A module M is said to have property (PJFC) if every closed subset of Prim (M) has a finite number of irreducible components.

Example 3.10. Let M be an R-module. Then M has property (PJFC) in each of the following cases:
(a) Prim (M) is a Noetherian topological space (see parts (b) and (c) of Remark 2.2);
(b) R is PID (see Proposition 3.2 and part (a));
(c) M is Noetherian (see Corollary 3.7 and part (a));
(d) M is semi local (see Remark 2.2 (e) and part (a)).
When M is the R-module R, then R has property (PJFC) if and only if every ideal of R has a finite number of PJ-components . Theorem 3.13(d) extends the this property for modules.
The proof of the following lemma is easy and is omitted.

Lemma 3.11. Let M be a Primary-surjective R-module. Then the following hold.
(a) If N is a submodule of M , then

$$PJ((N : M)) = (PJ(N) : M).$$

(b) If q is a PJ-radical ideal of R containing AnnR (M), then there exists a submodule Q of M such that (Q : M) = q.

Remark 3.12. If S is a commutative ring with non zero identity, then there exists a one-to-one correspondence between the PJ-radical primary ideals of ring S and irreducible closed subsets of Prim (S) .

Theorem 3.13. Let M be a Primary-surjective R-module. Then the following hold.
(a) If $Y \subseteq$ Prim (M), then Y is an irreducible closed subset of Prim (M) if and only if Y = VM(N) for some submodule N of M such that (N : M) is a PJ-radical primary ideal of R.
(b) If $W \subseteq$ Prim (M) and L is submodule of M , then W is an irreducible component of VM(L) if and only if W = VM(N′) for some PJ-component N′ of L.
(c) If $Z \subseteq$ Prim (M), then Z is an irreducible component of Prim (M) if and only if Z =VM(pM) for some PJ-component ideal p of AnnR (M).
(d) M has property (PJFC) if and only if every submodule of M has a finite number of PJ-components.

Proof. (a) (⇒) Let Y be an irreducible closed subset of Prim (M). Since Y is closed, Y = VM(N) for some submodule N of M . It turns out that φ(VM(N)) =$V\bar{R}$ (($\overline{N : M}$)) is an irreducible closed subset of Prim ($\bar{R}$) by Lemma 3.1 and Remark 2.2 (a).
Now by Remark 3.12, ($\overline{N : M}$) is a PJ-radical primary ideal of $\bar{R}$ so that (N : M) is a PJ-radical primary ideal of R. Conversely, let VM(K) be a closed subset of Prim (M), where K is a submodule of M such that (K : M) is a PJ-radical primary ideal of R. We show that VM(K) is irreducible. To see this, let E and E' be submodules of M with

$$VM(K) \subseteq VM(E) \cup VM(E')$$

Hence as in the proof of Lemma 3.1 (b), we have
$V\bar{R}$ (($\overline{K : M}$)) $\subseteq$ $V\bar{R}$ (($\overline{E : M}$)) $\cup$ $V\bar{R}$ (($\overline{E' : M}$)).

Since (K : M) is a PJ-radical primary ideal of R, it is easy to check that $(\overline{K : M})$ is a PJ-radical primary ideal of $\overline{R}$. Therefore $V\overline{R}$ $((\overline{K : M}))$ is an irreducible closed subset of Prim $(\overline{R})$ by Remark 3.12. Hence by Definition 2.1 (d),
$V\overline{R}((\overline{K : M})) \subseteq V\overline{R}((\overline{E : M}))$ or $V\overline{R}((\overline{K : M})) \subseteq V\overline{R}((\overline{E' : M}))$.
Suppose that $V\overline{R}((\overline{K : M})) \subseteq V\overline{R}((\overline{E : M}))$. This implies that VM(K) ⊆ VM(E).

By similar arguments, VM(K) ⊆ VM(E') when $V\overline{R}((\overline{K : M})) \subseteq V\overline{R}((\overline{E' : M}))$.
(a) (⇒) Let W be an irreducible component of VM(L). By Definition 2.1 (e) and Remark 2.2 (g),
W is an irreducible closed subset of Prim (M). So by part (a),W = VM(N1′) for some submodule N1′ of M such that (N1′ : M) is a PJ-radical primary ideal of R. We claim that N1 is a PJ-component of L or equivalently, (N1′ : M) is a PJ-component of (L : M).
Clearly (N1′ : M) ⊇ (L : M) by using Lemma 3.11 (a). So by the above arguments, it is enough to show that (N1′: M) is a minimal member of the family of PJ-radical primary ideals containing (L : M). To see this, let q be a PJ-radical primary ideal of R with (L : M) ⊆ q ⊆ (N1′ : M).
Since M is Primary-surjective, there exists a submodule Q of M such that q = (Q : M) by Lemma 3.11 (b). Hence
VM(L) ⊇ VM(Q) ⊇ VM(N1′).
Also VM(Q) is an irreducible closed subset of VM(L) by part (a), and Remark 2.2 (g).
Since W = VM(N1′) is an irreducible component of VM(L), by the above arguments, we have VM(Q) = VM(N1′).
Now by using Lemma 3.11 (a), q = (N1′ : M) as desired.
(⇐) Let N2′′ be a PJ-component of L. Then VM(N2′′) is an irreducible closed subset of VM(L) by part (a) and Remark 2.2 (g). Let L′ be a submodule of M such that
(L′ : M) is a PJ-radical primary ideal of R and
VM(N2′′) ⊆ VM(L′) ⊆ VM(L).
Since N2′′ be a PJ-component of L, by using Lemma 3.11 (a), we have VM(N2′′) =VM(L′)
as required.
(c) This follows from part (b) and Lemma 3.11 (b) and the fact that if N is a submodule of M , then
VM((N : M)M) = VM(N).
(d) Follows from part (b).

Let X be a topological space. We consider strictly decreasing chain $Z_0, Z_1,...,Z_r$ of length r of irreducible closed subsets Z_i of X . The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by dim(X). For the empty set, ∅, the combinatorial dimension of ∅ is defined to be −1.

Theorem 3.14. Let M be a Primary-surjective R-module. Then the following hold.
(a) Prim (M) is a Noetherian topological space if and only if Prim $(\overline{R})$ is a Noetherian topological space.

(b) Prim (M) is a connected topological space if and only if Prim $(\overline{R})$ is a connected topological space.
(c) Prim (M) is an irreducible topological space if and only if Prim $(\overline{R})$ is an irreducible topological space.
(d) Prim (M) is a quasi-compact topological space if and only if Prim $(\overline{R})$ is a quasi-compact topological space.
(e) dim(Prim (M)) = dim(Prim $(\overline{R})$).
Proof. Let φ : Prim (M) → Prim $(\overline{R})$ be the natural map of Prim (M).
(a) (⇒) Let $V\overline{R}(\overline{I}_1) \supseteq V\overline{R}(\overline{I}_2) \supseteq ... \supseteq V\overline{R}(\overline{I}_i) \supseteq ...$ be a descending chain of closed sets in Prim $(\overline{R})$, where each $\overline{I}_i$ is an ideal of $\overline{R}$. Since φ is continuous by Lemma 3.1 (a),
$\varphi^{-1}(V\overline{R}(\overline{I}_1)) \supseteq \varphi^{-1}(V\overline{R}(\overline{I}_2)) \supseteq .. \supseteq \varphi^{-1}(V\overline{R}(\overline{I}_i)) \supseteq ...$
is a descending chain of closed sets in Prim (M). By hypothesis, there exists a t ∈ N such that for all n > t, $\varphi^{-1}(V\overline{R}(\overline{I}_{t+n})) = \varphi^{-1}(V\overline{R}(\overline{I}_t))$. Hence for all n > t, we have $V\overline{R}(\overline{I}_t +n) = V\overline{R}(\overline{I}_t)$ because φ is surjective. Therefore, Prim $(\overline{R})$ is a Noetherian topological space.

To show the converse, by Theorem 3.6, it is enough to show that the ascending chain condition for PJ-radical submodules of M holds. To see this, let

$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$

be an ascending chain of PJ-radical submodules of M . Then by Lemma 3.11 (a), one can see that

$(\overline{N_1 : M}) \subseteq (\overline{N_2 : M}) \subseteq \cdots \subseteq (\overline{N_i : M}) \subseteq \cdots$

is an ascending chain of PJ-radical ideals of $\overline{R}$. So by Theorem 3.6, there exists a $k \in N$ such that for all $n > k$, $(\overline{N_{k+n} : M}) = (\overline{N_k : M})$. Hence for all $n > k$,

$VM(N_{k+n}) = VM((N_{k+n} : M)M) = VM((N_k : M)M) = VM(N_k)$.

So for all $n > k$, we have

$N_{k+n} = PJ(N_{k+n}) = PJ(N_k) = N_k$, as desired.

(b) First assume that Prim (M) is a connected topological space. Then Prim $(\overline{R}) = \varphi$(Prim (M)) is connected by Lemma 3.1 and Remark 2.2 (a). To see the reverse implication, we assume that Prim $(\overline{R})$ is a connected topological space. If Prim (M) is a disconnected topological space, then there exist submodules N and K of M such that

Prim (M) = VM(N) ∪ VM(K) and

VM(N) ∩ VM(K) = ∅,

where VM(N) = ∅, and VM(K) = ∅. Hence as in the proof of Lemma 3.1 (b), we have

Prim $(\overline{R}) = V\overline{R}((\overline{N : M})) \cup V\overline{R}((\overline{K : M}))$.

It is easy to cheek that

$V\overline{R}((\overline{N : M})) \cap V\overline{R}((\overline{K : M})) = \emptyset$, $V\overline{R}((\overline{N : M})) \neq \emptyset$, and $V\overline{R}((\overline{K : M})) \neq \emptyset$.

Therefore Prim $(\overline{R})$ is a disconnected topological space, a contradiction. Hence Prim (M) is a connected topological space.

(c) We have similar argument as in part (b).

(d) (⇒) This follows from Lemma 3.1 (a) and Remark 2.2 (a). To show the converse, let $\{VM(N_\alpha) : \alpha \in \Lambda\}$ be a family of closed subset of Prim (M) such that $\cap_{\alpha\in\Lambda} VM(N_\alpha) = \emptyset$, where Nα is a submodule of M for every $\alpha \in \Lambda$. Then $\{\varphi(VM(N_\alpha)) : \alpha \in \Lambda\}$ is a family of closed subset of Prim $(\overline{R})$ because φ is closed by Lemma 3.1 (b). Since φ is surjective, it is easy to see that

$\cap_{\alpha\in\Lambda}\varphi(VM(N_\alpha)) = \emptyset$.

As Prim $(\overline{R})$ is quasi compact, there exists a finite subset Γ of Λ such that $\cap_{\alpha\in\Gamma}\varphi(VM(N_\alpha)) = \emptyset$. This implies that $\cap_{\alpha\in\Gamma}VM(N_\alpha) = \emptyset$ and hence Prim (M)is quasi compact.

(e) Let $Z_0 \supset Z_1 \supset \ldots \supset Z_n$ be a descending chain of irreducible closed subset of Prim (M). Then by Theorem 3.13 (a), for i $(1 \le i \le n)$, there exists submodule Li of M such that $(L_i : M)$ is a PJ-radical primary ideal of R and $Z_i = VM(L_i)$. It follows that

$V\overline{R}((\overline{L_0 : M})) \supset V\overline{R}((\overline{L_1 : M}))\ldots \supset V\overline{R}((\overline{L_n : M}))$

is a descending chain of irreducible closed subset of Prim $(\overline{R})$ by Remark 3.12.

Hence dim(Prim (M)) ≤ dim(Prim $(\overline{R})$). Now let

$$A_0 \supset A_1 \supset \ldots \supset A_t$$

be a descending chain of irreducible closed subset of Prim $(\overline{R})$. By Remark 3.12, for each i $(1 \le i \le t)$, there exists a PJ-radical primary ideal $\overline{p}_i$ of $\overline{R}$ such that

$A_i = V\overline{R}(\overline{p}_i)$.

This yields that $p_0 \subset p_1 \subset \ldots \subset p_t$

is an ascending chain of PJ-radical primary ideal of R. Since M is Primary-surjective, by Lemma 3.11 (b), for every pi $(1 \le i \le t)$, there exists a submodule Qi of M such that pi = $(Q_i : M)$.

Hence by Theorem 3.13 (a),

$VM(Q_0) \supset VM(Q_1) \supset \ldots \supset VM(Q_t)$

is a descending chain of irreducible closed subset of Prim (M). It follows that dim(Prim (M)) ≥ dim(Prim $(\overline{R})$) and the proof is completed.

Corollary 3.15. Let M be a Primary-surjective R-module. Then the following hold.

(a) If R is Noetherian, then Prim (M) is a Noetherian topological space.

(b) If Ψ is the family of all PJ-radical primary ideal of R, then we have dim(Prim (M)) = sup$\{n | p_0 \subset p_1 \subset \ldots \subset$ pn is an ascending chain of Ψ }.

Proof. (a) Follows from Theorem 3.14 (a).
(b) Apply the technique of Theorem 3.14 (e).

Remark 3.16. We recall that an R-module M is a Hilbert module if every primary submodule in M is the intersection of all the Prim submodules containing it. For example, every finitely generated divisible module over an integral domain is a Hilbert module (see [8, p. 2]). Let M be a Hilbert R-module. If Prim (M) is connected (resp. irreducible) topological space, then $Spec_R$ (M) is connected (resp.irreducible) topological space. Since if M is Hilbert, by [2, Proposition 5.1] it is easy to see that Cl($Spec_R$ (M)) = Prim (M). Now the result follows from the Remark 2.2 (f).

Definition 3.17.We say that a topological space W is a Primary-spectral space if W is homeomorphic with the Primary ideal space of some ring S.

Remark 3.18. Primary-spectral spaces have been characterized by Hochster [4, p.57,Proposition 11] as the topological spaces W which satisfy the following conditions:
(a) W is a T1 space;
(b) W is quasi-compact.

Proposition 3.19. Let M be an R-module. Then the following are equivalent.
(a) M is Primary-injective.
(b) Prim (M) is a T_0 space.
(c) Prim (M) is a T_1 space.
(d) Prim (M) is a T_2 space.
Proof. Straightforward.

Corollary 3.20. Let M be an R-module.
(a) If Prim (M) is a Primary-spectral topological space, then M is Primary-injective.
(b) If M is primaryful and Prim (M) is a Primary-spectral topological space, then $Spec_R$ (M) = Prim (M).

Proof. This follows from Remark 3.18, Proposition 3.19, and [9, Theorem 4.3].

Let M be an R-module such that Prim (M) is a Primary-spectral topological space. For a submodule N of M , it is natural to ask the following question: Is Prim (M/N) a Primary-spectral topological space?
In Proposition 3.21 (c), we give a positive answer to this question under some additional conditions.

Proposition 3.21. Let M be an R module and let N be a submodule of M . Then the following hold.
(a) If Prim (M) is a T1 topological space, then so is Prim (M/N).
(b) If Prim (M) is a Noetherian topological space, then so is Prim (M/N).
(c) Let Prim (M) be a Primary-spectral space. Then Prim (M/N) is a Primary-spectral space in the following cases:
(i) The subspace H := {Q ∈ Prim (M)|Q ⊇ N } of Prim (M) is closed;
(ii) R is a ring such that the intersection of every infinite collection of Primary ideals is of R zero (for example, when R is PID or one dimensional Noetherian domain).

Proof. (a) Follows from Proposition 3.19 and the fact that if N is a submodule of M , then Prim (M/N) = {Q/N |Q ∈ Prim (M), Q ⊇ N }.
(b) We define the map f : Prim (M/N) → H , where H := {Q ∈ Prim (M)|Q ⊇N } and f (Q/N) = Q for every Q/N ∈ Prim (M/N).
Clearly f is a bijection map.
Now let VM(E) ∩ H be a closed set of H , where E is a submodule of M . Then
f^{-1}(VM(E) ∩ H) = f^{-1}(VM(E)) ∩ f^{-1}(H)
= f^{-1}(VM(E)) ∩ Prim (M/N)
= f^{-1}(VM(E)) = VM(K/N),
where K =(E : M)M + N. So f : Prim (M/N) → H is a continuous map. It is easy to check that
f (VM(L/N)) = VM(L) ∩ H
for every submodule L of M containing N .
Hence f : Prim (M/N) → H is a closed map so that
Prim (M/N) is homeomorphic with H . Now since Prim (M) is Noetherian, H is Noetherian by Remak 2.2 (b). Hence Prim (M/N) is a Noetherian space as desired.
(c)(i) As in the proof part (b), we see that M

axR (M/N) is homeomorphic with H . Now the result follows by part (a), Remark 3.18, and Remark 2.2 (d).
(c)(ii) This follows from Proposition 3.2, Remark 3.18, Remark 2.2 (b), and part(a).

The next theorem is an important result about an R-module M for which Prim (M) is Primary-spectral. This result is obtained by combining Lemma 3.1, Proposition 3.2, Theorem 3.6, Proposition 3.19, Remark 2.2 (e), and Remark 3.18.

Theorem 3.22. Let M be a Primary-injective R-module. Then Prim (M) is a Primary-spectral topological space in each of the following cases:

(a) M is Primary-surjective;
(b) Im(φ) is quasi compact, where φ : Prim (M) → Prim ($\overline{R}$) is the natural map of Prim (M);
(c) AnnR (M) is a Primary ideal of R;
(d) Prim (M) is a finite set;
(e) Prim (R) is a finite set;
(f) Prim ($\overline{R}$) is Noetherian, in particular when R is Noetherian;
(g) The intersection of every infinite of Primary ideals of R is zero, in particular when R is PID or one dimensional Noetherian domain;
(h) The ascending chain condition for PJ-radical submodules of M holds.

An R-module M is multiplication if for every submodule N of M , there exits an ideal I of R such that N = IM (see [10]).

Corollary 3.23. Let M be an R-module. Then Prim (M) is a Primary-spectral topological space in each of the following cases:
(a) M is finitely generated and multiplication;
(b) M is primaryful and top; (We refer the reader to [10] and [11] for the concept and properties of top modules.
(c) M is primaryful and X -injective;
(d) M is X -injective and R is PID.
Proof. This follows from parts (a) and (g) of Theorem 3.22 and taking into account the following facts from [10, Theorem 3.5], [9, Proposition 3.3], [12, Theorem 2.2 ,3.3], and [13, Proposition 3.3 (c)],
Fact 1. Let denote the class of multiplication, top, X -injective, and Primary-injective modules respectively by Γ_1, Γ_2, Γ_3, and Γ_4, then
$\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \Gamma_4$.
Fact 2. If we denote the class of finitely generated, primaryful, and Primary-surjective modules respectively by Ω_1, Ω_2, and Ω_3, then $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3$.

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حول فضاء الموديولات الجزئية الابتدائية والموديول الاساسي المتكامل

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الخلاصة :

لأي موديول M على حلقة R ، Prim (M) هي مجموعة كل الموديولات الجزئية الابتدائية للموديول M في هذا البحث سوف نختبر العلاقة بين الخواص التبلوجية لفضاء الموديولات الجزئية الابتدائية والخصائص النظرية للموديول M بالإضافة الى انواع مختلفة لموديول M ، سوف نحصل على بعض الشروط التي تجعل Prim (M) متشاكلة مع فضاء المثاليات الابتدائية لبعض الحلقات .

الكلمات المفتاحية : (موديول جزئي ابتدائي – فضاء زارسكي – موديول ابتدائي شامل)